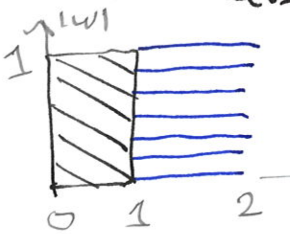
Propagation of Analyticity   August 27, 2020

We did the Hartogs lemma yesterday. It says that analyticity "propagates" along a family of parallel lines. This was a theorem in analysis — Osgood's Lemma used the Baire category theorem, and Hartogs' Theorem used Lebesgue measure.



How might we generalize this?

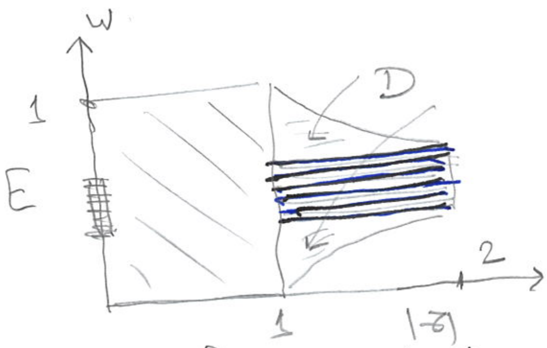
Q1 Analytical Generalization

Suppose $E \subset \Delta_w = \{|w|<1\}$ is given.

Suppose that for each $w \in E$, f extends analytically to $|z|<2$.

Suppose that f is analytic in $\{|z|, |w| < 1\}$.

Does f extend to a domain $D \supsetneq \{|z|, |w|<1\}$?



- we do not assume that E contains interior
- this question was in fact answered by Hartogs himself. We'll discuss it in class.

- this involves analytic continuation, not just analyticity of a pre-existing object.

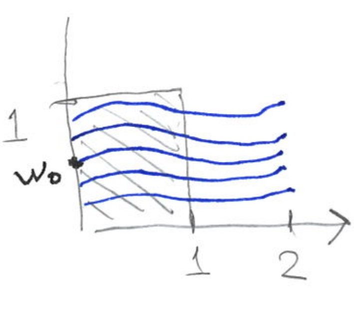
Q2 Geometric Generalization

Suppose that the "horizontals" are complex, but no longer flat? We assume that each blue curve is of the form $\{w = \varphi_{w_0}(z) : |z|<2\}$ where

$z \to \varphi_{w_0}(z)$ is analytic. But

$w_0 \to \varphi_{w_0}$ is continuous.

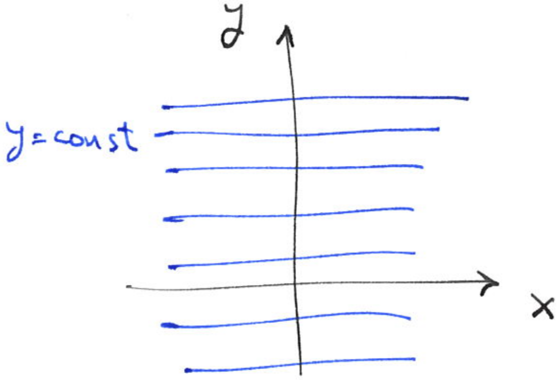
Does analyticity still propagate??

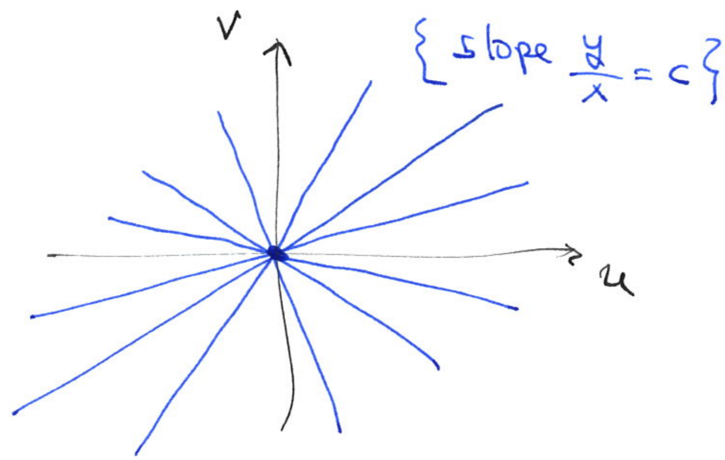


Propagation of Analyticity, cont'd.

If $\Gamma = \{ w = \varphi(z) \}$, (where φ is an analytic function) is an analytic curve, then we say that "$f(z,w)$"s analytic on Γ if $z \mapsto f(z, \varphi(z))$ is analytic.

Recall the map $(x,y) \longrightarrow (u = x,\ v = xy)$

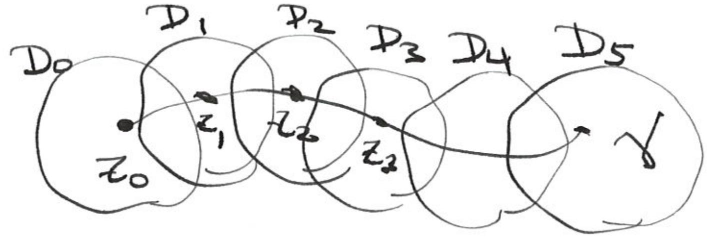




$f : \mathbb{C}^2 \to \mathbb{C}$

Question 3: Suppose $f(u,v)$ is analytic on every line through the origin. Does it follow that $f(u,v)$ is analytic? ??

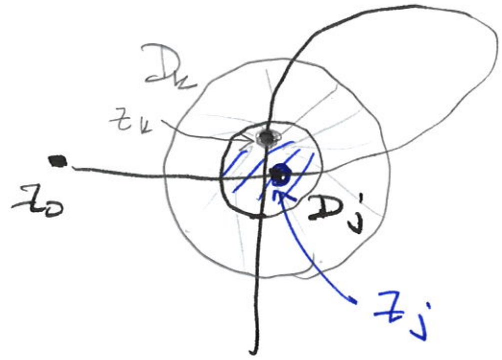
<u>Analytic Continuation</u> Let $D \overset{open}{\subset} \mathbb{C}^n$ be given, and let f be analytic on D. An <u>analytic continuation</u> of f is a path γ, starting at a point $z_0 \in D$, and there is a chain of disks centered at points of γ as pictured. On each disk D_j, there is an analytic function f_j.



Further:

1. $f_0 = f|_{D_0}$
2. $f_j = f_{j+1}$ on $D_j \cap D_{j+1}$.

- if γ crosses itself, it is not necessarily the case that the elements of the continuation agree



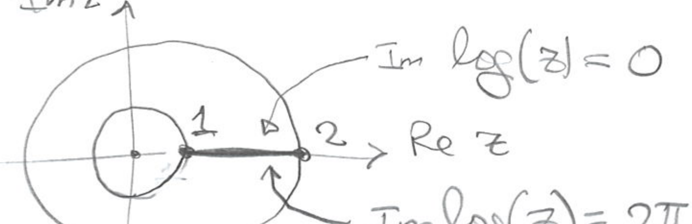
$k > j+1$

<u>Def</u>: $\xi \in \partial D$ is an <u>essential boundary point</u> (for f) if there is no analytic continuation along a path containing ξ.

- this definition is different from the one given by Bers. Recall the logarithm $\log(z)$ with a branch cut along the positive real axis:

Bers: $[1,2)$ consists of essential boundary points

Me: " not "



Exercise concerning Definition 2 (page 2 of Bers' notes) "holomorphic on K".

Let us define $K = \{(z,w) \in \mathbb{C}^2 : z^3 = w^2\} \subset \mathbb{C}^2$.

Let us parametrize by $\mathbb{C}$:

$$\mathbb{C} \ni t \longrightarrow (t^2, t^3) \in K.$$

Define $f, g, h : K \longrightarrow \mathbb{C}$ by: $\quad f(z,w) = t$

Show that g and h are holomorphic on K, but $\quad g(z,w) = t^2$, $\quad h(z,w) = t^3$.

f fails to be holomorphic on K near $(0,0)$.

Note that $f = h/g$ is continuous, but a continuous quotient may fail to be holomorphic. (Singular points can be tricky.)

What happens more generally if we take

$$K = \{ z^q = w^p \} \subset \mathbb{C}^2$$

and

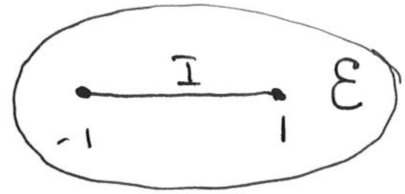
$$t \longrightarrow (t^p, t^q)$$

the book of Jarnicki & Pflug has a whole chapter on "Cross Theorems." Here is an example.

Let $I = [-1,1]$, $\mathcal{E}$ = ellipse $\subset \mathbb{C}$ with foci at $\{\pm 1\}$.

Let G denote the "relative Green function of $(I, \mathcal{E})$", i.e.

1. G is ~~clos~~ continuous on $\overline{\mathcal{E}}$.



2. G is harmonic on $\mathcal{E} - I$
3. $G = 0$ on I
4. $G = 1$ on $\mathcal{E}$.

(we may think of this as a capacitor, and G is the induced voltage.)

Given $f : (I \times \mathcal{E}) \cup (\mathcal{E} \times I) \longrightarrow \mathbb{C}$.

Theorem (Siciak). If f is separately analytic on $(I \times \mathcal{E}) \cup (\mathcal{E} \times I)$, then f extends analytically to $\{(z,w) \in \mathcal{E} \times \mathcal{E} : G(z) + G(w) < 1\}$.