Polynomial diffeomorphisms of C^2: currents, equilibrium measure and hyperbolicity

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1. Introduction

For fixed real numbers a and c the formula

$$g(x, y) = (y, y^2 + c - ax) \tag{1.1}$$

defines a polynomial diffeomorphism of $\mathbf{R}^2$. The dynamical study of this family of maps was initiated by Hénon. These diffeomorphisms provide examples of simple maps with complicated dynamics and have been studied intensively. The formula (1.1) can also be used to define a diffeomorphism of C^2 where a and c are now permitted to be complex numbers. These "complex Hénon maps" also display complicated dynamics but have received much less attention. In this paper we study the dynamics of a large class of polynomial diffeomorphisms of C^2 of which complex Hénon maps are prototypical examples.

Friedland and Milnor [FM, Theorem 6.5] have classified polynomial diffeomorphisms of C^2 up to conjugation in the group of polynomial diffeomorphisms. They show that every polynomial automorphism of C^2 is conjugate to a map in one of two classes. The first class contains the affine mappings and the "elementary" mappings E, which include the "shears", which have the form $h(x, y) = (x + p(y), y)$. The second class consists of the finite compositions of "generalized Hénon" mappings, which have the form

$$g(x, y) = (y, p(y) - ax) \tag{1.2}$$

for a monic polynomial $p(x)$ of degree at least 2; this is the class $\mathscr{G}$ defined in (2.4). The dynamics of the elementary maps are quite simple (see [FM] for more information). In this paper we will consider the dynamics of maps $g \in \mathscr{G}$.

The following terminology is taken from [HO]. Let

$$K^{\pm} = \{p \in C^2 : \{g^{\pm n}(p) : n = 0, 1, 2, \ldots\} \text{ is bounded}\}$$

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be the set of points with bounded forward (backward) orbits. Of interest also are the sets $J^\pm = \partial K^\pm$, $K = K^+ \cap K^-$, and $J = J^+ \cap J^-$. This terminology is meant to suggest an analogy with the study of polynomial mappings of $\mathbb{C}$. For a 1-variable polynomial map $f : \mathbb{C} \to \mathbb{C}$ the set K_f of points with bounded forward orbits is the filled-in Julia set, and $J_f = \partial K_f$ is the Julia set itself.

Methods of potential theory have been effective in the theory of polynomial iteration in $\mathbb{C}$. The harmonic measure μ_{K_f} is closely connected to the dynamics of f. The harmonic measure is invariant under f, and Brolin [Br] showed that for any $z_0 \in \mathbb{C}$ the average of point masses on the points of $f^{-n}(z_0)$ converges to μ_{K_f}. The link between dynamics and potential theory is provided by the fact that the Green function G of K_f has a dynamical characterization. One objective of this paper is to introduce the methods of plurisubharmonic functions and positive currents to the study of dynamics in $\mathbb{C}^2$.

We let $G^\pm$ denote the plurisubharmonic (psh) Green functions in $\mathbb{C}^2$ for the sets $K^\pm$. As in the one dimensional case these functions have an alternate dynamical description. The operator dd^c plays a role in several complex variables analogous to the role of the Laplacian in one complex variable. We define the *stable* and *unstable* currents $\mu^\pm := dd^c G^\pm$, which are supported on $J^\pm$ and have the property that $g^* \mu^\pm = \deg(g)^{\pm 1} \mu^\pm$. We show that $\mu := \mu^+ \wedge \mu^-$ is a well-defined g-invariant measure and coincides with the psh equilibrium measure of J in $\mathbb{C}^2$. One of our basic results (Theorem 4.7) is the analogue of Brolin's Theorem: *If V is a smooth algebraic curve in $\mathbb{C}^2$, then the currents $\deg(g)^{-n}[g^{-n}(V)]$ converge to a constant times* μ^+.

Hyperbolicity is the natural generalization to two dimensions of the one dimensional property of expansiveness on the Julia set. A second objective of this paper is to study the diffeomorphisms $g \in \mathscr{G}$ which are hyperbolic. To date the only diffeomorphisms $g \in \mathscr{G}$ which have been understood in detail are in fact hyperbolic. Consideration of polynomial dynamics in the one-dimensional case supports the intuition that while hyperbolic maps are easier to study than general maps, they nevertheless reflect the general behavior in many respects.

For technical reasons we adopt as our definition of hyperbolicity for diffeomorphisms in $\mathscr{G}$ the existence of a hyperbolic splitting of the tangent bundle over J. We show (Corollary 6.12) that this implies a hyperbolic splitting over the nonwandering set. In fact we show that if there is a hyperbolic splitting of the tangent bundle over J then the nonwandering set is the union of J and finitely many hyperbolic periodic orbits.

With the hypothesis of hyperbolicity, the Stable Manifold Theorem implies that $J^\pm$ are foliated by Riemann surfaces. Let $\mathscr{F}^\pm$ denote these foliations. We prove that the stable and unstable currents $\mu^\pm$ induce transversal measures on the foliations $\mathscr{F}^\pm$. These transverse measures are precisely the transverse measures introduced by Ruelle and Sullivan for hyperbolic maps. We show that $\mu^\pm$ are in fact foliation cycles in the sense of Sullivan. Using the foliation cycle description of $\mu^\pm$ and the transversality of the stable and unstable manifolds we obtain the result that the support of μ is J. The fact that J is the support of a finite invariant measure has dynamical implications. Combining this fact with results from hyperbolic dynamics we conclude (Corollary 6.13) that *periodic points are dense in J.* In other words hyperbolicity of g implies Axiom A.

The spectral decomposition theorem of Smale allows us to write the nonwandering set of an Axiom A diffeomorphism as a union of basic sets. We show that *J is the unique infinite basic set* and *g is topologically mixing on J.* This allows us

to show (Corollary 7.9) that μ is Bowen measure, the unique g-invariant ergodic measure of maximal entropy. It also implies that g satisfies the no cycle condition for basic sets. We also conclude that *hyperbolic polynomial diffeomorphisms are structurally stable on* J (Theorem 7.7) which implies that the set of hyperbolic diffeomorphisms is open in $\mathcal{G}$, the space of polynomial diffeomorphisms.

The following difference between real and complex cases may be a source of confusion. Let us call a set Λ an attractor for g if Λ is compact, invariant and if there is a neighborhood U of Λ such that $\lim_{n \to \infty} \text{dist}(g^n(p), \Lambda) = 0$ for every $p \in U$. Hénon diffeomorphism g of $\mathbf{R}^2$ can have a "strange attractor" Λ (see [BC]). This set Λ is an attractor but it is not a union of sink orbits. If g is considered as a diffeomorphism of $\mathbf{C}^2$ then $\Lambda \subset \mathbf{R}^2 \subset \mathbf{C}^2$ is compact and invariant but it is *not* an attractor. That is to say that there is no open set U in $\mathbf{C}^2$ consisting of points attracted to Λ. In fact a normal families argument shows that any attractor for g in $\mathbf{C}^2$ is a union of periodic sink orbits.

2. Filtrations

In this Section we define the mappings we will study, and we consider the sets $K^{\pm}$ on which orbits stay bounded in forward and backward time. A *generalized Hénon map* is a map of the form

$$g(x, y) = (y, p(y) - ax), \qquad (2.1)$$

where $p(y)$ is a monic polynomial of degree $d \geq 2$ and $a \neq 0$. Such a map defines an automorphism of $\mathbf{C}^2$; its inverse is given by

$$g^{-1}(x, y) = ((p(x) - y)/a, x), \qquad (2.2)$$

and the derivative is

$$Dg = \begin{pmatrix} 0 & 1 \\ -a & p'(y) \end{pmatrix}. \qquad (2.3)$$

In this paper we will consider the space $\mathcal{G}$ of finite compositions of generalized complex Hénon mappings:

$$\mathcal{G} = \{g = g_m \circ g_{m-1} \circ \ldots \circ g_1 : g_j(x, y) = (y, p_j(y) - a_j x), a_j \in \mathbf{C}, a_j \neq 0\}. \qquad (2.4)$$

As was noted in the Introduction, an arbitrary polynomial automorphism f of $\mathbf{C}^2$ is either conjugate to a finite product of Hénon mappings or to an "elementary" mapping. As was shown in [FM] any map $g \in \mathcal{G}$ is conjugate to a map in which each p_j is monic. We use the notation $p(y) = y^{d_j} + \ldots$, so that d_j is the degree of p_j, and $p(y) = p_m \circ \ldots \circ p_1(y)$ is the composition. We will also write $a = a_m \ldots a_1$; thus a is the complex jacobian determinant of g and is constant. g has multi-degree $(d_1, \ldots, d_m)$ and total degree $d = d_1 \ldots d_m$. The space $\mathcal{G}$ has a natural stratification into mappings of multi-degree $(d_1, \ldots, d_m)$, i.e. $\mathcal{G} = \cup \mathcal{G}(d_1, \ldots, d_m)$. For the rest of this Section, we will work with a fixed $g \in \mathcal{G}$, using the notation above.

Lemma 2.1 ([FM]). *For every generalized Hénon map*

$$g: (x, y) \mapsto (y, z) = (y, p(y) - ax)$$

there exists a constant $R > 0$ so that $|y| > R$ implies that either $|z| > |y|$ or $|x| > |y|$ or both.

Now choose R large enough that Lemma 2.1 holds for each g_j. Let

$$V^- = \{(x, y): |y| > R \text{ and } |y| > |x|\}$$
$$V^+ = \{(x, y): |x| > R \text{ and } |y| < |x|\}$$
$$V = \{(x, y): |x| \leq R \text{ and } |y| \leq R\}.$$

These domains will be useful because of their "trapping" properties, which are given in the following.

Lemma 2.2. *We have:*

(i) $g(V^-) \subset V^-$.

(ii) $g(V^- \cup V) \subset V^- \cup V$.

(iii) $g^{-1}(V^+) \subset V^+$.

(iv) $g^{-1}(V^+ \cup V) \subset V^+ \cup V$.

Proof. (i). If (x, y) is an element of V^-, then $|y| > |R|$ and $|y| > |x|$. By Lemma 2.1, $|z| > |y|$, and since $|y| > R$, we have $|z| > R$. This implies that $g(x, y) = (y, z)$ is in V^-.

(ii). By (i) it suffices to consider the case when (x, y) is an element of V. We will show that $g(x, y) = (y, z)$ is in $V \cup V^-$. Consider two cases. If $|z| \leq R$, then $(y, z) \in V$ since $|y| \leq R$. If $|z| > R$, then since $|y| \leq R$ we have $|z| > |y|$ so $(y, z) \in V^-$.

(iii). Let (y, z) be an element of V^+. We want to show that $g^{-1}(y, z) = (x, y)$ is in V^+. Since $|y| > R$ and $|y| > |z|$ Lemma 2.1 gives $|x| > |y|$. Since $|y| > R$ and $|x| > |y|$, we have $|x| > R$. This implies that $(x, y) \in V^+$.

(iv). By (iii) it suffices to consider the case when (y, z) is an element of V. We will show that $(x, y) \in V^+ \cup V$. If $|x| \leq R$, then since $|y| \leq R$ we have $(x, y) \in V$. If $|x| > R$, then since $|y| < R$ we have $|x| > |y|$, and so $(x, y) \in V^+$. This completes the proof.

By Lemma 2.2, $K^+ \subset V \cup V^+, K^- \subset V \cup V^-$, and $K \subset V$.

We let π_1 and π_2 denote projection to the first and second coordinates, respectively, so that $\pi_1 g_j = y$ and $\pi_2 g_j = p_j(y) - a_j x$.

Lemma 2.3. *For $\varepsilon > 0$ there exists R such that $g(V^-) \subset V^- \cap \{\varepsilon|y| > |x|\}$. In particular, $V^- \cap K^+ = \varnothing$ and $K \subset V$.*

Proof. Consider first $g_j(x, y)$ for $(x, y) \in V^-$. Since $|y| > |x|$ and $|y| > R$, we see that $c_j y^{d_j}$ dominates both the lower powers of y in p_j and $a_j x$ if R is large. Thus the Lemma holds for g_j. Clearly, then, it holds for the composition $g = g_m \circ \ldots \circ g_1$.

For a set $X \subset \mathbf{C}^2$, we define the *stable* and *unstable sets* W^s and W^u of X as

$$W^s(X) = \{q \in \mathbf{C}^2: \text{dist}(g^n(q), g^n(X)) \to 0 \text{ as } n \to +\infty\} \tag{2.5}$$

$$W^u(X) = \{q \in \mathbf{C}^2: \text{dist}(g^n(q), g^n(X)) \to 0 \text{ as } n \to -\infty\}.$$

Lemma 2.4. *The following hold*:

(i) $V^- \subset g^{-1} V^- \subset g^{-2} V^- \subset \ldots$, *and* $\cup g^{-n} V^- = \mathbb{C}^2 - K^+$.

(ii) $V^+ \subset g V^+ \subset g^2 V^+ \subset \ldots$, *and* $\cup g^n V^+ = \mathbb{C}^2 - K^-$.

(iii) *Let* $V_n = g^n V \cap g^{-n} V$. *Then* $V_1 \supset V_2 \supset \ldots$, *and* $\cap V_n = K$.

(iv) $W^s(K) = K^+$.

(v) $W^u(K) = K^-$.

(vi) $\{g^n\}$ *is a normal family on* int K^+.

Proof. (i) $g^{-n} V^- \subset g^{-n-1} V^-$ by Lemma 2.2. By Lemma 2.3 and the g-invariance of both sides of the equation in (i), we have $\subset$. On the other hand, if $q \notin K^+$, then we must show that $g^n q \in V^-$ for some large n. If this is not the case, then the sequence $\{g^n q\}$ is a subset of $V \cup V^+$, and it has a subsequence that tends to infinity. Thus we must have a subsequence of $\{|\pi_1 g^n q|\}$ tending to infinity. Let us "expand" this sequence by putting the terms $\{g_1 g^n q, g_2 g_1 g^n q, \ldots, g_{m-1} \cdots g_2 g_1 g^n q\}$ between $g^n q$ and $g^{n+1} q$. Thus a subsequence of this expanded sequence must tend to infinity. In particular, we have, for some $1 \le k \le m, |\pi_1 q'| < |\pi_1 g_k(q')|$. This is not possible, however, since $q' \in V^+$, and $\pi_1 g_k(x, y) = y$.

We prove (ii) by applying the same argument to g^{-1}.

For (iii) we show that $V_n \subset V_{n-1}$. We consider the inclusions $g^n V \subset g^{n-1} V \cup V^-$ and $g^{-n} V \subset g^{-n-1} V \cup V^+$. For $n = 1$, this is a consequence of (ii), (iv) of Lemma 2.2, and the result follows by induction. Since $K \subset V$, we see that $K \subset \cap V_n$. On the other hand, the orbits of points in $\bigcap V_n$ are clearly bounded in forward time, so the two sets coincide.

For (iv), we note that $W^s(K) \subset K^+$ is obvious. To prove the reverse inequality, it suffices to show that if $U \subset K^+$ is any neighborhood of K in K^+ and $q \in K^+$, then there is an M such that $g^n q \in U$ for $n > M$. For any $P \in V \cap (K^+ - U)$ there is an m such that $g^{-m} P \notin V$. The set $V \cap (K^+ - U)$ is compact, so there is a fixed number N such that this holds for $m > N$. Now choose N_0 such that $g^{N_0} \in V$. It follows that if $n > N + N_0$ then $g^n \in U$.

The statement (v) follows if we replace g by g^{-1}. For (iv), we let $\omega \subset \subset \mathbb{C}^2$ be open and bounded. In (iv) we showed that for any neighborhood U of K inside K^+ there is an n_0 such that $g^n(\omega \cap K^+) \subset U$ for $n \ge n_0$. In particular, $g^n|_{\omega \cap K^+}$ is bounded.

Lemma 2.5. *For $\delta > 0$ there exists R large enough that for $(x, y) \in V^-$*

$$(1 - \delta)|y^d| < |\pi_2 g(x, y)| < (1 + \delta)|y^d|. \tag{2.6}$$

Proof. First we consider the case where $g = g_j$. In this case, for given $\delta_j > 0$ we may choose R such that for $(x, y) \in V^-$

$$(1 - \delta_j)|y^{d_j}| < |\pi_2 g_j(x, y)| < (1 + \delta_j)|y^{d_j}|.$$

Applying this once, we get

$$|\pi_2 g_2 \circ g_1(x, y)| = |\pi_2 g_2((\pi_1 g_1(x, y), \pi_2 g_2(x, y))| < (1 + \delta_2)|\pi_2 g_1(x, y)|^{d_2}$$
$$< (1 + \delta_2)((1 + \delta_1)|y^{d_1}|)^{d_2}.$$

By induction, then, we have

$$|\pi_2 g(x,y)| < (1+\delta_m)(1+\delta_{m-1})^{d_m}(1+\delta_{m-2})^{d_m d_{m-1}} \ldots (1+\delta_1)^{d_m \cdots d_2}|y^d| .$$

Thus the Lemma follows by taking $\delta_m, \ldots, \delta_1$ sufficiently small.

We let $\{(x_n, y_n), n = 1, 2, 3, \ldots\}$ denote the forward orbit of $(x, y) \in \mathbf{C}^2$ under g.

Corollary 2.6. *If R is sufficiently large, then for $(x, y) \in V^-$, we have*

$$|(1-\delta)|^{d^{n-1}/d_m}|y|^{d^n/d_m} < |x_n| < |(1+\delta)|^{d^{n-1}/d_m}|y|^{d^n/d_m}$$

and

$$|(1+\delta)|^{d^{n-1}}|y|^{d^n} < |y_n| < |(1+\delta)|^{d^{n-1}}|y|^{d^n} .$$

A sequence $\{q_n\}$ is said to be an *ε-orbit* if $dist(q_{n+1}, g(q_n)) < \varepsilon$ holds for all $n = 1, 2, 3, \ldots$. A point q is *chain recurrent* if for any $\varepsilon > 0$ there is an ε-almost orbit $\{q_n\}$ with $q = q_1 = q_{1+jN}$ for some N and all $j = 1, 2, \ldots$. The set of chain recurrent points is denoted by $R(g)$.

Corollary 2.7. $R(g) \subset K$.

Proof. If $q \notin K^+$, then $g^M q \in V^-$ for some large M. Thus $\varepsilon > 0$ may be chosen small enough that any ε orbit enters V^- after M steps. By Corollary 2.6, it is clear that an ε orbit that enters V^- can never leave. (If R is chosen large, then we may take $\varepsilon = 1$.) Thus q cannot be chain recurrent.

A similar argument holds if $q \notin K^-$.

3. Currents on $K^\pm$

The results in this Section were developed with N. Sibony. We study the functions $G^\pm$, which are basic for the potential-theoretic approach, as are the currents $\mu^\pm := dd^c G^\pm$. We will show that the wedge product of these currents may be defined and gives the invariant measure $\mu = \mu^+ \wedge \mu^-$.

As in [Br] and [H], we define the functions

$$G^+(x,y) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|g^n(x,y)\| , \tag{3.1}$$

$$G^-(x,y) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|g^{-n}(x,y)\| , \tag{3.2}$$

which give the rate of escape of an orbit to infinity in positive and negative time. To simplify our treatment, we will discuss only G^+, since the analogous statements for G^- will be apparent. Here $\| \ \|$ denotes any norm on $\mathbf{C}^2$; it is evident that the definition of G^+ is independent of the choice of norm, and in fact by Corollary 2.6

$$G^+(x,y) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |y_n| . \tag{3.3}$$

Also by Corollary 2.6, we see that

$$G^+(x,y) = \log^+|y| + O(1) , \tag{3.4}$$

where the $O(1)$ holds for $|x| < R$ and y arbitrary, and a similar argument (which we omit) shows that

$$G^+(x, y) = (1/d)\log^+|x| + O(1) , \tag{3.5}$$

for $|y| < R$. It is immediate also that

$$\frac{1}{d}G^+ \circ g = G^+ . \tag{3.6}$$

When we want to let g depend on a parameter, we let $\mathscr{U} \subset \mathscr{G}(d_1, \ldots, d_m)$ be an open set, and we write $\mathscr{U} \ni (a, c) \mapsto g_{a,c}$, where we let $a = (a_1, \ldots, a_m)$, and in each g_j, the correspondence $c \mapsto p_{j,c} = y^{d_j} + \ldots$ depends holomorphically on c. We will use the notation $K_{a,c}$, etc.

Lemma 3.1. *The set $K_{a,c}$ depends semicontinuously on the variables a and c in the following sense: for $\varepsilon > 0$, $K_{a,c}$ lies in an ε-neighborhood of K_{a_0,c_0} if (a, c) is sufficiently close to (a_0, c_0).*

Proof. Let the domain V be given as above, and let a_0 and c_0 be fixed. For any $\varepsilon > 0$, there exists a number n with the property that if $dist(P, K) > \varepsilon$, then either $g^n(P)$ or $g^{-n}(P)$ lies outside of V. Since n must have this same property for (a, c) sufficiently close to (a_0, c_0) it follows that $K_{a,c}$ must lie within a ε-neighborhood of K.

Lemma 3.2. *If R is sufficiently large, the sequence defining G^+ converges uniformly on V^-. If $g_{a,c}$ depends holomorphically on $(a, c) \in \mathscr{U}$, then the limit is uniform also on $V^- \times \omega$ for any relatively compact $\omega \subset \mathscr{U}$, and $G_{a,c}^+(x, y)$ is pluriharmonic on $V^- \times \omega$.*

Proof. Since we may take $R > 1$, we have $|y| > 1$ for $(x, y) \in V^-$, so $\log^+|y_n| = \log|y_n|$. Thus

$$d^{-n}\log^+|y_n| - d^{n+1}\log^+|y_{n+1}| = d^{-n-1}\log|y_n^d/y_{n+1}| \leq d^{-n-1}\log|(1+\delta)c| .$$

It follows that this sequence converges uniformly on V^-. Clearly, the R defining V^- may be chosen large enough to work for all $(a, c) \in \omega$, so the convergence is uniform on this set, too. The functions $\log|y_n|$ are pluriharmonic, so it follows that the limit is pluriharmonic.

Proposition 3.3. $\{G^+ > 0\} = \mathbf{C}^2 - K^+$. *The limit in (3.3) is taken uniformly on compact subsets of $\mathbf{C}^2 \times \mathscr{U}$ and the function G^+ is pluriharmonic on $W = \{(x, y, a, c) \in \mathbf{C}^2 \times \mathscr{U} : G^+ > 0\}$, (in the variables (x, y, a, c)).*

Proof. It is clear that $G^+ = 0$ on K^+. Conversely, by Corollary 2.6, it follows that $G^+(x, y) > \log((|1 - \delta)c|^{1/d}|y|) > 0$ holds on V^-. By Lemma 2.3 and (3.4), we see that $G^+ > 0$ on $\mathbf{C}^2 - K^+$. By Proposition 2.9, we see that the limit (3.3) is taken uniformly on compact subsets of W. Thus the function is pluriharmonic there.

For any function h, we define the *upper semicontinuous (usc) regularization* h^* by

$$h^*(z) = \limsup_{\zeta \to z} h(\zeta) .$$

The function h is usc iff $h = h^*$. Let us recall that a function u on $\mathbf{C}^n$ is *plurisubharmonic (psh)* if it is usc on $\mathbf{C}^n$ and subharmonic on each complex line in $\mathbf{C}^n$. It is evident that G^{+*} is psh on $\mathbf{C}^2 \times \mathscr{G}$, and $G^{+*} = G^+$ on the set W.

Proposition 3.4. G^+ *is continuous on* $\mathbf{C}^2 \times \mathcal{G}$, *and the limit in* (3.3) *is taken uniformly on compact subsets of* $\mathbf{C}^2 \times \mathcal{G}$.

Proof. Since G^+ is pluriharmonic on the set W of Proposition 3.3, it is continuous there. It suffices to show that $G^{+*} = 0$ on K^+. If q_0 is any point of K^+ then it has a bounded orbit under g. Since G^{+*} is usc, it is bounded above by a number M on this orbit. On the other hand, G^{+*} satisfies (3.6) on this orbit, and so $G^{+*} \circ g^n(q_0) \leqq M d^{-n}$, which shows that G^+ is continuous.

To see that the limit in (3.3) is taken uniformly on compact subsets, we recall from Proposition 3.3 that the convergence is uniform on compact subsets of $\mathbf{C}^2 - K^+$. We now note that the functions are all nonnegative, so we can use the continuity of the limit and the maximum principle to obtain the uniform convergence.

Although the set K^+ is not compact, it is polynomially convex in the sense that if $X \subset \mathbf{C}^2$ is compact and polynomially convex, then $X \cap K^+$ is polynomially convex. This is seen, because by Lemma 2.4, there is a constant C such that

$$X \cap K^+ = X \cap \{q \in \mathbf{C}^2 : \|g^n(q)\| \leqq C \text{ for all } n = 1, 2, \ldots \} .$$

A set S is *pseudoconcave* if it is locally the complement of a domain of holomorphy. It is interesting that the complement of K^+ is also holomorphically convex.

Corollary 3.5. *The set* K^+ *is pseudoconcave.*

Proof. The function $-G^+$ is a psh exhaustion of the complement of K^+.

The current $\mu^+ = dd^c G^+$ is a positive, d-closed $(1,1)$-current supported on J^+, and thus by a Theorem of Lelong [L] it is representable by integration, i.e. it may be evaluated on continuous or even Borel-measurable forms with compact support. Sometimes it is convenient to consider μ^+ as a $(1,1)$-form whose coefficients are Borel measures. Lelong [L] is a good general reference for properties of positive currents.

Lemma 3.6. supp $\mu^+ = J^+$.

Proof. To see that the support of μ^+ is all of J^+, we suppose that U is an open set which intersects J^+. If μ^+ puts no mass on U, then G^+ is pluriharmonic there. But G^+ is not constant and has a local minimum of J^+, which is a contradiction.

Now we proceed to define the wedge product of $\mu^{\pm}$. If h_1, h_2 are continuous, psh functions, then $dd^c h_j$ is representable by integration for $j = 1, 2$. In general, multiplication of currents by continuous functions is not possible, but in this case $dd^c h_2$ is a current of order 0, so $h_1 dd^c h_2$ is well defined. The identity

$$dd^c h_1 \wedge dd^c h_2 = dd^c(h_1 dd^c h_2) \tag{3.7}$$

holds for smooth functions; and in general, we may use it to define the left hand side as a current in terms of the right. By the positivity of the $(1,1)$-current $dd^c h_j$, we see that the left hand side must be a nonnegative current. Again by the Theorem of Lelong, it and the right hand side are represented by integration.

We may show that this definition is justified in the sense that it extends the usual definition when applied to smooth, psh functions. To do this, we take a standard sequence of psh smoothings $\{h_\sigma^j\}$ which converge uniformly to h_σ. (This may be done by the usual operation of convolution with respect to a radial

smoothing kernel.) This implies that the sequence of currents $\{dd^c h_\sigma^j\}$ converges to $dd^c h_\sigma$ in the sense of currents representable by integration (which is equivalent to the weak convergence of measures), and so $\{h_1^j dd^c h_2^j\}$ converges in the sense of currents to $h_1 dd^c h_2$. Thus the sequence $\{dd^c h_1^j \wedge dd^c h_2^j\}$ converges to $dd^c h_1 \wedge dd^c h_2$.

As an example, we let $G_\varepsilon^+ = \max\{G^+, \varepsilon\}$. Then the current $dd^c G_\varepsilon^+$ is the $(1,1)$ current dual to $dG^+ \wedge d^c G^+ \wedge S_\varepsilon$, where S_ε denotes the surface area measure of the surface $\{G^+ = \varepsilon\}$. From this, we may see that $\mu_\varepsilon^+ \wedge \mu_\varepsilon^+ = 0$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we conclude that the current defined by (3.7) is

$$(dd^c G^+)^2 = (\mu^+)^2 = 0 . \tag{3.8}$$

We now define

$$\mu_K := \mu^+ \wedge \mu^-$$

by (3.7). Let us note here that the operation of taking wedge products of currents of the form $dd^c w$ is even well defined if w is psh and merely locally bounded (but not required to be continuous) (cf. [BT2]).

It follows from equation (3.6) that

$$\frac{1}{d} g^* \mu^+ = \mu^+, \quad \text{and} \quad g^* \mu^- = \frac{1}{d} \mu^- , \tag{3.9}$$

so we are led to the following observation.

Proposition 3.7. *The current μ_K is g-invariant.*

Proof. $g^*(\mu^+ \wedge \mu^-) = g^* \mu^+ \wedge g^* \mu^- = (d)\mu^+ \wedge (1/d)\mu^- = \mu^+ \wedge \mu^-$.

Let us recall the class of functions of logarithmic growth on $\mathbf{C}^n$

$$\mathscr{L} = \{u \text{ psh on } \mathbf{C}^n : u(z) \leq \log(|z| + 1) + O(1)\} .$$

By (3.4), $G^\pm \in \mathscr{L}$. The psh *Green function* of a set S is defined as L_S^*, where

$$L_S(z) = \sup\{u(z) : u \in \mathscr{L}, u \leq 0 \text{ on } S\} .$$

Proposition 3.8. *G^+ is the psh Green function of the set K^+ and of the set J^+.*

Proof. It follows from Corollary 2.6 that for fixed y, $G^+(x, y) = \log^+ |x| + O(1)$. Thus if $u \in \mathscr{L}$ and $u \leq 0$ on K^+, then for fixed y, we have $u(x, y) \leq G^+(x, y)$ by the maximum principle. Thus $L_{K^+} \leq G^+$, and the reverse inequality follows since $G^+ \in \mathscr{L}$.

We also consider the restricted class

$$\mathscr{L}_+ = \{u \text{ psh on } \mathbf{C}^n : u(z) = \log(|z| + 1) + O(1)\} .$$

If K is compact, then $L_K \in \mathscr{L}_+$. The complex Monge-Ampère operator $(dd^c)^2$ is well defined on $\mathscr{L}_+$, and the measure

$$\lambda_K := (dd^c L_K^*)^2 \tag{3.10}$$

is then well defined for any compact set K and is called the *complex equilibrium measure* of K. λ_K is supported on K. Conversely, the support of $(dd^c)^2$ may be used to characterize the Green function in the following sense (see [BT2]): if $u \in \mathscr{L}_+$ satisfies $u = 0$ on K and if $(dd^c u)^2$ is supported on K, then $u = L_K$.

Proposition 3.9. $L_K = \max\{G^+, G^-\}$.

Proof. Following a calculation in [BT1], we know that $(dd^c \max(u_1, u_2))^2 = 0$ for any pluriharmonic functions u_1 and u_2. Thus we see that if we set $u := \max(G^+, G^-)$, then $(dd^c u)^2$ is supported on K. Further, $u = 0$ holds on K, and it follows from Corollary 2.6 that

$$\max\{G^+, G^-\} = L_K + O(1) .$$

The Proposition then follows by the remarks above.

Since we know that the functions $G^\pm$ are continuous, it follows that K is *regular* in sense of psh functions. Thus the sets K and $K^\pm$ are not thin in the sense of psh functions, and in particular they are not locally pluri-polar at any of their points.

Corollary 3.10. *K is a perfect set.*

Although the Green function L_K is not invariant, the equilibrium measure λ_K is.

Proposition 3.11. *The equilibrium measure is given by $\lambda_K = \mu_K$ and is a measure of total mass $4\pi^2$ supported on J.*

Proof. We note that the total mass of any equilibrium measure of a compact subset of $\mathbf{C}^2$ is $4\pi^2$. It is also clear that the support of $\mu^\pm$ is contained in $J^\pm$, and so the support of μ_K is contained in J. Thus it remains to show that $\lambda_K = \mu_K$. To do this, we note that

$$\lambda_K = \lim_{\varepsilon \to 0} (dd^c \max(L_K, \varepsilon))^2 .$$

Further, if we set $G_\varepsilon^\pm = \max(G^\pm, \varepsilon)$, then we have

$$(dd^c \max(L_K, \varepsilon))^2 = (dd^c \max(G^+, G^-, \varepsilon))^2$$
$$= dd^c G_\varepsilon^+ \wedge dd^c G_\varepsilon^- .$$

This last identity follows because the functions $G^\pm$ are pluriharmonic where they are positive, and $(dd^c)^2$ of the maximum of two pluriharmonic functions is zero. Thus, to compute $(dd^c)^2$ of the maximum of three pluriharmonic functions we may wedge the dd^c's of the maximum of two pairs (this calculation is done in [BT1]). Now letting ε tend to zero, we see that $dd^c G_\varepsilon^\pm$ converges to $\mu^\pm$, and thus the right hand side of the second equation converges to μ_K, which completes the proof.

We note that by [NZ], [BT2] the precise support of λ_K is the Šilov boundary of K. In other words, the support S of λ_K is the smallest closed subset of K with the property that

$$\sup_{z \in S} |P(z)| = \sup_{z \in K} |P(z)|$$

for all polynomials $P(z_1, z_2)$. It also follows that $L_S = L_K$, and that S is regular. Thus S is not locally polar at any of its points.

If $f : X \to X$ is a continuous mapping, we say that a point $p \in X$ is *wandering* if there is an open set $U \subset X$ containing p such that U is disjoint from $f^n U$ for all $n > 1$. The *nonwandering set* of f, written $\Omega(f)$, is the set of all points which are not wandering. It is easily seen that if p is in the support of a finite, invariant measure, then p belongs to the nonwandering set.

Corollary 3.12. $\operatorname{supp} \mu_J \subset \Omega(g|J)$.

4. Convergence to $\mu^\pm$

In this Section, we give a more general way of defining G^+ (Proposition 4.2). This gives a version (Theorem 4.7) of Brolin's Theorem: the successive preimages of an algebraic curve converge to a constant times μ^+.

Lemma 4.1. *Let $h(x, y)$ be a polynomial such that cy^k is the unique term of highest total degree. Then $G^+(q) = \lim_{n \to \infty} k^{-1} d^{-n} \log^+ |h(g^n q)|$, and the convergence is uniform on compact subsets of $\mathbf{C}^2$.*

Proof. We may write $h(x, y) = cy^k + h_1(x, y)$ where h_1 has total degree $\leq k - 1$. Then

$$\log |h(x_n, y_n)| = \log |cy_n^k| + \log |1 + y_n^{-k} h_1(x_n, y_n)/c| \ .$$

It follows from Corollary 2.6 that the sequence converges uniformly on V^-. The Lemma then follows from Lemma 2.4.

Proposition 4.2. *Let $h(x, y)$ be a nonzero polynomial, and let $g \in \mathscr{G}$ be given. Then there exists n_0 such that $h \circ g^{n_0}$ has cy^k as the unique term of highest total degree. Thus*

$$G^+(q) = \lim_{n \to \infty} \frac{1}{kd^{n-n_0}} \log^+ |hg^n q|$$

with uniform convergence on compact subsets of $\mathbf{C}^2$.

The proof will follow from a series of four lemmas. In each of the Lemmas we will assume that g_1 is a generalized Hénon mapping of the form (2.1). Let $\mathbf{C}[x, y]$ denote the ring of polynomials in two variables. A *valuation on* $\mathbf{C}[x, y]$ is a function $\lambda: \mathbf{C}[x, y] \to \mathbf{N}$ such that

$$\lambda(p + q) \leq \max(\lambda(p), \lambda(q))$$

$$\lambda(pq) = \lambda(p) + \lambda(q) \ .$$

Let $d_1, \ldots, d_m > 1$ be positive integers. For $\sigma = 1, 2, \ldots, m$ we define the valuations: $\lambda_\sigma(x^i y^j) = i + d_\sigma j$ and $\mu_\sigma(x^i y^j) = d_\sigma i + j$. ($\lambda_\sigma$ and μ_σ extend uniquely to valuations on all polynomials.) In the following Lemmas it will be convenient to work only $\sigma = 1, 2$, but the application to the proof of Proposition 4.2 will be clear.

Lemma 4.3. *Let $r(x, y) = x^{m - d_1 i} y^i$ be a monomial of λ_1-weight m. Then the composition $r \circ g_1(x, y)$ is a polynomial of μ_1-weight m, and any monomial of μ_1-weight m in $r \circ g_1$ has the form $\binom{i}{j}(-a)^j x^j y^{m - d_1 j}$.*

Proof. Consider $r \circ g_1(x, y) = y^{m - d_1 i}(p_1(y) - a_1 x)^i$. The term $(p_1(y) - a_1 x)$ has μ_1-weight d_1, and y has μ_1-weight 1, so $r \circ g_1$ has μ_1-weight m. To evaluate the coefficients of the highest μ_1-weight terms in $r \circ g_1$, it suffices to replace $p_1(y)$ by its highest weight term y_1^d in the expansion for $r \circ g_1$:

$$(y_1^d - a_1 x)^i y^{m - d_1 i} = \left(\sum_{j=0}^i \binom{i}{j} y^{(i-j)d_1} (-a_1 x)^j \right) y^{m - d_1 i}$$

$$= \sum_{j=0}^i \binom{i}{j} (-a_1)^j x^j y^{m - d_1 j} \ .$$

Lemma 4.4. *If $0 < l_1 < \ldots < l_J$ are positive integers, and $a \neq 0$, then the matrix*
$A_{i,j} = \begin{pmatrix} l_i \\ j-1 \end{pmatrix}(-a)^{j-1}, 1 \leq i, j \leq J$ *is nonsingular.*

Proof. We will perform column operations to simplify the matrix A. Multiply
the column j by $(-a)^{-j+1}$ to get $A_{i,j} = \begin{pmatrix} l_i \\ j-1 \end{pmatrix}$. Now $\begin{pmatrix} l_i \\ j-1 \end{pmatrix} = l_i!/(j-1)!$
$(l_i - j + 1)!$. Multiplying the j-th column by $(j-1)!$ gives $A_{i,j} = l_i!/(l_i - j + 1)!$
$= l_i(l_i - 1) \ldots (l_i - j + 1)$. We claim that A is equivalent to the matrix B where
$B_{i,j} = (l_i)^{j-1}$. We prove this the induction on the columns. The first two columns of
A are the same as those of B. Assume that the matrices agree on columns 1 through
t. Now $A_{i,t+1} = P_t(l_i)$ where $P_t(X) = \prod_{s=0}^{t}(X - s) = \sum_{r=0}^{t} a_r X^r$. The polynomial
P is monic and of degree t. Subtracting a_r times column r from the column $t + 1$ for
$r = 1, \ldots, t$ gives the desired result. The matrix B is the Vandermonde matrix and
is invertible.

Lemma 4.5. *Let q be a polynomial with $\lambda_1(q) = m_1$ and suppose that q has exactly k
monomials of maximal weight. Then there exists an $s, 0 \leq s < k$ such that the
coefficient of $x^s y^{d_1(m_1 - s)}$ in $q \circ g_1$ is nonzero. Further, the λ_2-weight of $q \circ g_1$ is at least
$k - 1 + d_1 d_2(m - k + 1)$.*

Proof. Let $c_j x^{m - d_1 l_j} y^{l_j}, j = 1, \ldots, k$ be the nonzero monomials of maximal
λ_1-weight in q. If the first k monomials of μ_1-weight m_1 in $q \circ g_1$ vanish, then

$$\sum_{i=1}^{k} \begin{pmatrix} l_i \\ j \end{pmatrix}(-a)^j c^i = 0 \text{ for } j = 1, \ldots, k .$$

By Lemma 4.4, however, this is not possible since the c_i's do not all vanish.

The λ_2-weight of $x^s y^{d_1(m_1 - s)}$ is $s + d_1 d_2(m_1 - s)$, which is decreasing in s and
thus is $\geq k - 1 + d_1 d_2(m_1 - k + 1)$.

Lemma 4.6. *With the notation of Lemma 4.5, we conclude that there are at most
$1 + (k - 1)/d_2$ monomials of maximal λ_2-weight in $q \circ g_1$.*

Proof. By Lemma 4.3, the μ_1-weight of $q \circ g_1$ is $\leq m$. Thus $q \circ g_1 = \sum_{(s,t) \in T} c_{s,t} x^s y^t$,
where T is the set of lattice points $T = \{(s, t): s, t \geq 0, d_1 s + t \leq m\}$. This forms a
triangular region in the first quadrant.

The monomials $x^s y^t$ that can appear with λ_2-weight $\hat{\lambda}_2$ in $q \circ g_1$ correspond to
the lattice points of T which lie on the line $L = \{s + d_2 t = \hat{\lambda}_2\}$. If we take $\hat{\lambda}_2$ to be
the λ_2-weight of $q \circ g_1$, then by Lemma 4.5, $T \cap L \subset \{s = 0, \ldots, k - 1\}$. However,
the number of lattice points which can lie in $\{s = 0, \ldots, k - 1\} \cap L$ is bounded by
$1 + (k - 1)/d_2$, which completes the proof.

Proof of Proposition 4.2. We write $g = g_m \circ \ldots \circ g_1$, and we consider the sequence
of iterates $h, h \circ g_1, (h \circ g_1) \circ g_2 = h \circ (g_2 \circ g_1)$, etc. We let the valuations $\lambda_1, \ldots, \lambda_m$,
$\mu_1, \ldots, \mu_m$ be defined as above using $d_1, \ldots, d_m$ from the definition of $g_1, \ldots, g_m$.

Now let us suppose that h has exactly k terms of maximal λ_1-weight. Then by
Lemma 4.6, $h \circ g_1$ has at most $1 + (k - 1)/d_2$ terms of maximal λ_2-weight. Applying
Lemma 4.6 now to $q = h \circ g_1$, we see that $h \circ (g_2 \circ g_1)$ has at most $1 + (k - 1)/d_1 d_2$
terms of maximal λ_3-weight. Continuing this way, we reach the point where
$h \circ (g_{m-1} \circ \ldots \circ g_1 \circ g^n)$ has only one term of maximal λ_m-weight D. It follows from
Lemma 4.5 that the λ_1-weight of $h \circ g^n$ is $d_1 d_m D$. Further, by Lemma 4.3, the μ_m-

weight of $h \circ g^{n+1}$ is $d_m D$. Now $y^{d_m D}$ is the only monomial which has both μ_m-weight and λ_1-weight $d_m D$. Thus $c y^{d_m D}$ is also the unique monomial of $h \circ g^{n+1}$ with highest total degree.

The last statement follows by applying Lemma 4.1 to $h \circ g^{n_0}$.

Let us consider the convergence of iterates of a manifold M in the sense of currents. We use the notation $[M]$ for the current of integration over M. If $M = \{y = 0\}$ denotes the y-axis, then $g_*^{-n}[M] = [\{\pi_2 g^n = 0\}]$. More generally, if $h = h(x, y)$ is any polynomial, and if $M = \{h = 0\}$, then $g_*^{-n}[M] = [\{h(g^n) = 0\}]$.

In the case of the x-axis, this current is also given by the Lelong-Poincaré formula $[\{x = 0\}] = \frac{1}{2\pi} dd^c \log|x|$. Since $g^{-1}[M]$ is a current which projects to the x-axis with multiplicity d, we must divide by d at each iteration of preserve the total mass. In general, if $M = \{h = 0\}$, then $\frac{1}{2\pi} dd^c \log|h| = [M]$ only if $dh \neq 0$ on a dense subset of M. (Otherwise, it is necessary to introduce integer multiplicities of the irreducible components of M, corresponding to the order of vanishing of h.)

Theorem 4.7. *Let M be a nonsingular algebraic hypersurface. Then the sequence of iterates $d^{-n} g_*^{-n}[M]$ converges to the current $c\mu^+$ for some constant $c \gg 0$.*

Proof. Let h be a polynomial such that $M = \{h = 0\}$ and $dh \neq 0$ on M. Let n_0 and k be as in Proposition 4.2. Let us set $G_n := k^{-1} d^{n_0-n} \log|h \circ g^n|$.

Our first step is to show that G_n converges to G^+ on $C^2 - \partial K^+$. By Proposition 4.2, we know that $\lim_{n \to \infty} G_n = G^+$, uniformly on compact subsets of the set $C^2 - K^+$. Next we consider the behavior of G_n on int K^+. Let $\mathscr{B}$ denote a bounded, open subset of int K^+. By Lemma 2.4 the iterates $g^n(\mathscr{B})$ for $n \geq 0$ remain inside the polydisk $\Delta^2(R)$ for R sufficiently large. We will show that

$$\lim_{n \to \infty} \int_{\mathscr{B}} |G_n| = 0 . \tag{4.1}$$

Since $|h|$ is bounded above on $\Delta^2(R)$, we may assume that $G_n \leq 0$ on $\Delta^2(R)$. Thus it suffices to show

$$\lim_{n \to \infty} \int_{\mathscr{B}} G_n \geq 0 .$$

By the change of variables formula we have

$$
\begin{aligned}
\int_{\mathscr{B}} G_n &= k^{-1} d^{n_0-n} \int_{\mathscr{B}} \log|h \circ g^n| \\
&= k^{-1} d^{n_0-n} \int_{g^n(\mathscr{B})} \log|h| |Dg^{-n}|^2 \\
&= k^{-1} d^{n_0-n} |a|^{-2n} \int_{g^n(\mathscr{B})} \log|h| ,
\end{aligned} \tag{4.2}
$$

where $|Dg^{-n}|$ denotes the (real) jacobian determinant of g^{-n}, which is equal to $|a|^{-2n}$. This last integral may be bounded below by rearranging the set of integration so that $\log|h|$ is as negative as possible. In other words,

$$\int_{\{|h| < \varepsilon\} \cap \Delta^2(R)} \log|h| \leq \int_S \log|h| \tag{4.3}$$

holds for any open set S if ε is chosen so that the volume of $\{|h| < \varepsilon\} \cap \Delta^2(R)$ is equal to the volume of S.

Since $dh \neq 0$ on M, there are constants $0 < A_1 \leq A_2 < \infty$ such that

$$A_1 \varepsilon^2 \leq \mathrm{Vol}(\{|h| < \varepsilon\} \cap \varDelta^2(R)) \leq A_2 \varepsilon^2 . \tag{4.4}$$

The volume of $g^n(\mathscr{B})$ is $|a|^{2n}$, so

$$n \log|a| - (\log A_2)/2 \leq \log \varepsilon . \tag{4.5}$$

Applying (4.3) in the case $S = g^n(\mathscr{B})$, we have

$$\begin{aligned}
\int_{g^n(\mathscr{B})} \log|h| &\geq \int_{\{|h| < \varepsilon\} \cap \varDelta^2(R)} \log|h| \\
&\geq C(\log \varepsilon)\mathrm{Vol}(\{|h| < \varepsilon\} \cap \varDelta^2(R)) \\
&\geq \mathrm{Const.}\,|a|^{2n}(n \log|a| - \mathrm{Const.}) ,
\end{aligned} \tag{4.6}$$

where the second inequality follows from Lemma 4.8, and the last inequality follows from (4.5). Substituting the estimate (4.6) into (4.2), we see that G_n converges to 0 in $L^1(\mathscr{B})$.

The next step is to show that the sequence $\{G_n\}$ has a limit in L^1 locally. For this, we note that a function of the form (3.4) has the property that $\varDelta_y G_n(x_0, y)$ has total mass 2π; here $\varDelta_y$ denotes the Laplacian in the y-variable alone, and the Laplacian is considered as a measure on the set $\{(x_0 \cdot y): y \in \mathbf{C}\}$. Let us consider $\mu_{n,x_0} = \varDelta_y G_n(x_0, y)$ as a family of measures on $\mathbf{C}$ varying with the parameter x_0. Then the partial convolution $*_y$ in the y variable satisfies

$$G_n(x_0, y) := \frac{1}{2\pi} \log *_y \mu_{n,x_0} = \frac{1}{2\pi} \int \log|y - \eta| \mu_{n,x_0}(d\eta) .$$

For any $a < \infty$, we may choose $b < \infty$ such that $K^+ \cap \{|x| < a\} \subset \{|y| < b\}$. We consider the family $\mathscr{M}$ of Borel measures μ supported on the set $\mathscr{B} = \{|x| \leq a, |y| \leq b\}$ and such that if $A \subset \{|x| < a\}$ and $B \subset \{|y| < b\}$ are Borel sets, then $\mu(A \times B) \leq 2\pi|A|$, where $|A|$ denotes the 2-dimensional Lebesgue measure of A. We let μ_n denote the restriction of the measure $\varDelta_y G_n$ to the set $\mathscr{B}$, so by the remark above $\mu_n \in \mathscr{M}$. Since $\mathscr{B}$ is weakly compact set of measures, we may take a subsequence of $\{\mu_n\}$ which converges to a measure $\tilde{\mu} \in \mathscr{M}$. It is a standard result that we may disintegrate any measure $\mu \in \mathscr{M}$ with respect to the Lebesgue measure on the complex x-axis, and thus identify μ with a family of measures $x \mapsto \mu_x$.

To show that $\{G_n\}$ converges in $L^1(\mathscr{B})$, we let $\log|y| = K^N + E^N$, where $K^N(y) = \max(-N, \log|y|)$. We define $\tilde{G} = \log *_y \tilde{\mu}$, $\tilde{G}^N = K^N *_y \tilde{\mu}$, $G_n^N = K^N *_y \mu_n$, etc. Thus we may make the estimate

$$\begin{aligned}
\|\tilde{G} - G_n\|_{L^1(\mathscr{B})} &\leq \|\tilde{G} - \tilde{G}^N\|_{L^1(\mathscr{B})} + \|\tilde{G}^N - G_n^N\|_{L^1(\mathscr{B})} + \|G_n^N - G_n\|_{L^1(\mathscr{B})} \\
&= \|E_n^N *_y \tilde{\mu}\|_{\mathscr{L}^1(\mathscr{B})} + \|K^N *_y (\tilde{\mu} - \mu_n)\| + \|E^N *_y \mu_n\| .
\end{aligned}$$

The usual estimate on convolution gives $\|E^N *_y \mu_n\|_{L^1(\mathscr{B})} \leq \|E^N\|_{L^1(2\mathscr{B})} \|\mu_n\|$, where $\|\mu_n\|$ denotes the total mass of μ_n. Now if we choose N sufficiently large that $\|E^N\|_{L^1(2\mathscr{B})} < \varepsilon$, then the first and third terms are estimated by $2\pi^2|a|^2\varepsilon$. The second term tends to zero as $n \to \infty$, since K^N is continuous, and μ_n converges weakly to μ. We conclude, then, that G_n converges $L^1(\mathscr{B})$ to $\tilde{G}$.

Next we show that $\tilde{G} = G^+$. The space of psh functions is a closed subset of $L^1(\mathbf{C}^2, \mathrm{loc})$, so after possible modification on a set of measure zero, $\tilde{G}$ will be usc and thus psh. G^+ is continuous, and $G^+ = \tilde{G}$ on $\mathbf{C}^2 - \partial K^+$, so $\tilde{G} \geq G^+$. On the other hand, G^+ is a continuous function which vanishes on K^+. The line

$\{x = \text{const.}\}$ intersects K^+ in a compact set, and $\omega = \{G^+ < \varepsilon\} \cap \{x = \text{const.}\}$ is a relatively compact neighborhood of $K^+ \cap \{x = \text{const.}\}$ in this line. Since $G^+ = G$ on $\partial\omega$, we may apply the maximum principle to conclude that $\tilde{G} = 0$ on K^+.

Finally, convergence in $L^1(C^2, \text{loc})$ implies that G_n converges to G^+ in the sense of distributions, and thus we have the convergence of the currents:

$$\lim_{n \to \infty} d^{-n} g_*^{-n}[M] = \lim_{n \to \infty} \frac{1}{d^n} dd^c \log|h(g^n)|$$

$$= \frac{k}{d^{n_0}} dd^c G^+$$

$$= \frac{k}{d^{n_0}} \mu^+ .$$

Lemma 4.8. *Let h be holomorphic in a neighborhood of the closure of the polydisk $\Delta^2(R)$, and suppose that $dh \neq 0$ in a neighborhood of $\{h = 0\}$. Then there is a constant C such that for $0 < \varepsilon < 1/2$*

$$C\varepsilon^2 \log \varepsilon \leq \int_{\{|h| < \varepsilon\} \cap \Delta^2(R)} \log|h| .$$

Proof. It suffices to cover $\{|h| = 0\} \cap \Delta^2(R)$ with a finite number of open sets for which this estimate holds. At any point $(x_0, y_0) \in \{h = 0\}$ the set $\{h = 0\}$ has nonsingular projection to either the x- or y-axis. We may assume that π_x is nonsingular there. Thus near (x_0, y_0) the function h has the form $h = q(x, y)$ $(y - a(x))$ where q and a are holomorphic, and $q(x_0, y_0) \neq 0$. For $\eta > 0$ small we consider the open set $\mathscr{U} = \{|h| < \eta\} \cap \pi_x^{-1}\{|x - x_0| < \eta\}$. For fixed x, we may compute the integral

$$\int_{\{|y - a(x)| < \varepsilon\}} \log|y - a(x)| = 4\pi(2\varepsilon^2 \log \varepsilon - \varepsilon^2) .$$

Applying Fubini's Theorem, we may integrate over $\mathscr{U}$, and the Lemma follows.

5. Structure of hyperbolic maps

In this section we study polynomial diffeomorphisms $g \in \mathscr{G}$ for which J is a hyperbolic set. We will assume for convenience that the jacobian determinant a satisfies $|a| \leq 1$. First we show (Theorem 5.6) that the interior of K^+ (if it is nonempty) consists of basins of attraction of finitely many sink orbits $s_1, \ldots, s_k$. Then we show (Theorem 5.9) that J^+ and $J^- - \{s_1, \ldots, s_k\}$ are foliated by complex manifolds.

We start by recalling some definitions. (See Shub [S] for further information.) A point p is *periodic* if $g^n(p) = p$ for some $n \geq 1$. A periodic point is a *sink* if it attracts all nearby points, i.e. if $W^s(p)$ contains a neighborhood of p. A periodic point is *hyperbolic* if $Dg^n(p)$ has no eigenvalues on the unit circle. A *hyperbolic sink* is a periodic point for which all the eigenvalues of $Dg^n(p)$ lie inside the unit circle. If p is a sink, then $W^s(p) \subset K^+$. By considering the normal family $\{f^n\}$ on $W^s(p)$, we see that all sinks are hyperbolic sinks.

The following notion extends the definition of hyperbolicity from periodic points to invariant sets. Let $\Lambda \subset C^n$ be an invariant set for a diffeomorphism f. We say that Λ is a *hyperbolic set* for f if there is a continuous f-invariant splitting of the

tanget bundle of $\mathbf{C}^n$ over Λ, i.e. there are continuous subbundles E^u and E^s such that $T\mathbf{C}^n_\Lambda = E^s \oplus E^u$, and $Df(E^s) = E^s$ and $Df(E^u) = E^u$, and if there exist constants c and $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s}\| < c\lambda^n, \qquad n \geq 0 \tag{5.1}$$

$$\|Df^{-n}|_{E^u}\| < c\lambda^n, \qquad n \geq 0. \tag{5.2}$$

If the dimension of E^u is equal to i at every point of Λ we say that Λ has *index* i. If f is holomorphic then E^u is a complex subspace and the index will refer to the complex dimension of E^u.

We recall the stable set $W^s(x)$ of a point from (2.5). The following result says that if Λ is a hyperbolic set for f, then the set $W^s(x)$ is in fact a manifold (see [S], Chapter 6).

Stable Manifold Theorem. *Let Λ be a compact hyperbolic set for f. For every point $x \in \Lambda$, $W^s(x)$ is an immersed submanifold of dimension equal to that of E^s. Further, $T_x W^s(x) = E^s_x$. Analogous results hold for unstable manifolds.*

When f is a holomorphic diffeomorphism the stable manifolds are complex submanifolds.

Definition. We say g is *hyperbolic* if J is a hyperbolic set for g.

It would seem more natural from the point of view of dynamical system theory to define a map g to be hyperbolic if it were hyperbolic on its nonwandering set. We will see in Proposition 5.8 that if g is hyperbolic on J then in fact g is hyperbolic on its nonwandering set. In addition it seems easier to check a priori that a map g is hyperbolic on J than that it is hyperbolic on its nonwandering set.

Proposition 5.1. *If g is hyperbolic then J has index 1, $W^s(J) \subseteqq J^+$ and $W^u(J) \subseteqq J^-$.*

Proof. Let Λ_i be the set of points $p \in J$ such that the dimension of E^u_x is i. The set Λ_0 is a union of sink orbits. This implies that points in some neighborhood of Λ_0 are attracted to Λ_0 which implies that Λ_0 is contained in the interior of K^+. On the other hand J is contained in the boundary of K^+. Thus Λ_0 must be empty. The same argument applied to g^{-1} shows that Λ_2 is empty. We conclude that J has index 1.

We prove that $W^s(J) \subseteqq J^+$. The fact that $W^u(J) \subseteqq J^-$ follows by considering g^{-1}. It is clear that $W^s(J) \subset K^+$. We will show that $W^s(J)$ cannot intersect the interior of K^+. Let p be a point in int K^+. Lemma 2.4 shows that $\{g^n\}$ is a normal family in a neighborhood of p. It follows that for any tangent vector ξ at p the sequence $\|Dg^n(\xi)\|$ is bounded. On the other hand the following argument shows that if $p \in W^s(J)$ the sequence cannot be bounded.

A cone field $\mathscr{C}$ over a set $\mathscr{U}$ assigns to each point $p \in \mathscr{U}$ a homogeneous cone $\mathscr{C}_p \subset T_p$. We can construct a continous cone field over a neighborhood $\mathscr{U}_0$ of J such that at each point $p \in J$ the vector space E^u_p is contained in the interior of $\mathscr{C}_p$ and E^s_p is contained in the interior of the complement of $\mathscr{C}_p$. We can choose n sufficiently large that, for every $p \in J$, we have $Dg^n(\mathscr{C}_p) \subset \text{int } \mathscr{C}_{g^n(p)}$ and $\|g^n(\xi)\| \geqq 2\|\xi\|$ for every v in $\mathscr{C}_p$. These conditions will hold for p in some neighborhood $\mathscr{U}_0 \subset \mathscr{U}$ of J. The point p is in $W^s(J)$ so there is some m_0 so that for $m \geqq m_0$ we have $g^m(p) \in \mathscr{U}_0$. Choose $\xi \in T_p$ so that $Dg^n(\xi) \in \mathscr{C}_{g^{m_0}(p)}$. Then for any positive k, $\|Dg^{m_0 + kn}(\xi)\| \geqq 2^k \|Dg^{m_0}(\xi)\|$. In particular the norms are not bounded. This completes the proof.

We say an invariant set Λ has *a local product structure* for a mapping f if $W^s(p_1) \cap W^u(p_2) \subset \Lambda$ whenever p_1 and p_2 are in Λ.

Proposition 5.2. *If g is hyperbolic, then J has a local product structure.*

Proof. By Proposition 5.1 if p_1 and p_2 are in J then $W^s(p_1)$ is in J^+ and $W^u(p_2)$ is in J^-. Thus the intersection lies in J.

Corollary 5.3. *The set J is locally maximal. More precisely there is a neighborhood U of J so that every invariant set contained in U is contained in J.*

Proof. This is a consequence of local product structure and hyperbolicity. (See [S, Prop. 8.22].)

In the following Theorem we summarize the geometric properties of the foliations $\mathscr{F}^\pm$ in $J^\pm$ when g is hyperbolic.

Theorem 5.4. *If g is hyperbolic, then the foliations $\mathscr{F}^\pm$ intersect transversely at each point of J, and the leaves of $\mathscr{F}^\pm$ are biholomorphically equivalent to $\mathbf{C}$.*

Proof. By the Stable Manifold Theorem, there are complex manifolds $W^s(x)$ and $W^u(x)$ through every point $x \in J$. The manifolds $W^s(x)$ and $W^u(x)$ intersect transversely at x since their tangent spaces at x are E^s and E^u, respectively.

The stable manifold theorem shows that sets $W^s(x)$ and $W^u(x)$ are imbedded complex submanifolds diffeomorphic to $\mathbf{R}^2$. It remains to show that they are holomorphically copies of $\mathbf{C}$ rather than disks. We will show that each submanifold $W^u(x)$ contains an infinite increasing family of disjoint annuli $A(k)$ so that $A(1)$ surrounds x and $A(k+1)$ surrounds $A(k)$ and the moduli of the annuli are bounded below. This proves that $W^u(x)$ is not a disk. For each $x \in J$ the Euclidean metric on $\mathbf{C}^2$ induces a metric on $W^u(x)$. Let $A_x \subset W^u(x)$ be the annulus bounded by the circles of radius 1 and 2. The compactness of J implies that the moduli of the annuli A_x is bounded below. Hyperbolicity implies that there is an n such that g^{-n} takes all circles of radius 2 inside all circles of radius 1. Fix x. Let $A(k) = g^{nk}(A_{g^{-nk}(x)})$. These annuli satisfy the properties listed above.

We will make use of the following Lemma.

Lemma 5.5. *If $|\det Dg| = 1$ then int $K^+ = $ int $K^- = $ int K. If $|\det Dg| < 1$ then int $K^- = \varnothing$. If $|\det Dg| > 1$ then int $K^+ = \varnothing$.*

Proof. It is observed in [FM] Lemma 3.7 that if $|\det Dg| = 1$ then the symmetric difference of K^+ and K^- has Lebesgue measure zero. This implies that int $K^+ \subset K^-$ for otherwise int $K^+ - K^-$ would be a nonempty open set and therefore have positive Lebesgue measure. In particular int $K^+ \subset K$ which gives int $K^+ \subset $ int K. The opposite inclusion is immediate because $K \subset K^+$. Thus int $K^+ = $ int K. The equality int $K^- = $ int K is proved in the same way.

If $|\det Dg| < 1$ it is observed in [FM, Lemma 3.7] that K^- has Lebesgue measure zero. This implies that K^- has empty interior. The last statement is proved the same way.

The following result shows that our assumption that g is hyperbolic on J severely restricts the possible dynamics of g on the complement of J.

Theorem 5.6. *If g is hyperbolic then the interior of K^+ consists of the basins of finitely many hyperbolic sink orbits.*

Proof. We may assume that $|\det Dg| \leqq 1$ otherwise int K^+ is empty and there is nothing to prove. We consider components of $int(K)$. If such a component is not periodic we call it wandering. The theorem is a consequence of the following four assertions.

(1) There are no wandering components.
(2) Each periodic component is a basin of a sink.
(3) There are finitely many sink orbits.
(4) Every sink is hyperbolic.

Proof of assertion (1). If $|\det Dg| = 1$ then according to [FM, Lemma 3.7] the Lebesgue measure of int K^+ is finite. The Poincaré Recurrence Theorem implies that almost every point is recurrent. Thus there can be no wandering domains.

Assume now that $|\det Dg| < 1$. Let C be a wandering component of int K^+ and let p be a point in C. Let L be the set of limit points of the sequence $g^n(p)$. For n sufficiently large $g^n(p) \in V$. Since V is compact it follows that L is nonempty. We also have $L \subset V$. The set L is invariant under g^{-1} and bounded. It follows that $L \subset K$. Now $K \subset K^-$ and our assumption on the determinant implies that $K^- = J^-$. So $L \subset J^-$.

We will show that L is disjoint from int K^+. Assume there is a point $q \in L \cap$ int K^+. Let C_0 be the component of int K^+ that contains q. For n sufficiently large $g^n(p)$ is in C_0. In particular there are distinct numbers n_1 and n_2 so that $g^{n_1}(p) \in C_0$ and $g^{n_2}(p) \in C_0$. Thus $g^{n_1}(C) = g^{n_2}(C)$ but this contradicts the assumption that C is a wandering component. We conclude that $q \in J^+$. Since q is an arbitrary point of L we get $L \subset J^+$. Combined with the above result this gives $L \subset J$.

The fact that all limit points of the sequence $g^n(p)$ lie in J implies that $p \in W^s(J)$. Proposition 5.1 then implies that $p \in J^+$. This contradicts our assumption that p is in the interior of K^+ and proves the assertion.

Proof of assertion (2). Let C' be a component of period m. Replacing g by g^m we may assume that C' is taken itself by g. Let $C = C' \cap V$. We show g takes C into itself. Now $g(V) \subset V \cup V^-$ and $g(C') \subset C' \subset V \cup V^+$ so $g(C' \cap V) \subset V$. Clearly $g(C) \subset C'$, so $g(C) \subset C' \cap V = C$. The domain C is bounded and we can apply a result of [Be] showing that either iterates of points diverge to the boundary of C or there is an invariant submanifold of C to which points are attracted.

We show first that points in C do not diverge to ∂C. Let L be the set of limit points of forward orbits of point in C. Assume that L is contained in $\partial C \subset J^+$. The set L is bounded and invariant under g^{-1} so $L \subset K$. If $|\det Dg| < 1$ then $K \subset K^- = J^-$ so $L \subset J^-$. If $|\det Dg| = 1$ then since L is disjoint from int K^+ and int $K^+ =$ int K^- we deduce that L is disjoint from int K^-. But L is contained in K^- so $L \subset J^-$. In either case $L \subset J^+ \cap J^- = J$. We conclude that $C \subset W^s(J)$ but this contradicts Proposition 5.1. This contradiction shows that points in C do not converage to ∂C.

The result of [Be] shows that there is a connected subset $S \subset C$ which is a smooth complex submanifold with a metric so that $C = W^s(S)$ and $g|s$ is an isometry. The manifold S is properly embedded in C and S is a retract of C. In order to show that C is the basin of a sink it suffices to show that S has dimension zero. Assume that the dimension of S is greater than zero. We will derive a contradiction.

Since S is a complex submanifold of $\mathbf{C}^2$, S cannot be compact. Since S is properly embedded in C, $\partial S \subset \partial C \subset J^+$. Now ∂S is a compact invariant subset of

J^+ and the only compact invariant subsets of J^+ are contained in J so $\partial S \subset J$. By Corollary 5.4, J is locally maximal. Let U be an open set in which J is the maximal invariant set. The set $S - U$ is a compact subset of S. Since g acts isometrically on S, the set $Q = \text{closure} (\bigcup_n g^n(S - U))$ is compact. Since S is not compact, there is some $p \in S - Q$. Since Q is invariant the orbit of p is in the complement of Q. So the orbit of p is contained in U. Since p is not in J this contradicts the assumption that J is the maximal invariant set in U. This contradiction proves that S has dimension zero.

In the case that $|\det Dg| = 1$ there can be no sinks so we conclude that in this case the interior of K^+ is empty. For the remainder of the proof we assume that $|\det Dg| < 1$.

Proof of assertion (3). If there are infinitely many sink orbits there is some q which is a limit of sinks. Arguing as before q is not in int K^+. Thus $q \in J^+$. The set L of limit points of sequences of sinks is an invariant subset of V. Thus $L \subset K$. Now $K \subset K^-$ and given that $|\det Dg| < 1$ we have $K^- = J^-$ so $L \subset J^-$. Combining with the previous inclusion gives $L \subset J$.

Corollary 5.4 gives us a neighborhood U of J in which J is the maximal invariant set. If every cluster point of a sequence of sink orbits lies J then the sequence must eventually be contained in U. In particular there is a sink orbit in U. This contradicts fact that J is the maximal invariant set in U.

Proof of assertion (4). By replacing g by g^m we may assume that the sink orbit is a single fixed point p. Let B be an open ball centered at p. There is an n such that $g^n(\bar{B}) \subset B$. Let A be the affine map that fixes p and has linear part $(1 + \varepsilon)I$. Let $g_0 = Ag^n$. Choose ε small enough that $g_0(\bar{B}) \subset B$. Now the family of functions $g_0^k|B$ is normal because $g_0^k(B) \subset B$. On the other hand Dg_0 at p has an eigenvalue greater than one so the derivatives of the family $g_0^k|B$ are unbounded. This contradition proves the assertion. (Note that this assertion is true for all holomorphic diffeomorphisms in any dimension.)

Corollary 5.7. *If g is hyperbolic and $|\det Dg| = 1$ then* int $K^+ = $ int $K^- = $ int $K = \varnothing$.

Proof. By Theorem 5.7 the set int K^+ consists of basins of sinks. When $|\det Dg| = 1$ there can be no sinks so int $K^+ = \varnothing$. By Lemma 5.6 we have int $K^+ = $ int $K^- = $ int K.

Proposition 5.8. *If $|\det Dg| \leq 1$ and g is hyperbolic then the chain recurrent set of g is contained in the union of J and the finitely many sink orbits.*

Remark. We will show in §6 that the chain recurrent set is in fact equal to the union of J and the finite set of sink orbits.

Proof. The existence of a filtration shows that $R(g)$ cannot contain points which do not have bounded orbits. Furthermore $R(g)$ cannot contain points in the basins of sinks other than the sinks themselves. Thus $R(g)$ is contained in the union of J and the sinks.

Theorem 5.9. *If g is hyperbolic and $|\det Dg| \leq 1$, then $W^s(J) = J^+$. If $s_1, s_2, \ldots, s_k$ are the sinks of g then $W^u(J) = J^- - \{s_1, \ldots, s_k\}$.*

Proof. Proposition 5.1 gives $W^s(J) \subset J^+$. To prove the first statement we will show that $J^+ \subset W^s(J)$. Lemma 2.4 gives $J^+ \subset W^s(K)$. Since J^+ is a closed set we have $J^+ \subset W^s(K \cap J^+)$. We observe that $K^- = J^-$. If $|\det Dg| < 1$ then this

follows from Lemma 5.5. If $|\det Dg| = 1$ then this follows from Corollary 5.7. Using this fact we see that $K \cap J^+ = K^- \cap J^+ = J^- \cap J^+ = J$. Thus $J^+ \subset W^s(J)$.

We now prove the second statement of the theorem. If $p \in J^- - \operatorname{int} K^+$ then since $p \in W^u(K)$ and since $J^- - \operatorname{int} K^+$ is closed we have $p \in W^u((J^- - \operatorname{int} K^+) \cap K^+)$. But $(J^- - \operatorname{int} K^+) \cap K^+ = (J^- \cap (K^+ - \operatorname{int} K^+)) = J^- \cap J^+ = J$ so we have $p \in W^u(J)$. If $p \in J^- \cap \operatorname{int} K^+$ then by Theorem 5.8 p is in the basin of a sink. If p is not itself a sink then as $n \to -\infty$ we have $g^n(p) \to \partial K^+ = J^+$. So $p \in W^u(J^+)$. But $p \in W^u(J^-)$ so $p \in W^u(J^+ \cap J^-) = W^u(J)$.

6. Transversal measures

Throughout this section we will assume that J is a hyperbolic set for g. In this case the currents $\mu^\pm$ define transversal measures on the foliations $\mathscr{F}^\pm$ of $J^\pm$ (see Theorem 6.5). Using this, we are able to show (Theorem 6.7) that $\mu^\pm$ have the structure of foliation cycles as in [Su]. As a consequence of this, we see (Theorem 6.9) that the support of the invariant measure μ is all of J. An interesting dynamical consequence of this is the fact that periodic points are dense in J (Corollary 6.10).

If $T \subset \mathbf{C}^2$ is any 1-dimensional complex submanifold, we may define a measure $\mu^+|_T$ on T as follows: if φ is a test function, we set

$$\int \varphi(\mu|_T) = \int \varphi(dd^c)_T(G^+|_T)$$

$$= \int G^+|_T(dd^c)_T\varphi \tag{6.1}$$

where $(dd^c)_T$ denotes the operator dd^c intrinsic to T. Since $\mu^+|_T$ is evidently positive, (6.1) serves to define $\mu^+|_T$ as a measure. Since G^+ is continuous, it follows that the correspondence $T \mapsto \mu^+|_T$ is continuous in the sense that if $\{T_j\}$ is a sequence of complex manifolds converging to T, then $\{\mu^+|_T\}$ converges to $\mu^+|_T$ in the weak (vague) topology of measures.

Let us note some properties of $\mu^+|_T$ which are independent of hyperbolicity. We may smoothly foliate a neighborhood of T by complex manifolds $\{T_\alpha\}$ such that $T_0 = T$.

Proposition 6.1 *If T is a 1-dimensional complex submanifold, and if $c \in \mathbf{C}$, then for almost any α, we may estimate the number of points of intersection of $g^n T_\alpha$ and the vertical and horizontal lines by*

$$\lim_{n \to \infty} d^{-n} \#(g^n T_\alpha \cap \{y = c\}) = \frac{1}{2\pi}\mu^+|_{T_\alpha}(T_\alpha)$$

and

$$\lim_{n \to \infty} d^{-n+1} \#(g^n T_\alpha \cap \{x = c\}) = \frac{1}{2\pi}\mu^+|_{T_\alpha}(T_\alpha) \ .$$

Proof. The number of points of intersection is given by

$$d^{-n} \#(g^n T_\alpha \cap \{y = c\}) = d^{-n} \#(T_\alpha \cap g^{-n}\{y = c\})$$

$$= \frac{d^{-n}}{2\pi} \int_{T_\alpha} dd^c \log|(g^n)_y - c|,$$

where $(g^n)_y$ denotes the y-component of g^n. By Theorem 4.7, $\log|(g^n)_y - c|$ converges in L^1_{loc} to G^+. Thus for almost every α, the restriction of this function to T_α

converges $G^+|_{T_\alpha}$. Passing to the limit as $n \to \infty$, the integrand converges to $G^+|_{T_\alpha}$ so the right hand side of the second equation converges to $\mu^+|_{T_\alpha}(T_\alpha)/2\pi$.

Corollary 6.2. *Let* $\pi_y(x, y) = y$ *denote projection to the y-axis, and let* $D \subset \mathbf{C}$ *be an open set. Let* A_n *denote the area (with multiplicity) of the projection of* $\pi_y^{-1}(D) \cap g^n(T_\alpha)$ *to the y-axis. Then* $\lim_{n \to \infty} d^{-n} A_n = \dfrac{1}{2\pi} \mu^+|_{T_\alpha}(T_\alpha)$ Area (D).

Proof. Without loss of generality, we may assume that $D = \{|y| < 1\}$ is the unit disk. Then as in the proof of Proposition 6.1, we have

$$A_n = \int_D \mathscr{L}^2(dc) \int_T dd^c \log|(g^n)_y - c|$$

$$= \int_T dd^c \left(\int_D \log|(g^n)_y - c| \mathscr{L}^2(dc) \right) .$$

We define $\lambda(s)$ to be the increasing function of s such that

$$\lambda(\log|t|) = \int_D \log|t - c| \mathscr{L}^2(dc)$$

so that

$$d^{-n} A_n = d^{-n} \int_T dd^c \lambda(\log|(g^n)_y|) .$$

We note that $\lim_{s \to \infty} \lambda(s)$ exists and is greater than $-\infty$, so $\lambda(s)$ is effectively equivalent to $\max(0, s \, \text{Area}(D))$, and thus we may apply Proposition 3.3 to conclude that the limit

$$\lim_{n \to \infty} d^{-n} \lambda(\log|(g^n)_y|) = \text{Area}(D)G^+$$

converges uniformly on compact sets. Thus $d^{-n} A_n$ converges to the desired limit.

Corollary 6.3. *If* $\mathscr{B} \subset \mathbf{C}^2$ *is any bounded set, then the area of* $g^n T_\alpha \cap \mathscr{B}$ *is bounded by* const. d^n.

We will say that a manifold $T \subset \mathbf{C}^2$ is *transversal* to $\mathscr{F}^+$ if T intersects the leaves of $\mathscr{F}^+$ transversally. If L is a leaf of $\mathscr{F}^+$, and if T_1 is a transversal, we may assume that we have smooth complex coordinates (x, y) such that $\Delta^2 = \{|x|, |y| < 1\}$ is a coordinate neighborhood, and $L = \{y = 0\}$, and $T_1 = \{x = 0\}$. We will write $\mathscr{F}_0^+(\Delta^2)$ for the leaves of $\mathscr{F}^+ \cap \Delta^2$ which have the form $\{y = \varphi(x): x \in \Delta\}$ for some smooth function φ. Thus we see that $\mathscr{F}_0^+(\Delta^2)$ contains a neighborhood of L in $\mathscr{F}^+ \cap \Delta^2$. Let us call such a neighborhood a *transverse box*. If T is a complex submanifold then the coordinate system may also be assumed to be holomorphic when this is convenient. In this case φ will be holomorphic.

Now let T_2 be another transversal to $\mathscr{F}^+$ in Δ^2. By shrinking our neighborhood, if necessary, we may assume that $T_2 = \{x = \psi(y): y \in \Delta\}$ is a graph of a smooth function, and $T_2 \cap \partial \Delta \times \Delta = \varnothing$. Let us write $E_j = T_j \cap \mathscr{F}_0^+(\Delta^2)$. Then there is the natural homeomorphism

$$\chi_{T_1, T_2}: E_1 \to E_2 \tag{6.2}$$

given by moving along a leaf M of $\mathscr{F}_0^+(\Delta^2)$ from the intersection point $M \cap T_1$ to the (unique) intersection point $M \cap T_2$.

In particular, we see that the μ^+ defines a measure $\mu^+|_T$ for any complex transversal T to the foliation $\mathscr{F}^+$. Following [RS], we say that the family $\{\mu^+|_T\}$ defines a *transversal measure* on $\mathscr{F}^+$ if the measures $\mu^+|_T$ are compatible with the homeomorphism (6.2), i.e. if

$$(\chi_{T_1,T_2})_* \mu^+|_{T_1} \, \llcorner \, E_1 = \mu^+|_{T_2} \, \llcorner \, E_2 \, , \tag{6.3}$$

where the notation $\mu^+|_{T_j} \, \llcorner \, E_j$ indicates the restriction of the measure $\mu^+|_{T_j}$ to the set E_j. If (6.3) holds, then the family $\{\mu^+|_T\}$ assigns a well-defined number

$$\mu^+|_{T_1}(T_1 \cap S) =: \mu^+|_{T_2}(T_2 \cap S) \tag{6.4}$$

to any Borel set S of leaves of $\mathscr{F}_0^+(\Delta^2)$. The equivalence class of these measures defines a measure $\tilde{\mu}^+$ on $\mathscr{F}_0^+(\Delta^2)$. In defining a transversal measure Ruelle and Sullivan use the family of smooth transverse disks. In our case, where the leaves of $\mathscr{F}^+$ are complex submanifolds, there are sufficiently many complex transversals that a transversal measure in the sense of [RS] is determined by its values on complex transversals. Conversely any transversal measure defined on complex transversals has a (unique) extension to a transversal measure defined on all smooth transversals.

For a point $p_0 \in J$, let T_1 denote a portion of the unstable manifold $W^u(g, p_0)$. Thus T_1 is a transversal near p_0, and we may construct a transversal box $\mathscr{B} = \Delta^2$ as above such that $p_0 = (0; 0)$ and $T_1 \subset \{x = 0\}$. Let Σ_1 denote a compact subset of $T_1 \cap \mathscr{F}_0^+(\mathscr{B})$. If T_2 is another transversal close to T_1, we let $\Sigma_2 = T_2 \cap \mathscr{F}_0^+(\mathscr{B})$, and we may assume that $\chi: \Sigma_1 \to \Sigma_2$, defined in (6.2), is a homeomorphism. Since a point $\sigma \in \Sigma_1$ and $\chi(\sigma) \in \Sigma_2$ lie in a stable manifold, we see that $\text{dist}(g^n(\sigma), g^n(\chi(\sigma)))$ goes to zero. In other words, $g^n(\Sigma_1)$ and $g^n(\Sigma_2)$ approach each other very rapidly as $n \to \infty$. In fact, the portions of $g^n(T_2)$ and $g^n(T_1)$ which remain close to J also approach each other as $n \to \infty$. This is made precise in Theorem 6.4 which is a version of the "Lambda Lemma."

We refer the reader to [S] for details of the following constructions. Let Λ be a hyperbolic invariant set contained in a manifold M. Choose an adapted metric on Λ and extend it to M. For each $p \in \Lambda$ we can define the exponential map with respect to the adapted metric $\exp: T_p \to M$. The vector space T_p is canonically identified with $E_p^s \oplus E_p^u$. Let D_p^u and D_p^s be the ε balls in E_p^u and E_p^s. We define an ε-box in T_p to be the product of D_p^u and D_p^s. Let $B(p, \varepsilon) \subset M$ be the image of the ε-box in T_p. We will assume that ε is chosen to be small enough so that the exponential map defines a local coordinate system for $B(p, \varepsilon)$. We say that a subset of the ε-box at p is a graph if it can be written in local coordinates as the graph of a function $\psi: D_p^u \to D_p^s$. If ε is sufficiently small then the ε-boxes satisfy an "overflowing condition" which implies that if σ is a graph in $B(\varepsilon, p)$ then $g(\sigma) \cap B(\varepsilon, g(p))$ is a graph in $B(\varepsilon, g(p))$.

Theorem 6.4. *Let T_1 and T_2 be transversals to the stable foliation and assume that T_1 is contained in a leaf of the unstable foliation. Let $\Sigma_i \subset T_i$ be contained in the stable set of Λ and be homeomorphic via χ_{T_1, T_2}. Let ε be as above. For each $p \in \Sigma_1$ assume that T_2 intersects $B(p, \varepsilon)$ in a graph.*

(i) *Let $p \in \Sigma_1$. The component of $g^n(T_1) \cap B(g^n(p))$ that contains $g^n(p)$ is the graph of a function $\psi_{1,p}^n: D_p^u \to D_p^s$.*

(ii) *Let $p \in \Sigma_1$ and let $p' = \chi(p)$. The component of $g^n(T_2) \cap B(g^n(p))$ that contains $g^n(p')$ is the graph of $\psi_{2,p}^n: D_p^u \to D_p^s$.*

(iii) $\lim_{n \to \infty} \sup_{p \in \Sigma_1} \sup_{y \in D_p^u} |\psi_{1,p}^n(y) - \psi_{2,p}^n(y)| = 0$.

Proof. Our choice of ε insures that, for each $p \in \Sigma_1$, the set $T_2 \cap B(p, \varepsilon)$ is a graph which contains $p' = \chi(p)$. This is the content of statement (ii) when $n = 0$. Call this graph σ. The overflowing condition implies that $g(\sigma) \cap B(g(p), \varepsilon)$ is again a graph. This set is the component of $g(T_2) \cap B(g(p), \varepsilon)$ that contains $g(p')$. This proves statement (ii) when $n = 1$. Repeating this argument proves statement (ii) in general. Statement (i) is proved similarly.

To prove (iii) we can use a version of the Lambda Lemma for the Banach space of bounded functions from A to M which we write $\Gamma(A, M)$. (The Lamda Lemma is commonly given near a hyperbolic fixed point, and in order to deal with the general case, we work near the identity mapping, which is a hyperbolic fixed point in this Banach space.) There is a natural action of g on $\Gamma(A, M)$ for which the inclusion $\iota : A \to M$ is a hyperbolic fixed point ([S]). Let $B(\varepsilon)$ be the neighborhood of ι consisting of the maps σ such that $\sigma(p) \in B(p, \varepsilon)$. Let D^s and D^u be the bundle of ε-disks in E^s and E^u. By using the exponential map the box $B(\varepsilon)$ can be identified $\Gamma(A, D^s) \oplus \Gamma(A, D^u)$. Relative to these coordinates we can define a graph in $B(\varepsilon)$ to be a set which can be written as the graph of a function $\Psi : \Gamma(A, D^u) \to \Gamma(A, D^s)$.

We can use the disks T_1 and T_2 to define graphs Ψ_1 and Ψ_2 of $B(\varepsilon)$ as follows. We begin by constructing $\Psi_1 : \Gamma(A, D^u) \to \Gamma(A, D^s)$. For each $p \in \Sigma_1$, the set $T_1 \cap B(p, \varepsilon)$ is a graph of a function $\psi_{1,p} : D_p^u \to D_p^s$. Let $\tau \in \Gamma(A, D^u)$ then $\Psi(\tau)(p) = \psi_{1,p}(\tau(p))$ for all $p \in A$. We will construct Ψ_2. For each $p \in \Sigma_1$ the set $T_2 \cap B(p, \varepsilon)$ is the graph of a $\psi_{2,p} : D_p^u \to D_p^s$. Let τ be an element of $\Gamma(A, E^u)$ then define $\Psi_2(\tau)(p)$ to be equal to $\psi_{2,p}(\tau(p))$ for $p \in \Sigma_1$ and to be equal to $\tau(p)$ otherwise. The graph transform map on graphs in $B(\varepsilon)$ describes how they transform under g. It is obtained by applying g to a graph in $B(\varepsilon)$ and then restricting to $B(\varepsilon)$. The graph transform in an overflowing neighborhood of a hyperbolic fixed point is a uniform contraction with respect to the sup norm on graphs (see [S]). The graph Ψ_1 is the unique fixed point. It follows that Ψ_2 converges to Ψ_1. After unraveling definitions this gives (iii).

Theorem 6.5. *The family of measures* $\{\mu^+|_T\}$ *defines a transversal measure* $\tilde{\mu}^+$ *on* $\mathscr{F}^+$.

Proof. If T_1 and T_2 are two transversals as above, then $g^n(T_j \cap J^+)$ approaches J as $n \to \infty$. Thus without loss of generality, we may assume that T_1 and T_2 are arbitrarily close to J. Further, there is no loss of generality if we assume that T_2 is an open subset of an unstable manifold.

We need to show that if $\Sigma_1 \subset T_1$ is compact, then

$$\mu^+|_{T_1}(\Sigma_1) = \mu^+|_{T_2}(\chi(\Sigma_1)) .$$

For this, we consider continuous functions φ_j on T_j with support in Σ_j and such that $\varphi_2(\chi) = \varphi_1$; it will suffice to show that

$$\int \varphi_1 \mu^+|_{T_1} = \int \varphi_2 \mu^+|_{T_2} .$$

By the change of variables formula,

$$\int \varphi_j \mu^+|_{T_j} = \int (\varphi_j \circ g^{-n})((g^{-n})_* \mu^+)|_{g^n T_j}$$
$$= d^{-n} \int (\varphi_j \circ g^{-n}) \mu^+|_{g^n T_j} .$$

holds for $j = 1, 2$.

For each $p \in g^n(\Sigma_i)$ let $D_{j,p}^n$ be the component of $g^n(T_j)$ that contains p. We can find a finite set of points $p_k \in \Sigma_1$ so that the disks D_{1,p_k}^n cover Σ_1 and the disks $D_{2,p_k'}^n$

cover Σ_2 where $p'_k = \chi(p_k)$. We will write $D^n_{1,k}$ for D^n_{1,p_k} and $D^n_{2,k}$ for D^n_{2,p'_k}. Each $D^n_{j,k}$ is a graph as in (iii) of Theorem 6.4.

We can choose partitions of unity $\{\rho^n_{j,k}\}$ on $g^n(T_j)$ subordinate to the cover $\{D^n_{j,k}\}$. Multiplying by these partitions of unity we may restrict our attention to $\rho^n_{j,k}(\varphi_j \circ g^{-n})$.

By (iii) of Theorem 6.4, we have

$$\lim_{n \to \infty} \sup_k \sup_{y \in D^u} |\psi^{(n)}_{1,k}(y) - \psi^{(n)}_{2,k}(y)| = 0 , \qquad (6.5)$$

where $\{k\} = \{k_n\}$ can be chosen arbitrarily.

Since the mapping g^{-n} is contracting on $D^n_{j,k}$, the function $\varphi \circ g^{-n}$ has a modulus of continuity on all of the sets $D^n_{j,k}$ which is independent of n. Thus we conclude from (6.5) and the continuous dependence of $\mu^+|_T$ on T that

$$\left| \int_{D^n_{1,k}} \rho^n_{1,k}\varphi(g^{-n})\mu^+|_{D^n_{1,k}} - \int_{D^n_{2,k}} \rho^n_{2,k}\varphi(g^{-n})\mu^+|_{D^n_{2,k}} \right|$$
$$= o(\sup|\varphi|(\mu^+|_{D^n_{1,k}}(D^n_{1,k}) + \mu^+|_{D^n_{1,k}}(D^n_{1,k}))) \quad (6.6)$$

as $n \to \infty$, where again we let $1 \leq k \leq K_n$ be arbitrary.

Now we observe that

$$\left| \int \varphi \mu^+|_{T_1} - \int \varphi \mu^+|_{T_2} \right| \leq d^{-n} \sum_{k=1}^{K_n} \left| \int_{D^n_{1,k}} \rho^n_{1,k}\varphi(g^{-n})\mu^+|_{D^n_{1,k}} \right.$$
$$\left. - \int_{D^n_{2,k}} \rho^n_{2,k}\varphi(g^{-n})\mu^+|_{D^n_{2,k}} \right| .$$
$$(6.7)$$

By Corollary 6.3,

$$\sum_{k=1}^{K_n} \mu^+|_{D^n_{j,k}}(D^n_{j,k}) = O(d^n) .$$

So if we combine this with (6.6), we conclude that the right hand side of (6.7) tends to 0 as $n \to \infty$, which completes the proof.

If T is a transversal, we may take a holomorphic mapping $\pi_T : \mathbf{C}^2 \to \mathbf{C}$ of rank one such that $T = \pi_T^{-1}(0)$. The sets $T_\zeta := \pi_T^{-1}(\zeta)$ will be transversals for ζ near 0. Now let us recall some results from the theory of slicing (cf. [F]). If S is a $(1,1)$ current on $\mathbf{C}^2$, then we may define the *slicing* of S with respect to the mapping π_T as a family of measures $\langle S, \pi_T, \zeta \rangle$ supported on T_ζ which have the following property: for any test function φ

$$\int S \lfloor \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right)(\varphi) = \int \mathscr{L}^2(d\zeta) \langle S, \pi_T, \zeta \rangle(\varphi) \qquad (6.8)$$

where $\mathscr{L}^2$ is Lebesgue measure on $\mathbf{C}$, and $\langle S \pi_T, \zeta \rangle(\varphi)$ denotes the integral of φ over T_ζ with respect to the measure $\langle S, \pi_T, \zeta \rangle$.

In the case of our current $S = \mu^+$, thus means that

$$\int \varphi dd^c G^+ \wedge \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) = \int \mathscr{L}^2(d\zeta) \langle \mu^+, \pi_T, \zeta \rangle(\varphi) \qquad (6.9)$$

By the change of variables formula, this means that

$$\langle \mu^+, \pi_T, \zeta \rangle = dd^c G^+|_{T_\zeta}$$
$$= \mu^+|_{T_\zeta} \qquad (6.10)$$

Now let $\mathscr{B}$ be a transverse box as above, and let $\mathscr{V}$ be the union of the leaves $M \in \mathscr{F}_0^+(\mathscr{B})$, and let $\pi_0 \colon \mathscr{V} \to E$ be the projection which takes the set M to the point $E \cap M$. Since T_ζ is transversal to $\mathscr{F}^+$ for $\{|\zeta| < \varepsilon\}$, it follows that π_0 and π_T define coordinates on $\mathscr{V}$ in a neighborhood of $T \cap \mathscr{V}$, i.e.

$$\pi := (\pi_0, \pi_T) \colon \mathscr{V} \cap \pi_T^{-1}(|\zeta| < \varepsilon) \to E \cap \{|\zeta| < \varepsilon\} \tag{6.11}$$

is a homeomorphism. Thus a neighborhood of T in $\mathscr{V}$ has a (topological) product structure.

Lemma 6.6. $\mu^+ \llcorner \pi_T^* \left(\dfrac{i}{2} d\zeta \wedge \overline{d\zeta} \right)$ *coincides with the product measure under the mapping* π.

Proof. If $A \subset E$ and $B \subset \{|\zeta| < \varepsilon\}$ are Borel sets, then by the slicing formula

$$\mu^+ \llcorner \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) \pi^{-1}(A \times B) = \int_B \mathscr{L}^2(d\zeta) \langle \mu^+, \pi_T, \zeta \rangle (\pi_T^{-1} A)$$

$$= \int_B \mathscr{L}^2(d\zeta) \mu^+ |_{T_\zeta}(\pi_T^{-1} A)$$

$$= \tilde{\mu}^+(A) \int_B \mathscr{L}^2(d\zeta)$$

$$= \tilde{\mu}^+(A) \mathscr{L}^2(B)$$

where the second from the last equality arises because $\tilde{\mu}^+$ is a transversal measure.

For $a \in E$, we will use the notation $M_a := \pi_0^{-1}(a)$ for the leaf of $\mathscr{F}^+$ passing through a. $[M_a]$ will denote the current of integration over M_a.

Theorem 6.7. $\mu^+ = \int_{a \in E} \tilde{\mu}^+(da) [M_a]$.

Proof. Let us write $S = \int_{a \in E} \tilde{\mu}^+(da) [M_a]$. It will suffice to show that

$$\mu^+ \llcorner \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) = S \llcorner \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) \tag{6.12}$$

holds for an open set of transversals T. Given the special form of S, we have

$$S \llcorner \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) = \int_{a \in E} \tilde{\mu}^+(da) [M_a] \wedge \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right).$$

We may identify

$$[M_a] \wedge \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) = \alpha \mathscr{L}^2 \llcorner M_a \tag{6.13}$$

where $\mathscr{L}^2 \llcorner M_a$ is the restriction of Hausdorff 2-dimensional measure to M_a, and α satisfies

$$iv \wedge \bar{v} \wedge \pi_T^* \left(\frac{i}{2} d\zeta \wedge \overline{d\zeta} \right) = \alpha \beta_{2,2}$$

where v is a $(1,0)$ form normal to M_a, $|v| = 1$, and $\beta_{2,2}$ is the Euclidean volume form on C^2. We see, then that if we push the measure in (6.8) forward under π_T, then we obtain Lebesgue measure on C, i.e.

$$(\pi_T)_* (\alpha \mathscr{L}^2 \llcorner M_a) = \mathscr{L}^2.$$

Thus if $A \subset E$ and $B \subset \{|\zeta| < \varepsilon\}$ are Borel sets, then

$$S \llcorner \pi_T^*\left(\frac{i}{2}d\zeta \wedge d\bar{\zeta}\right)(\pi_0^{-1}(A) \cap \pi_T^{-1}(B)) = \int \tilde{\mu}^+(da)\alpha \mathscr{L}^2 \llcorner M_a(\pi_0^{-1}(A) \cap \pi_T^{-1}(B))$$

$$= \int_A \tilde{\mu}^+(da)(\alpha \mathscr{L}^2 \llcorner M_a(\pi_T^{-1}(B)))$$

$$= \int_A \tilde{\mu}^+(da)\mathscr{L}^2(B)$$

$$= \tilde{\mu}(A)\mathscr{L}^2(B) .$$

The Theorem then follows from Lemma 6.3.

Lemma 6.8. *Let* $\{T_j\}$ *be a sequence of positive, closed* $(1,1)$ *currents converging weakly to* T. *If* G *is continuous and psh, then* $\{T_j \wedge dd^cG\}$ *converges to* $T \wedge dd^cG$.

Proof. Since T_j is positive and closed, we may define $T_j \wedge dd^cG$ paired with a test form as

$$\int \chi \wedge T_j \wedge dd^cG = \int Gdd^c\chi \wedge T_j . \qquad (6.14)$$

(See [BT1, Proposition 2.1].) Since $\{dd^c\chi \wedge T_j\}$ converges to $dd^c\chi \wedge T$ in the sense of currents, and since T_j is positive, we conclude that the convergence actually holds in the sense of currents representable by integration, i.e. in the weak sense of measures. Thus the convergence continues to hold even when we multiply by a continuous function G.

Theorem 6.9. *If* g *is hyperbolic, then* supp $\mu = J$.

Proof. Since we know already that supp $\mu_k \subset J$, it will suffice to prove the reverse inclusion. To do this, we will show that if p is a point of J, and if Δ^2 is a transverse box centered at p, then $\mu_k(\mathscr{V}^+ \cap \mathscr{V}^-) > 0$.

Since the support of $\mu^\pm$ is $J^\pm$, and since $\mathscr{V}^\pm$ is an open subset, we have $\mu^\pm(\mathscr{V}^\pm) > 0$. By Theorem 6.7, we conclude that $\mu^+|_{T^+}(E^+) = \varepsilon^+ > 0$ and $\mu^-|_{T^-}(E^-) = \varepsilon^- > 0$. Let $b \in E^-$ be fixed, and let M_b^- be a disk of $\mathscr{F}_0^-(E^-)$. Then

$$[M_b^-] \wedge (\mu^+|_{\mathscr{V}^+}) = \int_{E^+} \lambda^+(da) [M_b^-] \wedge [M_a^+]$$

is a current with total mass $\lambda^-(E^-) = \varepsilon^-$, since $[M_b^-] \wedge [M_a^+] = [M_b^- \cap M_a^+]$ is the current of integration over a point.

Now let $\lambda_j^- = \sum c_j^i \delta_{b_i}$ be a sequence of discrete measures converging weakly to λ^-, and let $T_j^- = \sum c_j^i [M_{b_i}^-]$ be the corresponding $(1, 1)$ currents. Then the currents T_j^- converge weakly to $\mu^-|_{\mathscr{V}^-}$, and so by Lemma 6.8 we have

$$\mu|_{\mathscr{V}^+ \cap \mathscr{V}^-} = \lim_{j \to \infty} T_j^- \wedge \mu^+|_{\mathscr{V}^+}$$

$$= \lim_{j \to \infty} \sum_i c_j^i [M_{b_i}^-] \wedge \left(\int_{E^+} \mu^+|_{T^+}(da)[M_a^+]\right)$$

$$= \lim_{j \to \infty} \sum_i c_j^i \int_{E^+} \mu^+|_{T^+}(da)[M_{b_i}^-] \wedge [M_a^+]$$

$$= \lim_{j \to \infty} \int_{E^+} \mu^+|_{T^+}(da) \sum_i c_j^i [M_{b_i}^-] \wedge [M_a^+],$$

and so we have

$$\mu|_{\mathscr{V}^+ \cap \mathscr{V}^-} = \int_{E^+} \mu^+|_{T^+}(da) \int_{E^-} \mu^-|_{T^-}(db) \, [M_b^-] \wedge [M_a^+] \, . \tag{6.15}$$

From this we conclude that $\mu(\mathscr{V}^+ \cap \mathscr{V}^-) = \varepsilon^+ \varepsilon^- > 0$, which completes the proof.

Remark. Formula (6.15), derived in the proof of Theorem 6.9, is a local product description for the measure μ.

From Corollary 3.6 and Theorem 6.9 it follows immediately that

Corollary 6.10. $J = \Omega(g|_J)$.

Corollary 6.11. *If g is hyperbolic on J and if $s_1, \ldots, s_k$ are the sinks of g, then $R(g) = J \cup \{s_1, \ldots, s_k\}$ and thus g is hyperbolic on the chain recurrent set.*

Proof. Since $\Omega(g|_J) \subset R(g|_J) \subset J$, it follows from Corollary 6.10 that $J = R(g|_J)$. Thus by Proposition 5.8 we conclude that $R(g) = J$.

And from the remarks after Proposition 3.10, we have

Corollary 6.12. *J is the Shilov boundary of K.*

Corollary 6.13. *If g is hyperbolic, then the periodic points are dense in J, i.e.*

$$\overline{Per(g|_J)} = J \, .$$

Proof. By Corollary 6.11 g is hyperbolic on the chain recurrent set, and so by the hyperbolic closing lemma (Proposition 8.8 of Shub [S]), it follows that the periodic points are dense in J.

A diffeomorphism f is said to satisfy *Axiom A* if it is hyperbolic and periodic points are dense in the nonwandering set.

Corollary 6.14. *If g is hyperbolic, then it satisfies Axiom A.*

7. Mixing

In this Section, we show that if g is hyperbolic, then it is topologically mixing on J. We derive some consequences of this fact that follow from the theory of Axiom A diffeomorphisms.

Let us recall the filtration $\{V, V^\pm\}$ from §2. The relative homology groups $H_2(V \cup V^\pm, V^\pm)$ are generated by horizontal/vertical complex disks $\gamma^\pm$ contained in V and with $\partial\gamma^\pm \subset \partial V^\pm$. Thus these groups are isomorphic to $\mathbf{Z}$. The topological behavior of g with respect to this filtration is that g (resp. g^{-1}) acts as multiplication by d on the generator γ^- of $H_2(V \cup V^-, V^-)$ (resp. γ^+ of $H_2(V \cup V^+, V^+)$).

Lemma 7.1. *Let M^+ and M^- be Riemann surfaces in V with $\partial M^+ \subset \bar{V}^+$ and $\partial M^- \subset \bar{V}^-$. Then $M^+ \cap M^- \neq \varnothing$.*

Proof. Let the complex disks $\gamma^\pm$ be generators of the homology groups. We may use $\gamma^\pm$ to define generators of the dual cohomology groups. Now the currents of integration $[M^\pm]$ represent elements of the dual cohomology groups $H^2(V, \partial V^\pm)$, and so there are positive integers $d^\pm$ such that $[M^\pm] = d^\pm \gamma^\pm$. The number of intersection points of M^+ and M^- is given by the cup product of the cohomology classes $[M^+]$ and $[M^-]$ and is equal to $d^+ d^- > 0$. Thus the intersection is nonempty.

Remark. This result may also be applied to the image $g^n(V)$ of the polydisk if $\partial M^{\pm} \subset \partial g^n V^{\pm}$.

The following result is independent of hyperbolicity. The connectedness of $K^{\pm}$ was conjectured by Friedland and Milnor [FM].

Theorem 7.2. K^+, K^-, J^+, and J^- are connected.

Proof. We begin by proving that K^- is connected. Replacing g by g^{-1} interchanges K^- and K^+ and the connectivity of K^+ will follow from our proof of the connectivity of K^-. Let U be an open set that contains K^-. Let $p_i \in U \cap K^-$ for $i = 1, 2$. We will show that p_1 and p_2 are in the same path component of U. This will prove that K^- is topologically connected. Write $p_i = (x_i, y_i)$ and let $D_i = \{(x, y_i) : |x| \leqq R\}$. Now $K^- = \bigcap_{n=1}^{\infty} g^n(V \cup V^-)$.

$$K^- \cap V = \bigcap_{n=0}^{\infty} g^n(V \cup V^-) \cap V = \bigcap_{n=1}^{\infty} g^n(V) \cap V .$$

Using the compactness of V we can find an N_1 such that

$$\bigcap_{n=1}^{N_1} g^n(V) \cap V \subset U .$$

Let C_i be the component of $D_i \cap g^{N_1}(V)$ containing p_i. By construction $C_i \subset U$. Using the compactness of $g^{N_1}(V)$ we can find an N_2 so that $g^{N_2}(V) \cap g^{N_1}(V) \subset U \cap g^{N_1}(V)$. Let $D' = \{(0, y) : |y| \leqq R\}$. Let C' be any component of $g^{N_2}(D') \cap g^{N_1}(V)$. By construction C' is contained in U. Applying Lemma 7.1 to the bidisk $g^{N_1}(V)$, we see that C' meets D_1 and D_2. This proves that K^- is connected.

We now prove that J^- is connected. The connectivity of J^+ follows as before. The argument is the same as the previous argument with two changes. The first change is that in choosing N_1 we make use of the following lemma:

Lemma 7.3. Let $X_0 \supset X_1 \ldots$ be a decreasing sequence of compact sets. Let $X_{\infty} = \bigcap_{n=0}^{\infty} X_n$. If an open set U contains ∂X_{∞}, then there is an N so that for all $n \geqq N$, $\partial X_n \subset U$.

Proof. $U \cup \text{int}(X_{\infty})$ is open and contains X_{∞}. For some N, $U \cup X_{\infty} \supset X_n$ for $n \geqq N$. So $U \supset X_n - \text{int}(X_{\infty})$. The result now follows since $X_n \supset X_{\infty}$, we have $\text{int}(X_n) \supset \text{int}(X_{\infty})$ and $X_n - \text{int}(X_{\infty}) \supset X_n - \text{int}(X_n) = \partial X_n$.

The second modification we need to make in the proof is to choose D' to be in the boundary of V. For example set $D' = \{(R, y) : |y| \leqq R\}$. Now the proof proceeds as before

Theorem 7.4. If g is hyperbolic then $g|_J$ is mixing.

Proof. By Corollary 6.14 g is Axiom A. Smale's Spectral Theorem for Axiom A diffeomorphisms implies that J can be written as a disjoint union of a finite number of "mixing components", C_i, which are open and closed subsets of J that are permuted by g. Assume that $g|_J$ is not mixing. This implies that there are at least two mixing components. For some n, g^n will fix these sets C_i. We can decompose J into disjoint, open and nonempty g^n-invariant sets J_1 and J_2. Let ε be a small constant to be determined later. Let $U_i = \{x : \min_{k \in \mathbb{Z}} d(g^{nk}(x), J_i) < \varepsilon\}$ for $i = 1, 2$. The set $J^+ = W^s(J)$ is contained in $U_1 \cup U_2$. The sets U_i are open. If we can show

that they are disjoint we will have produced a disconnection of J^+, a contradiction which will prove this Theorem.

We state the following version of the shadowing lemma (see [S, Prop. 8.2.1]) for the set J making use of the assumption of hyperbolicity and the existence of a local product structure (Proposition 5.2): *For γ sufficiently small there is a constant α and a neighborhood U of J such that every α pseudo-orbit in U is γ shadowed by a point in J.*

Let δ be the separation distance between J_1 and J_2. Choose $\gamma < \delta/10$. Let α be chosen so that the above shadowing lemma applies. If we choose $\varepsilon = \min(\alpha, \delta/10)$ we claim that the sets U_1 and U_2 will be disjoint. Assume to the contrary that there is an x such that $d(f^{n_1}(x), p_1) < \varepsilon$ and $d(f^{n_2}(x), p_2) < \varepsilon$ for points $p_i \in J_i$. We may assume for notational simplicity that $n_1 \leq n_2$. We construct an α pseudo-orbit:

$$\ldots, f^{-2}(p_1), f^{-1}(p_1), f^{n_1}(x), \ldots, f^{n_2}(x), f(p_2), f^2(p_2), \ldots$$

The shadowing lemma implies that this pseudo-orbit is γ shadowed by a real orbit $\{f^n(z)\}$ in J. Since the pseudo-orbit contains points in J_1 the orbit $\{f^n(z)\}$ contains points within distance $\gamma < \delta/10$ of J_1. Since these points are in J and are closer to J_1 than the separation distance between J_1 and J_2, these points must actually be in J_1. Since J_1 is invariant the entire orbit is contained in J_1. By a similar argument we see that this orbit is contained in J_2. But this contradicts the assumption that J_1 and J_2 are disjoint, thus proving the Theorem.

Corollary 7.5. *Assume that g hyperbolic. Let p be a periodic point of g. Then the stable manifold of g at p is dense in J^+, and the unstable manifold at p is dense in J^-.*

Proof. We will prove that the unstable manifold is dense in J^-. The proof for the stable manifold is identical. By Proposition 5.9 $J^- = W^u(J)$. For a mixing basic set of an Axiom A diffeomorphism such as J the foliation of $W^u(J)$ by the unstable manifolds of points is a minimal foliation. This means that there are no proper nonempty closed subsets which are unions of leaves. In fact Bowen and Marcus prove the stronger result that this foliation is uniquely ergodic ([BM]). It follows from minimality that the closed set $cl(W^u(p))$ is all of J^-.

In the following Theorem, we will need a sharper version of part (iii) of Lemma 2.4 in the case where g is hyperbolic.

Lemma 7.6. *There is a neighborhood U of J so that the sets $U(m) = g^{-m}(U) \cap g^m(U)$ form a decreasing sequence of neighborhoods of J whose intersection is J. Each $U(m)$ has the property that points which leave $U(m)$ in forward time never return in forward time and points which leave $U(m)$ in backwards time never return in backwards time.*

Proof. Let us take $U = V - \cup B_i$, where the B_i are neighborhoods of the sinks such that $g(\cup B_i) \subset (\cup B_i)$. By Lemma 2.4, the set U has the property that if a point leaves U in forward (backward) time it never returns to U in forward (backward) time. The sets $U(m) = g^{-m}(U) \cap g^m(U)$ also have this property. By Theorem 5.7, the interior of K^+ is the union of the basins of attraction containing B_i, so the $U(m)$ approach J as $m \to \infty$.

Theorem 7.7. *If g is hyperbolic and $f \in \mathscr{G}$ is a polynomial diffeomorphism sufficiently close to g, then*

(1) *f is hyperbolic on J_f.*
(2) *$f|_{J_f}$ is topologically conjugate to $g|_{J_g}$.*

Proof. Let $\{V, V^\pm\}$ denote the filtration for g. If $f \in \mathcal{G}$ is sufficiently close to g in the uniform norm on V, then $\{V, V^\pm\}$ is a filtration for f. Let U be the open set as in Lemma 7.6, so that $J_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$. Let $M = V - \bigcup B_i$. For f sufficiently close to g it is still true that $J_f = \bigcap_{n \in \mathbb{Z}} f^n(U)$. As in [S, Proposition 8.22], for f close to g, f will be hyperbolic on $\bigcap_{n \in \mathbb{Z}} g^n(U)$. This set contains J_f so f will be hyperbolic on J_f. This proves (1).

As in [S, Prop. 8.23] $g|_{J_g}$ is topologically conjugate to f restricted to $\bigcap_{n \in \mathbb{Z}} g^n(U)$. In particular this latter set is the closure of the set of periodic points which are not sinks. Since f is J-hyperbolic we know that J_f is the closure of the set of non-sink periodic points. We conclude that $J_f = \bigcap_{n \in \mathbb{Z}} g^n(M)$ and that $g|_{J_g}$ is topologically conjugate to $g|_{J_f}$, which proves (2).

As we have seen, when g is hyperbolic it satisfies Axiom A. Axiom A basic sets have naturally associated measures and transverse measures, which we will describe. Let Fix_n be the set of fixed points of g^n. We say that a measure ν *describes the distribution of periodic points* if

$$\nu = \lim_{n \to \infty} \frac{1}{\# Fix_n} \sum_{x \in F_n} \delta_x .$$

It is a result of Bowen that Axiom A diffeomorphisms possess a measure, called *Bowen measure*, which describes the distribution of periodic points.

In various contexts Sinai, Margulis and Ruelle-Sullivan introduced transverse measures for hyperbolic dynamical systems. In the case of Axiom A diffeomorphisms these measures were introduced by Ruelle and Sullivan ([RS]). For a basic set B there is a stable transverse measure which is transverse to the stable foliation of $W^s(B)$ and an unstable transverse measure which is transverse to the unstable foliation of $W^u(B)$. These transverse measures are well defined up to scalar multiplication. Bowen measure and the Ruelle-Sullivan transverse measures are closely connected. With respect to the local product structure of an Axiom A basic set Bowen measure is locally the product of the stable and unstable Ruelle-Sullivan measures (up to a scalar multiple).

For the sake of stating the following results let us write $\mu = (1/4\pi^2)\mu_K$. This makes μ a probability measure.

Theorem 7.8. *If g is hyperbolic then μ^+ and μ^- are the Ruelle-Sullivan stable and unstable transverse measures and μ is Bowen measure.*

Proof. It is a result of Bowen and Marcus that the Ruelle-Sullivan transverse measures are the unique transverse measures (up to scalar multiplication) on the stable and unstable manifolds of mixing basic sets. Since μ^+ and μ^- define transverse measures on the stable and unstable manifolds of J by Theorem 6.5, they must be the Ruelle-Sullivan transverse measures.

Since Bowen measure is locally the product of the Ruelle-Sullivan stable and unstable transverse measures and μ is locally the product of μ^+ and μ^- it follows that Bowen measure and the measure μ are the same up to a scalar multiple. We have normalized the measure μ so that the scalar multiple is 1.

Corollary 7.9. *If g is hyperbolic on J, then:*

(i) *μ is mixing and in fact Bernoulli,*
(ii) *μ is unique measure of maximal entropy,*
(iii) *μ describes the distribution of periodic points.*

Proof. All these results follow from the identification of μ with Bowen measure. We have adopted (iii) as our definition of Bowen measure. The equivalence of (ii) and (iii) is discussed in ([Bo2)] Chapter 6). Item (i) is a property of Bowen measure for topologically mixing basic sets [(Bo2)] Chapter 6).

Remark. It would be interesting to know the behavior of general (non hyperbolic) diffeomorphisms in $\mathscr{G}$. J.H. Hubbard originally raised the question as to whether the periodic points are dense in J. He also asked whether the stable manifolds of hyperbolic points are dense in J^+. N. Sibony raised the question whether the complex equilibrium measure of the set J describes the distribution of periodic points. This paper was motivated in part by the desire to answer these questions in the hyperbolic case.

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