PREFACE

These notes reproduce almost <u>verbatim</u> a course taught during the academic year 1962/63. The original notes, prepared by Joan Landman and Marion Weiner, were distributed to the class during the year. The present edition differs from the original only in that many mistakes have been corrected. I am indebted to Miss Weiner who prepared this edition and to several colleagues who supplied lists of errata.

I intended the course as an introduction to the modern theory of several complex variables, for people with background mainly in classical analysis. The choice of material and the mode of presentation were determined by this aim. Limitations of time necessitated omitting several important topics.

Every account of the theory of several complex variables is largely a report on the ideas of Oka. This one is no exception.

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Zurich, July 8, 1964.

CONTENTS

Chapter 1. Basic Facts about Holomorphic Functions

§1. Preliminaries

We introduce the following notation:

$\mathbb{R}$ denotes the field of real numbers.

$\mathbb{C}$ denotes the field of complex numbers or the complex plane.

$\mathbb{C}^n$ denotes the space of n-tuples of complex numbers $(z_1, \ldots, z_n) = z$. $\mathbb{C}^n$ may be considered as an n-dimensional vector space over $\mathbb{C}$ or a 2n-dimensional vector space over $\mathbb{R}$. $\mathbb{C}^n$ may therefore be identified with $\mathbb{R}^{2n}$, which induces a topology in $\mathbb{C}^n$.

A. By function, we will mean a complex-valued function f unless otherwise stated, for instance $f\colon \mathbb{C}^n \to \mathbb{C}$.

<u>Definition 1.</u> Let $D \subset \mathbb{C}^n$ be open and $f(z_1, \ldots, z_n)$ a function defined in D. f is said to be <u>holomorphic in D</u> if, for every $(z_1, \ldots, z_n) \in D$ and each $j = 1, 2, \ldots, n$,

$$\lim_{|h| \to 0} \frac{f(z_1, \ldots, z_j + h, \ldots, z_n) - f(z_1, \ldots, z_n)}{h} \equiv \frac{\partial f}{\partial z_j}$$

exists and is finite.

We remark that f is holomorphic (in D) if it is holomorphic in each variable <u>separately</u>. Note that f is <u>not</u> assumed to be continuous. Hence we obtain immediately:

<u>Property 1.</u> If f_1 and f_2 are holomorphic (in D) then $f_1 + f_2$, $f_1 - f_2$, $f_1 f_2$, and, if $f_2 \neq 0$, f_1/f_2 are holomorphic (in D).

<u>Property 2.</u> If $\{f_j\}$ is a uniformly convergent sequence of holomorphic functions in D converging to f, then f is holomorphic in D. That is, functions holomorphic on an open set form a ring which is closed under uniform convergence.

Property 3. Maximum Principle. If f is holomorphic in D and has a local maximum at a point $p \in D$ then f is identically constant in the component of D containing p.

Definition 2. Let $K \subset \mathbb{C}^n$ be any set. Let f be defined in K. Then f is holomorphic in K if and only if to every point p of K there exists an open set $D \subset \mathbb{C}^n$ such that $p \in D$, $D \cap K$ is closed in D(i. e. D-K is open in $\mathbb{C}^n$), and there exists a function F holomorphic in D such that $F = f$ on $D \cap K$.

B. We remark that for functions $f: \mathbb{R}^n \to \mathbb{R}$, the existence of all partial derivatives $\partial f / \partial x_i$, $i = 1, \ldots, n$ does not imply f is continuous. However, the corresponding theorem for functions of several complex variables is true.

Theorem 1 (Hartogs). Every holomorphic function is continuous (in all variables simultaneously).

Proof. Given in § 3.

For the remainder of this section and Section 2, assume Theorem 1.

Definitions 3. A closed polydisc about z^0 in $\mathbb{C}^n$ is the set $\{(z_1, \ldots, z_n) \mid |z_j - z_j^0| \leq r_j,\ j = 1, \ldots, n;\ 0 < r_j < \infty\}$, denoted $\{|z_j - z_j^0| \leq r_j\}$.

An open polydisc in $\mathbb{C}^n$ is the interior of a closed polydisc, i. e. the set $\{(z_1, \ldots, z_n) \mid |z_j - z_j^0| < r_j,\ j = 1, \ldots, n;\ 0 < r_j < \infty\}$, denoted $\{|z_j - z_j^0| < r_j\}$.

The boundary of a polydisc is the set $\{(z_1, \ldots, z_n) \mid |z_j - z_j^0| \leq r_j,\ j = 1, \ldots, n, \text{ and } |z_j - z_j^0| = r_j \text{ for some } j\}$.

The distinguished boundary of a polydisc is the set $\{(z_1, \ldots, z_n) \mid z_j = z_j^0 + r_j e^{i\theta_j},\ 0 \leq \theta_j < 2\pi\}$.

Note that the dimension (over $\mathbb{R}$) of the boundary of a polydisc is 2n-1, while the dimension of the distinguished boundary is n.

We remark that proofs will be exhibited for the case n = 2. The proofs in the general case are similar and may be completed via an induction argument.

Theorem 2 (Cauchy Formula). Let $f(z_1, \ldots, z_n)$ be holomorphic in the closed polydisc $\{|z_j - z_j^0| \leq r_j\}$. Then for $\{|z_j - z_j^0| < r_j\}$,

$$f(z_1, \ldots, z_n) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_1 - z_1^0| = r_1} \cdots \int_{|\zeta_n - z_n^0| = r_n} \prod_{j=1}^{n} \frac{1}{\zeta_j - z_j} f(\zeta_1, \ldots, \zeta_n) d\zeta_n \ldots d\zeta_1 .$$

Proof. (for n = 2). We may assume $z^0 = 0$. $f(z_1, z_2)$ is, for fixed z_2, $|z_2| \leq r_2$, a holomorphic function of z_1 in $|z_1| \leq r_1$. Thus by Cauchy's Formula

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1| = r_1} \frac{1}{\zeta_1 - z_1} f(\zeta_1, z_2) d\zeta_1 .$$

Similarly, $f(\zeta_1, z_2)$ is a holomorphic function of z_2 for each ζ_1. Applying Cauchy's Formula to $f(\zeta_1, z_2)$ gives

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1| = r_1} \frac{1}{\zeta_1 - z_1} \left\{ \frac{1}{2\pi i} \int_{|\zeta_2| = r_2} \frac{1}{\zeta_2 - z_2} f(\zeta_1, \zeta_2) d\zeta_2 \right\} d\zeta_1 .$$

By Theorem 1 we may write this as a double integral

$$f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{|\zeta_1| = r_1} \int_{|\zeta_2| = r_2} \frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)} f(\zeta_1, \zeta_2) d\zeta_2 d\zeta_1 .$$

Corollary 1. A holomorphic function has derivatives w. r. t. x_i and y_i, $i = 1, \ldots, n$, of all orders, i. e. holomorphic implies C^∞.

Proof. Differentiate under integrals in Cauchy Formula as many times as desired.

Corollary 2. f holomorphic implies $\partial f / \partial \bar{z}_j$ holomorphic.

Corollary 3. Let f be holomorphic in the polydisc $P = \{|z_j - z_j^0| \leq r_j\}$, and let $|f| \leq M$ on P, then

$$\left|\left(\frac{\partial^{\nu_1+\dots+\nu_n} f}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}\dots\partial z_n^{\nu_n}}\right)_{z^0}\right| \le \frac{\nu_1!\dots\nu_n!\,M}{r_1^{\nu_1}\dots r_n^{\nu_n}} ,$$

where z^0 denotes the center of the polydisc.

In Corollary 3, f is assumed to be holomorphic in the open polydisc and continuous on its closure. Without assuming continuity on the boundary the inequality still holds with $M = \sup |f|$.

Corollary 4. If f_μ are holomorphic functions in $D \subset \mathbb{C}^n$ and $f_\mu \to f$ uniformly in D then $\partial f_\mu / \partial z_j \to \partial f / \partial z_j$ normally, i. e. uniformly on compact subsets of D.

Corollary 5. Power Series Expansion. Let f be holomorphic in int P, then

$$f(z_1,\dots,z_n) = \Sigma\, a_{\nu_1\dots\nu_n}(z_1-z_1^0)^{\nu_1}\dots(z_n-z_n^0)^{\nu_n}, \qquad \nu_i \ge 0,$$

which converges absolutely and normally in the open polydisc.

Proof. Same as in one variable. Expand $1/(\zeta_j - z_j)$ etc.

Note. Determining the domain of convergence of a given power series is not trivial. We will do it later.

Corollary 6. (Continuation of 5)

$$a_{\nu_1\dots\nu_n} = \frac{1}{\nu_1!\dots\nu_n!}\left(\frac{\partial^{\nu_1+\dots+\nu_n} f}{\partial z_1^{\nu_1}\dots\partial z_n^{\nu_n}}\right)_{z^0} .$$

Proof. By Corollaries 4 and 5.

Corollary 7. The power series expansion of a holomorphic function is unique.

Proof. Corollary 6 gives the coefficients.

Corollary 8. If f is holomorphic in a domain D and $f \equiv 0$ in a neighborhood of a point of D, then $f \equiv 0$ in D.

This is the basis of analytic continuation.

§ 2. An inequality

Definition 4. A continuous real-valued function u, of two real variables, in a plane domain $\mathcal{D}$ is said to be subharmonic if, for any closed disc Δ in $\mathcal{D}$, and for the corresponding function ϕ such that $\phi = u$ on the boundary of Δ, and ϕ harmonic in Δ, $u \leq \phi$ in Δ.

Properties of subharmonic functions.

1. If u and v are subharmonic then max (u, v) is subharmonic.
2. Being subharmonic is a local property, i. e., if a function is subharmonic in a neighborhood of every point of a domain then it is subharmonic in the domain.
3. A harmonic function is subharmonic.

Remark. If f(z) is a holomorphic function and $\epsilon > 0$ then $\log \max (|f(z)|, \epsilon)$ is subharmonic.

Proof. At a point where $|f| < \epsilon$, and thus in a neighborhood of this point, $\log \max (|f(z)|, \epsilon) = \log \epsilon = \log$ (constant) is subharmonic. At a point where $|f| > \epsilon$, $\log \max (|f|, \epsilon) = \log |f|$ which is harmonic since $f \neq 0$ and therefore subharmonic. At a point where $|f| = \epsilon$, and in a disc such that $|f| > \epsilon/2$, $\max \log (|f|, \epsilon)$ is also subharmonic and log max = max log.

Theorem 3. Under the hypothesis of Theorem 2 and for $r_j = 1$, $z_j = \rho_j e^{i\phi_j}$

$$\log |f(z_1, \dots, z_n)| \leq \left(\frac{1}{2\pi}\right)^n \underbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}_{\text{(n-fold)}} \prod_{j=1}^{n} \frac{1-\rho_j^2}{1-2\rho_j \cos(\phi_j - \theta_j) + \rho_j^2} \log |f(e^{i\theta_1}, \dots, e^{i\theta_n})| \, d\theta_1 \dots d\theta_n .$$

Proof. Take $\epsilon > 0$. Let $g_\epsilon = \log \max (|f(z_1, \dots, z_n)|, \epsilon)$. If we fix all variables but one, g_ϵ is a subharmonic function. We will prove the theorem for n = 2. Thus

$g_\epsilon = \log \max(|f(z_1, z_2)|, \epsilon)$. Fix z_2, then $g_\epsilon(z_1, z_2)$ is a subharmonic function of z_1. Thus for $z_1 = \rho_1 e^{i\phi_1}$

$$g_\epsilon(z_1, z_2) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho_1^2}{1-2\rho_1 \cos(\phi_1-\theta_1)+\rho_1^2}\, g_\epsilon(e^{i\theta_1}, z_2)\,d\theta_1 .$$

For fixed θ_1, $g_\epsilon(e^{i\theta_1}, z_2)$ is a subharmonic function of $z_2 = \rho_2 e^{i\phi_2}$. Thus

$$g_\epsilon(z_1, z_2) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho_1^2}{1-2\rho_1 \cos(\phi_1-\theta_1)+\rho_1^2} \cdot \left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho_2^2}{1-2\rho_2 \cos(\phi_2-\theta_2)+\rho_2^2}\, g_\epsilon(e^{i\theta_1}, e^{i\theta_2})\,d\theta_2\right\} d\theta_1 .$$

Using the continuity of f, write this as a double integral. But $\log|f(z_1, z_2)| \leq g_\epsilon(z_1, z_2)$, obtaining

$$\log|f(z_1, z_2)| \leq \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \prod_{j=1}^{2} \frac{1-\rho_j^2}{1-2\rho_j \cos(\theta_j-\phi_j)+\rho_j^2}\, g_\epsilon(e^{i\theta_1}, e^{i\theta_2})\,d\theta_1 d\theta_2 .$$

Let $\epsilon \to 0$, then $g_\epsilon \downarrow \log|f|$. Therefore by the Lebesgue Monotone Convergence Theorem

$$\log|f(z_1, z_2)| \leq \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \prod_{j=1}^{2} \frac{1-\rho_j^2}{1-2\rho_j \cos(\theta_j-\phi_j)+\rho_j^2}\, \log|f(e^{i\theta_1}, e^{i\theta_2})|\,d\theta_1 d\theta_2 .$$

§ 3. Proof of Hartogs' Theorem 1

Use induction on n. The theorem is trivially true for $n = 1$. Assume the theorem, and therefore all corollaries and following theorems for n-1 variables.

Osgood's Lemma. Let $f(z_1, \ldots, z_n)$ be holomorphic for $|z_j| \leq R_j$, $0 < R_j < \infty$. Then there exists ζ such that $|\zeta| < R_1$, and $\rho > 0$ such that $\rho < R_1 - |\zeta|$, and a number M, such that $|f(z_1, \ldots, z_n)| < M$ for $|z_1 - \zeta| \leq \rho$ and $|z_j| \leq R_j$ for $j \geq 2$.

Hartogs' Lemma. Let $f(z_1, \ldots, z_n)$ be holomorphic in $|z_j| \leq R_j$ and bounded for $|z_1 - \zeta| \leq \rho$ and $|z_j| \leq R_j$, $j \geq 2$. Then f is continuous in $|z_j| < R_j$.

Proof of Osgood's Lemma. Define for $|z| \leq R_1$,

$$m(z) = \max_{\substack{|z_j| \leq R_j \\ j \geq 2}} |f(z, z_2, \ldots, z_n)|.$$

This maximum exists by induction hypothesis. Denote by $\lambda_N = \{z \mid |z| \leq R_1 \text{ and } m(z) \leq N\}$. Then $\{z \mid |z| \leq R_1\} = \bigcup_{N=1}^{\infty} \lambda_N$. Now λ_N is closed, for if $a_r \in \lambda_N$ and $a_r \to a$,

$$m(a) = \max |f(a, z_2, \ldots, z_n)| = \max \lim_{r \to \infty} |f(a_r, z_2, \ldots, z_n)| \leq \max N = N.$$

For instance, by the Baire Category Theorem, one of the λ_N contains a disc. For z_1 in this disc f is bounded.

Proof of Hartogs' Lemma. We may assume $R_j = 1$, $j = 1, \ldots, n$, $\zeta = 0$, and $M = 1$. Let $D_1 = \{(z_1, \ldots, z_n) \mid |z_1| \leq \rho < 1, |z_2| \leq 1, \ldots, |z_n| \leq 1\}$. Let $z^0 = (z_1^0, \ldots, z_n^0)$ and $z^1 = (z_1^1, \ldots, z_n^1)$ be interior points of D_1. Then

$$|f(z^0)-f(z^1)| \le |f(z_1^0, z_2^0, \ldots, z_n^0)-f(z_1^1, z_2^0, \ldots, z_n^0)| + \ldots$$

$$+ |f(z_1^1, z_2^1, \ldots, z_{n-1}^1, z_n^0)-f(z_1^1, \ldots, z_n^1)|$$

$$\le |z_1^0-z_1^1| \frac{1}{(\rho-|z_1^0|)(\rho-|z_1^1|)} + \ldots$$

$$+ |z_n^0-z_n^1| \frac{1}{(1-|z_n^0|)(1-|z_n^1|)}$$

by the Cauchy integral representation on 1 variable and the existence of the maximum of $|f|$ in D_1. Since all the denominators are bounded away from zero as $z^1 \to z^0$, $|f(z^0)-f(z^1)| \to 0$ as $z^1 \to z^0$, proving the continuity of f at z^0. The arbitrariness of the choice of z^0 implies the continuity of f in the interior of D_1.

Now fix $z_2, \ldots, z_n$; $|z_i| < 1$. Then

$$f(z_1, z_2, \ldots, z_n) = \sum_{\nu=0}^{\infty} a_\nu(z_2, \ldots, z_n)z_1^\nu \quad \text{for} \quad |z_1| \le 1 ,$$

where

$$a_\nu(z_2, \ldots, z_n) = \frac{1}{\nu!}\left(\frac{\partial^\nu f}{\partial z_1^\nu}\right)_{z_1=0} ,$$

by the holomorphicity of f in z_1. Near $z_1 = 0$, $(z_1, \ldots, z_n)$ lies in D_1 and here f is bounded and continuous. Thus

$$a_\nu(z_2, \ldots, z_n) = \frac{1}{2\pi i} \int_{|\zeta|=\rho_1<\rho} \frac{f(\zeta, z_2, \ldots, z_n)}{\zeta^{\nu+1}} d\zeta$$

and therefore the a_ν are holomorphic. By our induction hypothesis, the a_ν are continuous, and the proof will be complete once we show that the power series converges normally for $|z_1| < 1$. Now,

$$\limsup_{\nu\to\infty} | a_\nu(z_2, \ldots, z_n)|^{\frac{1}{\nu}} < 1$$

and

$$|a_\nu(z_2,\ldots,z_n)| \leq \frac{1}{\rho^\nu},$$

since the series converges for $|z_1| < 1+\eta$, for some $\eta > 0$. Set

$$\Lambda = \left\{ z_j = e^{i\phi_j} \middle| 0 \leq \phi_j < 2\pi,\ j = 2,\ldots,n \right\},$$

$$\Lambda_\nu = \left\{ (z_2,\ldots,z_n) \middle| |a_\nu(z_2,\ldots,z_n)|^{\frac{1}{\nu}} > 1,\ (z_2,\ldots,z_n) \in \Lambda \right\}.$$

Then Λ_ν is open in Λ, and $\lim_{\nu\to\infty} m(\Lambda_\nu) = 0$, where $m(\Lambda_\nu)$ denotes the measure of Λ_ν. For, if we let

$$Q_j = \bigcup_{\nu=j}^{\infty} \Lambda_\nu ,$$

and note that

$$Q_j \supset Q_{j+1}$$

and

$$m(\cap Q_j) = \lim_{j\to\infty} m(Q_j)$$

then

$$\cap Q_j = \phi \text{ (the empty set)}.$$

Thus

$$m(\cap Q_j) = 0 = \lim_{j\to\infty} m(Q_j).$$

Since $Q_j \supset \Lambda_j$, $m(\Lambda_j) \to 0$.

Now let $z_j = r_j e^{i\phi_j}$, $r_j \leq r < 1$, $j = 2,\ldots,n$. By Theorem 3,

$$\frac{1}{\nu}\log|a_\nu(z_2,\ldots,z_n)|$$

$$\leq \frac{1}{\nu(2\pi)^{n-1}} \int \cdots \int_{\Lambda} \prod_{j=2}^{n} \frac{(1-r_j^2)\log|a_\nu(e^{i\theta_2},\ldots,e^{i\theta_n})|}{1-2r_j\cos(\phi_j-\theta_j)+r_j^2}\, d\theta_2\cdots d\theta_n$$

$$\leq \frac{1}{\nu(2\pi)^{n-1}} \int \cdots \int_{\Lambda_\nu} \prod_{j=2}^{n} \frac{(1-r_j^2)\log|a_\nu(e^{i\theta_2},\ldots,e^{i\theta_n})|}{1-2r_j\cos(\phi_j-\theta_j)+r_j^2}\, d\theta_2\cdots d\theta_n.$$

Furthermore

$$\frac{1-r_j^2}{1-2r_j\cos(\phi_j-\theta_j)+r_j^2} \leq \frac{1-r_j^2}{(1-r_j)^2} = \frac{1+r_j}{1-r_j} \leq \frac{1+r}{1-r}$$

and $|a_\nu| \leq 1/\rho^\nu$. Hence,

$$\frac{1}{\nu}\log|a_\nu| \leq \frac{1}{(2\pi)^{n-1}}\frac{\nu\log\frac{1}{\rho}}{\nu}\left(\frac{1+r}{1-r}\right)^{n-1}\int\ldots\int_{\Lambda_\nu} d\theta_2\ldots d\theta_n$$

$$= \frac{1}{(2\pi)^{n-1}}\left(\frac{1+r}{1-r}\right)^{n-1}\log\frac{1}{\rho}\, m(\Lambda_\nu).$$

Therefore for $\epsilon > 0$ there exists a number $N(\epsilon) > 0$ such that if $\nu > N(\epsilon)$ then

$$\frac{1}{\nu}\log|a_\nu| \leq \epsilon\log\frac{1}{\rho} = \log\left(\frac{1}{\rho}\right)^\epsilon ,$$

i. e.

$$|a_\nu(z_2,\ldots,z_n)| < \frac{1}{\rho^{\epsilon\nu}} \text{ for } |z_j| \leq r < 1, \quad j = 2,\ldots,n, \quad \nu > N(\epsilon).$$

Hence, the series converges normally for $|z_j| \leq r < 1$, $j = 2,\ldots, n$ and $|z_1| < \rho^\epsilon$.

§ 4. Holomorphic Mappings

In this section we make several remarks about the mappings determined by holomorphic functions.

Note that: f is holomorphic if $f \in C^\infty$ and

$$\frac{\partial f}{\partial \bar{z}_j} = 0 , \qquad j = 1,\ldots,n,$$

where

$$2\frac{\partial f}{\partial \bar{z}_j} = \frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j} .$$

We remark that, for $n > 1$, these criteria for holomorphicity form an overdetermined system of equations. Many of the phenomena associated with functions of several complex variables arise from just this fact.

Observe that as an easy consequence of the above remark, a holomorphic function of holomorphic functions is holomorphic.

Definition 5. Given $f_j: D \to \mathbb{C}$, $D \subset \mathbb{C}^n$, $j = 1, \ldots, n$; denote $f_j(z_1, \ldots, z_n) = \zeta_j$, $f = \{f_j\}$ is a holomorphic mapping if each f_j is holomorphic in D.

Note also that if

$$\zeta_j = \xi_j + i\eta_j \, ; \, z_j = x_j + iy_j ,$$

then

$$J = \frac{\partial(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)}{\partial(x_1, \ldots, x_n; y_1, \ldots, y_n)} = \left| \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(z_1, \ldots, z_n)} \right|^2 .$$

In particular, $J \geq 0$, i.e., a holomorphic mapping preserves orientation. If $J \neq 0$, the map f is locally 1-1, and has an inverse, f^{-1}, which is holomorphic. Hence, a holomorphic mapping such that $J \neq 0$ carries holomorphic functions into holomorphic functions.

However, under a holomorphic mapping it is possible to map a bounded domain onto all of $\mathbb{C}^n$. There is no simple geometric characterization of a holomorphic map (e.g. angle-preserving), and no Riemann's mapping theorem. Hence, there is no canonical domain, like the disc, and we are forced to consider arbitrary domains.

Chapter 2. Domains of Holomorphy

§1. Examples and definitions

Definition 6. Let $D_1 \overset{\text{open}}{\subset} D_2 \overset{\text{open}}{\subset} \mathbb{C}^n$. If f holomorphic in D_1 implies that there exists an F holomorphic in D_2 such that $F|D_1 = f$, then D_1 and D_2 are said to exhibit Hartogs' phenomenon.

Note that this does not occur for $n = 1$.

A. Example 1 ($n = 2$). Let

$$D = \left\{(z, w) \,\middle|\, |z| < 1,\ |w| < 1 \ \text{ and not } \ |z| \le \tfrac{1}{2} \ \text{ and } \ |w| \le \tfrac{1}{2}\right\}$$

and

$$D^* = \left\{(z, w) \,\middle|\, |z| < 1,\ |w| < 1\right\}.$$

We claim that D and D^* exhibit Hartogs' phenomenon. Let f(z, w) be holomorphic in D. Let

$$D' = \left\{(z, w) \,\middle|\, \tfrac{1}{2} < |z| < 1,\ |w| < 1\right\}.$$

$D' \subset D$. Hence

$$f(w, z) = \sum_{-\infty}^{\infty} a_j(w) z^j \quad \text{for fixed } w,\ |w| < 1 \ \text{ and } \tfrac{1}{2} < |z| < 1 .$$

Now the $a_j(w)$ are holomorphic in w, for

$$a_j(w) = \frac{1}{2\pi i} \int_{|z| = \frac{1}{2} + \frac{1}{10}} \frac{f(w, z)}{z^{j+1}} \, dz .$$

Let

$$D'' = \left\{(z, w) \,\middle|\, \tfrac{1}{2} < |w| < 1,\ |z| < 1\right\},$$

$D'' \subset D$, here:

$$f(w, z) = \sum_{j=0}^{\infty} b_j(w) z^j.$$

For w, z such that $\frac{1}{2} < |w| < 1$ and $\frac{1}{2} < |z| < 1$, the series must agree. Therefore

$$a_j = b_j \quad , \quad j \geq 0$$

$$a_j = 0 \quad \text{for} \quad j < 0 \ .$$

Thus $f(w, z) = \sum_0^\infty a_j(w) z^j$, for every fixed w, $|w| < 1$ and for every z, $|z| < 1$, i.e. in D^*.

A similar proof gives the holomorphicity in w.

Corollary. (Hartogs' Theorem). A holomorphic function of at least two variables cannot have isolated non-removable singularities.

Exercise. Consider the domain $0 < r < |z_1|^2 + \ldots + |z_n|^2 < 1$. Show that, if $f(z_1, \ldots, z_n)$ is holomorphic in such a spherical shell, then it is holomorphic for $0 \leq |z_1|^2 + \ldots + |z_n|^2 < 1$.

Theorem 4. (Hartogs' 2nd Theorem). Let $D \subset \mathbb{C}^n$ be any domain homeomorphic to a ball, and bounded by a sufficiently smooth surface Σ. Then if F is holomorphic in a neighborhood of Σ, F can be continued analytically over D.

We remark that this is a theorem in overdetermined systems, and shall not be proven here.

Example 2. Let $D = \left\{(z, w) \,\middle|\, |w| < \frac{1}{2}, |z| < 1\right\} \cup \left\{(z, w) \,\middle|\, \frac{1}{2} < |z| < 1, |w| < 1\right\}$. Then if f is holomorphic in D, f is holomorphic in $D^* = \left\{(z, w) \,\middle|\, |w| < 1, |z| < 1\right\}$. (Proof as above.) Note that D is a cell.

B. Definition 7. Let $D^{\text{open}} \subset \mathbb{C}^n$ and $\xi \in$ boundary of D. ξ is said to be an essential boundary point of D if and only if there exists an f, holomorphic in D and singular at ξ (i.e., f is not the restriction of a function holomorphic in a domain $D_1 \supset D$ such that $\xi \in D_1$.

Definition 8. $D^{\text{open}} \subset \mathbb{C}^n$ is called a region of holomorphy if and only if there exists a function f, holomorphic in D, and singular at every boundary point.

We shall show that, if every boundary point of D is essential, then D is a region of holomorphy.

§ 2. Convexity with respect to a family of functions

A. Let X denote a topological space and $\mathcal{F}$ a family of real or complex valued continuous functions defined in X. Let $K \subset X$.

Definition 9. The $\mathcal{F}$-hull, $\hat{K}_{\mathcal{F}}$, of K is the set of points p of X, such that for each function $f \in \mathcal{F}$ which is bounded by 1 on K, $|f(p)| \leq 1$.

We remark that $\hat{K}_{\mathcal{F}}$ is closed in X.

Note: If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families such that $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\hat{K}_{\mathcal{F}_1} \supset \hat{K}_{\mathcal{F}_2}$.

Definition 10. X is called $\mathcal{F}$-convex if K compact implies that $\hat{K}_{\mathcal{F}}$ is compact.

Example. Suppose $X = D^{\text{open}} \subset \mathbb{R}^n$, and $\mathcal{F}$ is the family of functions which are linear on every component of D. Then D is $\mathcal{F}$-convex if and only if every component of D is convex, in the ordinary sense.

If $\mathcal{F}$ consists of linear functions, then D is $\mathcal{F}$-convex if it is convex.

Lemma 1. Let $K \subset \mathbb{C}^n$. Then K bounded implies $\hat{K}_{\mathcal{F}}$ bounded, where $\mathcal{F}$ is the family of monomials in $z_1, \ldots, z_n$.

Proof. Let $M_i = \sup\limits_{z \in K} |z_i|$, which exists and is finite as K is bounded. The functions $g_i(z) = \frac{z_i}{M_i} \in \mathcal{F}$, $i = 1, \ldots, n$. But $|g_i(z)| \leq 1$ for every $z \in K$, hence $|g_i(\hat{z})| \leq 1$ for every $\hat{z} \in \hat{K}_{\mathcal{F}}$. Hence, if $\hat{z} \in \hat{K}_{\mathcal{F}}$, $||\hat{z}|| \leq \max M_i < \infty$, i. e. $\hat{K}_{\mathcal{F}}$ is bounded.

B. Note. $\mathbb{C}^n$ is a normed vector space under the norm, $||z|| = \max(|z_1|, \ldots, |z_n|)$ (and under many norms too, of course).

Definition 11. Let $D^{\text{open}} \subset \mathbb{C}^n$. The distance of a point z from the boundary of D, denoted by $\Delta_D(z)$ or simply $\Delta(z)$, is $\Delta(z) \equiv \inf\limits_{\zeta \in \text{bndry } D} ||z - \zeta||$.

$\Delta(z)$ satisfies a Lipschitz condition with constant 1.

Proof. $\Delta(z') = \inf_{\zeta} ||z'-\zeta|| \leq \inf_{\zeta} (||z'-z''|| + ||z''-\zeta||)$

$$\leq ||z'-z''|| + \inf_{\zeta} ||z''-\zeta|| = ||z'-z''|| + \Delta(z'')$$

Therefore $|\Delta(z')-\Delta(z'')| \leq ||z'-z''||$.

Definition 12. For any set $K \subset D^{\text{open}} \subset \mathbb{C}^n$.

$$\Delta_D(K) \equiv \inf_{z \in K} \Delta_D(z) \ .$$

C. Note. $K \subset\subset D$, read "K is relatively compact in D", is defined to mean that the closure of K is compact and contained in D; hence $\Delta(\text{cl } K) > 0$.

Theorem 5. (Cartan-Thullen). If $D^{\text{open}} \subset \mathbb{C}^n$, then the following conditions are equivalent:

(i) D is a region of holomorphy.

(ii) All boundary points of D are essential.

(iii) If $K \subset\subset D$, then $\Delta(K) = \Delta(\hat{K})$, where $\hat{K} = \hat{K}_{\mathcal{F}}$ and $\mathcal{F}$ is the family of all functions holomorphic in D.

(iv) D is holomorphically convex.

i.e. $K \subset\subset D$ implies $\hat{K} \subset\subset D$.

Corollary 1. If $n = 1$, then every open set is a region of holomorphy.

Corollary 2. D is a region of holomorphy if and only if every component of D is a domain of holomorphy.

Corollary 3. If $D_1 \subset \mathbb{C}^p$ and $D_2 \subset \mathbb{C}^q$ are domains of holomorphy, then $D_1 \times D_2 \subset \mathbb{C}^{p+q}$ is a domain of holomorphy.

Corollary 4. If D_α is a region of holomorphy for every α in some set A, and if $\cap D_\alpha$ is open, then $\cap D_\alpha$ is a region of holomorphy.

Proof. Assume $K \subset\subset (\cap D_\alpha)$; then $K \subset\subset D_\alpha$ for every α. Let K_α denote the hull of K in D_α with respect to all functions holomorphic in D_α. K_α is compact by assumption and $\hat{K} \subset K_\alpha$. Thus $\hat{K} \subset \bigcap_\alpha K_\alpha$, and hence $\hat{K}$ is compact. Since $\hat{K} \subset \cap D_\alpha$, $\hat{K} \subset\subset (\cap D_\alpha)$.

Note. In general, a union of regions of holomorphy is not a region of holomorphy. However, if the regions are nested, i.e. $D_1 \subset D_2 \subset \ldots$, then their union is a region of holomorphy. This, however, we will prove much later.

Corollary 5. (Exercise) If D is geometrically convex then D is a region of holomorphy.

D. Proof of Theorem 5. If $D = \mathbb{C}^n$ the theorem is easily verified. Therefore, assume that D is a proper subset of $\mathbb{C}^n$.

(i) implies (ii) by the definition of a region of holomorphy.

(ii) implies (iii). Assume (iii) does not hold. Then $K \subset\subset D$ and $\Delta(K) \neq \Delta(\hat{K})$. Let $\Delta(K) = M$ and $\Delta(\hat{K}) = m$. Then $m < M$ as $\hat{K} \supset K$. Choose r, R such that $m < r < R < M$. Thus, there exists a $\hat{z} \in \hat{K}$ such that $\Delta(\hat{z}) < r$. Let $\tilde{K} = \bigcup_{z' \in K} \{z \mid \|z - z'\| \leq R\}$. Then $\tilde{K}$ is compact. Let f be a holomorphic function in D. $\max_{z \in \tilde{K}} |f(z)| = \mu$ exists and is finite as f is continuous and $\tilde{K}$ compact. If $z \in K$ then

$$\left| \left(\frac{\partial^{\nu_1 + \ldots + \nu_n} f}{\partial z_1^{\nu_1} \ldots \partial z_n^{\nu_n}} \right)_z \right| \leq \frac{\nu_1! \ldots \nu_n! \mu}{R^{\nu_1 + \ldots + \nu_n}} .$$

Since this inequality holds for all $z \in K$ it holds at every point of $\hat{K}$, and in particular, at $\hat{z}$. Thus if f is expanded in a power series about $\hat{z}$, the series will converge inside a polydisc of radius R about $\hat{z}$. However, this polydisc contains, in its interior, points on the boundary of D. Therefore, these boundary points are not essential.

(iii) implies (iv). Assume that $K \subset\subset D$. Then K is bounded, and hence $\hat{K}$ is bounded, by Lemma 1 above. $\hat{K}$ is closed relative to D, but since $\Delta(K) = \Delta(\hat{K}) > 0$, $\hat{K}$ is compact.

(iv) implies (i). Let $\tilde{D}$ be any component of D. Construct a sequence of sets D_j, $j = 1, 2, \ldots$ such that $D_1 \subset\subset D_2 \subset\subset D_3 \ldots \subset\subset \tilde{D}$ and $\tilde{D} = \bigcup_j D_j$, (e.g. $D_j = \left\{ z \mid z \in \tilde{D}, \|z\| < j, \Delta_{\tilde{D}}(z) > \frac{1}{j} \right\}$). Let $\{p_j\}$ be dense in $\tilde{D}$. ($\{p_j\}$ could be chosen as the set of points of $\tilde{D}$ whose coordinates in $\mathbb{R}^{2n}$ are rational). For each j we find a point $z_j \in \tilde{D}$ such that $||z_j - p_j|| < \Delta_{\tilde{D}}(p_j)$ and $z_j \notin \hat{D}_j$. We can find such a point since $\hat{D}_j$ doesn't come arbitrarily close to the boundary of $\tilde{D}$. Now, for each j, there exists a function f_j holomorphic in $\tilde{D}$, such that $|f_j| \leq 1$ in D_j and $|f_j(z_j)| = A_j > 1$. Next choose integers $N_j > 0$ such that

$$\sum_1^\infty jA_j^{-N_j} < \infty .$$

Let

$$g(z) = \prod_{j=1}^{\infty} \left[1 - \left(\frac{f_j(z)}{f_j(z_j)} \right)^{N_j} \right]^j .$$

Each term of g(z) is holomorphic. In fact, the infinite product converges normally to a function not indentically zero. For $\left| \frac{f_j(z)}{f_j(z_j)} \right| < 1$ for each j, in D_j, implies that $\left| \frac{f_j(z)}{f_j(z_j)} \right| < 1$ for all j, in D_1. Furthermore, recall that a necessary and sufficient condition for the absolute and uniform convergence of the product $\prod_1^\infty (1+\phi_j)$ is the uniform convergence of the tail end of the series $\sum_1^\infty |\phi_n|$. Now for $z \in D_{j'}$, $j \geq j'$, $\left| \frac{f_j(z)}{f_j(z_j)} \right|^{N_j} \leq A_j^{-N_j}$, and by construction $\sum_{j \geq j'} jA_j^{-N_j} < \infty$. Therefore g is a holomorphic function in $\tilde{D}$.

We claim that g cannot be continued holomorphically to a domain $D^1 \supset \tilde{D}$. For, at z_j, g has a zero of at least the j^{th} order. Thus

$$g(z) = O(||z - z_j||^j) \quad \text{as} \quad z \to z_j .$$

So, the derivatives of g up to order j vanish at z_j. On the other hand, if q is a boundary point of $\tilde{D}$, then there exists a subsequence $\{p_{j_\nu}\}$ of $\{p_j\}$ which converges to q. This means that the p_{j_ν} come arbitrarily close to the boundary, i.e. $\Delta(p_{j_\nu}) \to 0$. This implies that $||z_{j_\nu} - p_{j_\nu}|| \to 0$ and hence that $z_{j_\nu} \to q$. Therefore if g is holomorphic at q, all the derivatives of g vanish at q. This can only happen if $g \equiv 0$, a contradiction. Hence g is not holomorphic at any boundary point.

Note. Under a holomorphic mapping, a region of holomorphy is mapped into a region of holomorphy.

§ 3. Domains of Convergence of Power Series

A. In this section we consider power series of the form

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1 \ldots k_n} (z_1-\zeta_1)^{k_1} \ldots (z_n-\zeta_n)^{k_n} \qquad (1)$$

for $\zeta = (\zeta_1, \ldots, \zeta_n)$ fixed. We say that the series converges at a point $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_n)$ if there is some arrangement of terms for which the series converges.

Note. In the sequel, we take $\zeta = 0$; i.e. we deal with series of the form

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n} \qquad (1')$$

Abel's Lemma. If the series (1') converges at some point $\hat{z} = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_n)$, then the series converges uniformly and absolutely in every compact subset of the polydisc $\{|z_j| < |\hat{z}_j|\}$.

Definition 13.

(i) A point ζ is said to be a point of normal convergence if there is a neighborhood of ζ in which the series (1')converges absolutely and normally

(ii) The convergence domain of the series (1') is the set of points of normal convergence.

From the definition, it is clear that a convergence domain is an open, connected set, star-shaped about the origin.

Corollary. If D is a domain of convergence of the series (1') and $(z_1, z_2, \ldots, z_n) \in D$ then $(\alpha_1 z_1, \alpha_2 z_2, \ldots, \alpha_n z_n) \in D$ where $|\alpha_i| \leq 1$, $i = 1, \ldots, n$.

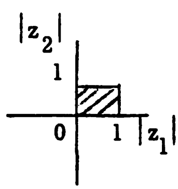
Definition 14. A domain with the above property is called a complete circular domain.

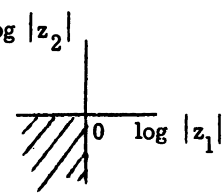
If $|\alpha_i| = 1$, $i = 1, \ldots, n$, the domain is called a circular domain.

Intuitively, a domain is circular if it is rotation-invariant, and complete circular if it has no "holes".

To any complete circular domain $D \subset \mathbb{C}^n$ we associate a set $D^* \subset \mathbb{R}^n$ as follows: D^* is the image of D under the mapping $(z_1, z_2, \ldots, z_n) \to (|z_1|, |z_2|, \ldots, |z_n|)$. To D^* we associate the set $\log D^*$, which is the image of D^* under the map $(|z_1|, \ldots, |z_n|) \to (\log|z_1|, \ldots, \log|z_n|)$, defined for $|z_i| \neq 0$, $i = 1, \ldots, n$.

Example. If $D \subset \mathbb{C}^2$ is the complete circular domain $\{|z_1| < 1, |z_2| < 1\}$, then D^* is the unit square and $\log D^*$ the third quadrant, i. e.

D^* is 

and $\log D^*$ is 

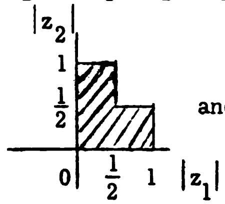
B. Theorem 6. Let D be a complete circular domain. Then the following conditions are equivalent:

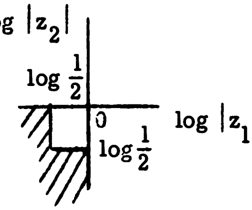
(i) D is a domain of convergence.

(ii) $\log D^*$ is convex.

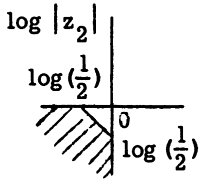
(iii) D is a domain of holomorphy.

Example. Let $D \subset \mathbb{C}^2$ be the complete circular domain $(|z_1| < 1, |z_2| < \frac{1}{2}$ or $|z_1| < \frac{1}{2}, |z_2| < 1)$. D^* is then

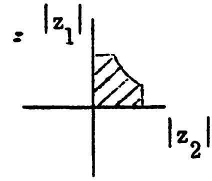


and $\log D^*$ is 

But, by Theorem 6, if there is a power series which converges normally in D, log D* must be convex. Therefore the power series must actually converge in



, and thus in



Hence D exhibits Hartogs' Phenomenon.

<u>Proof of Theorem 6.</u>

(i) implies (ii). Let $(r_1e^{i\phi_1}, \ldots, r_ne^{i\phi_n})$ and $(R_1e^{i\theta_1}, \ldots, R_ne^{i\theta_n})$ be arbitrary points of D. Then log D* is convex if and only if $(r_1^\alpha R_1^{1-\alpha}e^{i\psi_1}, \ldots, r_n^\alpha R_n^{1-\alpha}e^{i\psi_n}) \in D$ for each α, $0 < \alpha < 1$. Since $(r_1e^{i\phi_1}, \ldots, r_ne^{i\phi_n})$ is a point of normal convergence $|a_{k_1 \ldots k_n}| < \dfrac{A}{(r_1+\epsilon_1)^{k_1} \ldots (r_n+\epsilon_1)^{k_n}}$ where $\epsilon_1 > 0$ and A is some constant. Therefore for $0 < \alpha < 1$,

$|a_{k_1 \ldots k_n}|^\alpha < \dfrac{A^\alpha}{(r_1+\epsilon_1)^{\alpha k_1} \ldots (r_n+\epsilon_1)^{\alpha k_n}}$. Similarly, since $(R_1e^{i\phi_1}, \ldots, R_ne^{i\phi_n})$ is a point of normal convergence, we have for $\epsilon_2 > 0$, B some constant, and $0 < \alpha < 1$, $|a_{k_1 \ldots k_n}|^{1-\alpha} < \dfrac{B^{1-\alpha}}{\left((R_1+\epsilon_2)^{k_1} \ldots (R_n+\epsilon_2)^{k_n}\right)^{1-\alpha}}$. Let $\rho_i = r_i^\alpha R_i^{1-\alpha}$, $i = 1, \ldots, n$. Then, $|a_{k_1 \ldots k_n}|^\alpha |a_{k_1 \ldots k_n}|^{1-\alpha} = |a_{k_1 \ldots k_n}|$ $\leq \dfrac{\text{constant}}{\hat{\rho}_1^{k_1} \ldots \hat{\rho}_n^{k_n}}$, where $\hat{\rho}_i > \rho_i$, $i = 1, \ldots, n$.

(ii) implies (iii). By Theorem 5, a sufficient condition for (iii) to hold is that D is convex with respect to the family $\mathcal{F}$ of all the functions holomorphic in D. If D is convex with respect to a subfamily of $\mathcal{F}$ then D is convex with respect to $\mathcal{F}$. But, (ii) implies that D is convex with respect to monomials, as follows:

Let $K \subset\subset D$, and denote by $\hat{K}$ the hull of K w.r.t. monomials. $\hat{K}$ is closed in $\mathbb{C}^n$ and by Lemma 1, bounded. It remains to prove $\hat{K} \subset D$ or equivalently that $\log (\hat{K})^* \subset \log D^*$. Assume $D \neq \mathbb{C}^n$, as we already know $\hat{K} \subset \mathbb{C}^n$.

Let $p(z) = az_1^{m_1} \ldots z_n^{m_n}$ be an arbitrary monomial, $m_i \geq 0$ integers. Then $\log |p(z)| = \log |a| + m_1 \log |z_1| + \ldots + m_n \log |z_n|$, and $\log$ (hull of K w.r.t. $\log |p(z)|)^*$ is simply the closed half space $\supset \log K^*$ defined by the hyperplane P with coefficients $(m_1, \ldots, m_n)$ such that $P \cap \overline{\log K^*} \neq \phi$. Therefore the intersection of all such half spaces is $\log \left\{z \in \mathbb{C}^n \,|\, \log |p(z)| \leq \sup_{\zeta \in K} \log |p(\zeta)\right\}^*$, which is $\log (\hat{K})^*$, by the monotonicity of the log.

Now, if a subset S of a complete circular domain D stays away from ∂D, then $\overline{\log S^*}$ stays away from $\partial \log D^*$. To prove this it suffices to show that (1) $z \in \partial D$ implies $\alpha z = (\alpha_1 z_1, \ldots, \alpha_n z_n) \in \partial D$ for all $\alpha \in \mathbb{C}^n$ with $|\alpha_i| = 1$ for all i and (2) every $z \in \mathbb{C}^n$ with $(\log|z_1|, \ldots, \log|z_n|) \in \partial \log D^*$ belongs to ∂D. For then if $\overline{\log S^*}$ comes arbitrarily close to $\partial \log D^*$, given $\epsilon > 0$ there are points $s \in S$ and $d \in \partial D$ such that (using sup norm) for every i, $|\log|s_i| - \log|d_i|| < \epsilon$, which implies $||s_i| - |d_i|| < \epsilon'$, $\epsilon' \to 0$ as $\epsilon \to 0$; and by (1) then, there is a point $d' \in \partial D$ with $|s_i - d'_i| < \epsilon'$ for all i: a contradiction. (1) is easily established once we note that $z \notin D$ implies for all α with $|\alpha_i| = 1$, $\alpha z \notin D$; because then if $z^o \in \partial D$ and $\epsilon > 0$ is given, there are points $\lambda \in D$ and $\nu \notin D$ such that $|\lambda_i - z_i^o| < \epsilon$ and $|\nu_i - z_i^o| < \epsilon$ for all i. Hence for all α, $|\alpha_i| = 1$, $|\alpha_i \nu_i - \alpha_i z_i^o| < \epsilon$ and $|\alpha_i \lambda_i - \alpha_i z_i^o| < \epsilon$, $\alpha\nu \notin D$ and $\alpha\lambda \in D$ so that $\alpha z^o \in \partial D$, as claimed. (2) is proved similarly by considering the preimage in $\mathbb{C}^n$ of a neighborhood of a boundary point of $\log D^*$.

Since cl K is compact, there is a ball B in $\mathbb{C}^n$, $\underline{(|z_i| \leq r)}$, $\supset$ cl K, and a $\delta > 0$ such that dist $(\overline{\log K^*}, \partial(\log B^* \cap \log D^*)) > \delta$: for $B \cap D$ is a complete circular domain and $K \subset\subset (B \cap D)$.

Let $T = \log B^* \cap \log D^*$. T is convex and $\subset \log D^*$ We will show that $\log (\hat{K})^* \subset T$, by considering the convex hull T_1 of T, i.e. the intersection of all closed half-spaces in $\mathbb{R}^n$ containing T. Now, if $\xi \notin T_1$, then there is a hyperplane P in $\mathbb{R}^n$ separating ξ from T_1 and such that $P \cap (T_1 \cup \{\xi\}) = \emptyset$ Because $B \cap D$ is a complete circular domain, if $\xi^o \in T$, then $\{\xi \in \mathbb{R}^n | \xi_i \leq \xi_i^o\} \subset T$. Thus T can only be contained in a closed half space expressible as $\{\xi \in \mathbb{R}^n | m_1 \xi_1 + \ldots + m_n \xi_n \leq c;\ c \in \mathbb{R},\ \text{all } m_i \geq o\}$. Suppose $\xi \notin T_1$, if $\log|\xi_i| > \log r$ for at least one i, say i = 1, then we may take P to be the hyperplane : $\log|z_1| = \log r + \epsilon$, where $\epsilon > o$ is such that $\log|\xi_1| > \log r + \epsilon$. If $\log|\xi_i| \leq \log r$ for all i, either the separating hyperplane P can already be given with rational (and hence integral) coefficients; or since $P \cap \log B^*$ is then compact in $\mathbb{R}^n$ and the distances of ξ and T_1 to P depend continuously on the coefficients of P, we can find a P with rational coefficients effecting the separation. Therefore, the convex hull of T, which is cl T, is the intersection of those closed half spaces $\supset T$ defined by hyperplanes P with non-negative, integral coefficients and $P \cap \text{cl}\,T = \emptyset$.

Thus, if P is any hyperplane to be considered in (the intersection giving cl T) then if we translate P parallel to itself into the half space $\supset T$ until it intersects $\overline{\log K^*}$, P becomes a hyperplane to be considered in (the intersection giving $\log (\hat{K})^*$), and we have translated P a distance $> \delta$. Therefore $\log (\hat{K})^* \subset \text{cl}\,T$, and the distance of any point of $\log (\hat{K})^*$ to such a hyperplane P is $> \delta$, so that the closed ball of radius δ about any point of $\log (\hat{K})^* \subset$ the closed half space $\supset \text{cl}\,T$ defined by every such hyperplane P and therefore $\subset \text{cl}\,T$. By the convexity of T, it follows that $\log (\hat{K})^* \subset T$, as claime

(iii) implies (i). (iii) implies that there exists a function f holomorphi in D and in no larger domain. At any point $z \in D$, we may expand f in a power series which converges normally in a neighborhood of z. Thus there is a power series which converges in D and in no larger domain.

§ 4. Bergman Domains

A. Definition 15. Let $f(z_1, \ldots, z_n, \lambda) \in \mathbb{C}$, $z_i \in \mathbb{C}$, $i = 1, \ldots, n$ and $\lambda \in \mathbb{R}$, such that f is continuous in all variables simultaneously and holomorphic in some domain $D \subset \mathbb{C}^n$ for each fixed $\lambda \in I \subset \mathbb{R}$. Then $\left\{ z \,\middle|\, f(z_1, \ldots, z_n, \lambda) = 0,\ z = (z_1, \ldots, z_n) \in D,\ \lambda \in I \right\}$ is called a Bergman Surface.

This is a surface of (real) dimension 2n-1.

Definition 16. A Bergman domain is a domain bounded by a finite number of Bergman surfaces.

Example. Let $n = 2$. Let $B = \left\{ (z_1, z_2) \,\middle|\, |z_1| < 1,\ z_2 \in B(z_1) \right\}$ where the $B(z_1)$ are domains parametrized by z_1 and bounded by Jordan curves: $z_2 = g(z_1, e^{i\lambda})$, where g is continuous in all variables and holomorphic in z_1 for $|z_1| < 1 + \epsilon$. Then B is a Bergman domain.

For example, we might choose

$$g(z_1, e^{i\lambda}) = \frac{1}{2+z_1} e^{i\lambda} + \frac{z_1^5}{100} e^{-i\lambda} .$$

For n arbitrary, $D = \left\{ (z_1, \ldots, z_n) \,\middle|\, |z_1| < 1,\ z_2 \in B(z_1),\ z_3 \in B(z_1, z_2), \ldots \right\}$. These domains will be called quasi-product domains. They reduce to product domains if $B(z_1) = B_1$ independent of z_1, $B(z_1, z_2) = B_2$, etc. We list the following properties, stated for any Bergman domain and proven for quasi-product domains in $\mathbb{C}^2$.

Property 1. Every Bergman domain is a domain of holomorphy.

Proof. If $(\hat{z}_1, \hat{z}_2)$ is a boundary point then either

(a) $|\hat{z}_1| = 1$, and then $\dfrac{1}{z_1 - \hat{z}_1}$ is singular at $(\hat{z}_1, \hat{z}_2)$ and regular inside

or (b) $\hat{z}_2 = g(\hat{z}_1, e^{i\lambda_0})$ and then $\dfrac{1}{z_2 - g(z_1, e^{i\lambda_0})}$ is singular at $(\hat{z}_1, \hat{z}_2)$ and regular elsewhere.

Property 2. A Bergman domain has a distinguished boundary surface, defined to be the set of those points of the boundary at which at least n Bergman surfaces intersect. (The distinguished boundary in our example is $\left\{ (e^{i\phi}, g(e^{i\phi}, e^{i\lambda})) \,\middle|\, \phi, \lambda \in [0, 2\pi] \right\}$).

Property 3. If f is holomorphic in a neighborhood of a Bergman domain D then the maximum of $|f|$ in D is achieved on the distinguished boundary.

Proof. For each fixed z_1, $|f(z_1, z_2)|$ has its maximum on the boundary of $B(z_1)$; hence, consider $|f(z_1, g(z_1, e^{i\lambda}))|$. But for each fixed λ, this function has its maximum on $|z_1| = 1$.

Corollary. If f is known on the distinguished boundary of a Bergman domain B, it is known throughout B.

In fact, we have the following:

Bergman Generalization of Cauchy's Formula.

$$f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \iint\limits_{\substack{\text{distinguished}\\ \text{bndry}}} \frac{f(e^{i\phi}, g(e^{i\phi}, e^{i\lambda}))}{(e^{i\phi}-z_1)(g(e^{i\phi}, e^{i\lambda})-z_2)}\, dg(e^{i\phi}, e^{i\lambda})\, de^{i\phi}$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \frac{f(e^{i\phi}, g(e^{i\phi}, e^{i\lambda}))}{(e^{i\phi}-z_1)(g(e^{i\phi}, e^{i\lambda})-z_2)}\, e^{i\phi} e^{i\lambda} \frac{\partial g}{\partial \lambda}(e^{i\phi}, e^{i\lambda})\, d\phi\, d\lambda .$$

§ 5. Analytic Polyhedra

In this section we shall define analytic polyhedra. We shall show that every holomorphy region is a limit of an increasing sequence of analytic polyhedra.

A. Definition 17. Let $D^{\text{open}} \subset \mathbb{C}^n$; $f_1, \ldots, f_k$ holomorphic in D, and $A = \{z \mid z \in D \text{ and } |f_j(z)| < 1;\ j = 1, \ldots, k\}$. If $A \subset\subset D$, A is called an analytic polyhedron (of dimension n).

Corollary 1. Every analytic polyhedron is a region of holomorphy.

Proof. Let B be an analytic polyhedron, $\zeta \in$ bndry B. Then $|f_j(\zeta)| = 1$ for some j, say $j = 1$. But then

$$g(z) = \frac{1}{f_1(z) - f_1(\zeta)}$$

is holomorphic at every point of B and singular at ζ.

Corollary 2. Every connected analytic polyhedron is a Bergman domain.

Proof. Let B be a connected analytic polyhedron, $\zeta \in$ bndry B. Then $|f_j(\zeta)| = 1$ for some j; i.e. every boundary point satisfies an equation of the form $f_j(z_1, \ldots, z_n) - e^{i\lambda} = 0$.

B. Theorem 7. Let D be a region of holomorphy, and $K \subset\subset D$. Choose D_o such that $\hat{K} \subset\subset D_o^{open} \subset\subset D$. Then there exists an analytic polyhedron A in D such that $\hat{K} \subset\subset A \subset\subset D_o$.

Proof. Let $\zeta \in$ bndry D_o. There exists a holomorphic function g_ζ in D such that $|g_\zeta(\zeta)| > 1$, and $|g_\zeta(z)| < 1$ in $\hat{K}$. Hence, there exists a neighborhood N_ζ of each $\zeta \in$ bndry D_o such that $|g_\zeta(N_\zeta)| > 1$. But bndry D_o, being compact, is covered by a finite number of such neighborhoods, say $N_{\zeta_1}, \ldots, N_{\zeta_k}$. Let $A = \left\{ z \,\middle|\, |g_{\zeta_j}(z)| < 1;\ j = 1, \ldots, k \right\}$.

Corollary. Let D be a region of holomorphy. Then there exists a sequence A_j, $j = 1, 2, \ldots$ of analytic polyhedra in D such that $A_j \subset\subset A_{j+1} \subset\subset D$, and $D = \bigcup_{j=1}^{\infty} A_j$.

Proof. Choose $D_1^{open} \subset\subset D_2^{open} \subset\subset \ldots \subset\subset D$ such that $\bigcup_{i=1}^{\infty} D_i = D$. Consider the sequence $\{\hat{D}_j\}$; there exists a subsequence, say $\{\hat{D}_j\}$, such that

$$\hat{D}_1 \subset\subset D_2 \subset\subset \hat{D}_2 \subset\subset \ldots \subset\subset D; \quad D = \bigcup_{i=1}^{\infty} \hat{D}_i .$$

Then by theorem 7, there exist A_i analytic polyhedra, such that

$$\hat{D}_1 \subset\subset A_1 \subset\subset \hat{D}_2 \subset\subset A_2 \subset\subset \ldots \subset\subset D .$$

Chapter 3. Pseudoconvexity

§1. Plurisubharmonic and pseudoconvex functions

A, We have already introduced the notion of a continuous subharmonic function (I, § 2). We now extend this definition as follows:

Definition 18. Let D be open in $\mathbb{C}$, $\phi : D \to \mathbb{R}$. Then ϕ is said to be subharmonic if:

i) $-\infty \leq \phi < \infty$, $\phi \not\equiv -\infty$

ii) ϕ is upper semi-continuous; i. e. $\limsup_{p' \to p} \phi(p') \leq \phi(p)$

iii) for any domain $D_0 \subset\subset D$; if h is harmonic in D_0 and continuous on ∂D_0, then $h \geq \phi$ on ∂D_0 implies $h \geq \phi$ in D_0.

Property 1 (Mean Value Property, I). Let $\{|z-z_0| \leq r\} \subset D$, ϕ subharmonic in D. Then

$$\phi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta})\, d\theta .$$

Property 2 (Mean Value Property, II). Let $\{|z-z_0| \leq r\} \subset D$, ϕ subharmonic in D. Then

$$\phi(z_0) \leq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \phi(z_0 + re^{i\theta})\, r\, dr d\theta .$$

Property 3 (Strong Maximum Principle). Let ϕ be subharmonic in D. Let $M = \sup_D \phi$. Then, in each component of D either $\phi(z) < M$ or ϕ is constant.

Property 4. If ϕ satisfies i), ii) and the integral condition of Property 1 or 2, then ϕ is subharmonic in D.

Property 5. If ϕ and ψ are subharmonic, then $\max(\phi, \psi)$ is subharmonic.

Property 6. If $\phi \in C^2$, ϕ is subharmonic if and only if $\Delta\phi \geq 0$.

Definition 19. Let $D \subset \mathbb{C}^n$, $\phi : D \to \mathbb{R}$. Then ϕ is said to be plurisubharmonic in D if:

i) $-\infty \leq \phi < \infty$, $\phi \not\equiv -\infty$

ii) ϕ is upper semicontinuous

iii) if $(z_1, \ldots, z_n) \in D$ and $a_i \in \mathbb{C}$ arbitrary, $i = 1, \ldots n$, then $\Phi(\zeta) = \phi(z_1 + \zeta a_1, \ldots, z_n + \zeta a_n)$ is subharmonic for small $|\zeta|$.

ϕ is said to be pseudoconvex if it is plurisubharmonic and continuous.

Remark. Statement iii) above is equivalent to the following:

$\phi \circ T(z_1, \ldots, z_n)$ is subharmonic in each variable separately, for all linear transformations T.

Corollary 1. Let $D \subset \mathbb{C}^n$, $f : D \to \mathbb{C}$. If f is holomorphic in D, then $\log|f|$ is plurisubharmonic. Furthermore, if $f \neq 0$ on D, $\log|f|$ is pseudoconvex.

Corollary 2. Plurisubharmonic functions satisfy the strong maximum principle.

Corollary 3. If $D \subset \mathbb{C}^n$ and $\phi, \psi : D \to \mathbb{R}$ are plurisubharmonic, then so are $\max(\phi, \psi)$, $\phi + \psi$, and $\lambda\phi$, $\lambda > 0$.

Definition 20. Let $\phi \in C^2$. The the Hessian of ϕ is defined to be the following matrix:

$$H = \left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \right).$$

Note that H is Hermitian, if ϕ is real valued.

Proposition 1. Let $D \subset \mathbb{C}^n$, $\phi : D \to \mathbb{R}$, $\phi \in C^2$. Then ϕ is pseudoconvex if and only if the Hessian of ϕ is positive semidefinite.

Proof. Consider $\Phi(\zeta) = \phi(z_1 + \zeta a_1, \ldots, z_n + \zeta a_n)$, $a_i \in \mathbb{C}$, $i = 1, \ldots, n$. Now $\Phi(\zeta)$ is subharmonic if and only if $\Delta\Phi \geq 0$. But

$$\Delta\bar{\Phi} = 4\frac{\partial^2\bar{\Phi}}{\partial\zeta\partial\bar{\zeta}}$$

$$= 4\frac{\partial}{\partial\zeta}\sum_{j=1}^{n}\frac{\partial\phi}{\partial\bar{z}_j}\bar{a}_j$$

$$= 4\sum_{j,k=1}^{n}\frac{\partial^2\phi}{\partial z_k\partial\bar{z}_j}a_k\bar{a}_j .$$

Proposition 2. Let $D \subset \mathbb{C}^n$, $D_o \subset\subset D$, $\phi : D \to \mathbb{R}$ such that ϕ is pseudoconvex in D. Then there exists a sequence $\{\phi_j\}$ of pseudoconvex, C^∞ functions in D_o such that $\phi_j \to \phi$ uniformly in D_o.

Proof. Define the "smoothing functions" K_ϵ, $\epsilon > 0$ as follows:

$$K_\epsilon : \mathbb{C}^n \to \mathbb{R}$$

$$K_\epsilon(\zeta) \geq 0$$

Support $K_\epsilon \subset \{||\zeta|| < \epsilon\}$; i.e. $K_\epsilon(\zeta) = 0$ for $||\zeta|| \geq \epsilon$

$$\int_{\mathbb{C}^n} K_\epsilon(\zeta)d\xi_1 d\eta_1 \ldots d\xi_n d\eta_n = 1$$

$$K_\epsilon \in C^\infty$$

Define: $\phi_\epsilon(z) = \int_{\mathbb{C}^n} K_\epsilon(z-\zeta)\,\phi(\zeta)\,d\xi_1 \ldots d\eta_n$, where we take $\phi = 0$ where it is undefined. Then $\phi_\epsilon(z) \in C^\infty$. Furthermore, $\phi_\epsilon \to \phi$ uniformly in D_o, as follows:

$$\phi(z) = \int_{\mathbb{C}^n} K_\epsilon(\zeta)\,\phi(z)d\xi_1 \ldots d\eta_n$$

$$\phi_\epsilon(z) = \int_{\mathbb{C}^n} K_\epsilon(\zeta)\,\phi(z-\zeta)d\xi_1 \ldots d\eta_n .$$

Therefore

$$|\phi(z)-\phi_\epsilon(z)| \leq \int_{\mathbb{C}^n} K_\epsilon(\zeta)\, |\phi(z)-\phi(z-\zeta)|\, d\xi_1 \ldots d\eta_n$$

$$\leq \max_{||\zeta|| < \epsilon,\, z \in D_0} |\phi(z)-\phi(z-\zeta)| \to 0 \text{ as } \epsilon \to 0$$

since ϕ is uniformly continuous on D_0.

Now ϕ_ϵ is plurisubharmonic since

$$\phi_\epsilon(z) = \int_{\mathbb{C}^n} K_\epsilon(\zeta)\, \phi(z-\zeta)\, d\xi_1 \ldots d\eta_n ,$$

is essentially a linear combination of plurisubharmonic functions with positive coefficients.

Proposition 3. Let $\Delta^{\text{domain}} \subset \mathbb{C}$, $g:\Delta \to D \subset \mathbb{C}^n$; g holomorphic. Then ϕ is pseudoconvex in D if and only if $\phi(g(\zeta))$ is subharmonic and continuous for all such g.

Proof. Assume ϕ is pseudoconvex. By proposition 2, we may assume $\phi \in C^\infty$. Then $\phi g \in C^\infty$. Now,

$$g(\zeta) = (g_1(\zeta), \ldots, g_n(\zeta))$$

and let $\Phi(\zeta) = \phi g(\zeta)$.

Then

$$\frac{\partial^2 \Phi}{\partial \zeta \partial \bar{\zeta}} = \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\, g_j'(\zeta) \overline{g_k'(\zeta)} \geq 0 ,$$

as the Hessian of ϕ is positive definite.

The converse is trivial, as the class of holomorphic $g:\Delta \to D$ contains all linear transformations.

Corollary. The image of a pseudoconvex function under a holomorphic mapping is pseudoconvex.

§ 2. Pseudoconvex domains

A. Definition 21. Let $\{|\zeta| \leq 1\} \subset \mathbb{C}$. Then an analytic disc in D is a mapping $g: \{|\zeta| < 1\} \to D \subset \mathbb{C}^n$, continuous on $\{|\zeta| \leq 1\}$ and holomorphic in the interior.

The boundary of the analytic disc is the mapping g restricted to $|\zeta| = 1$. Set $g(\{|\zeta| \leq 1\}) = \Sigma$.

Abusing terminology by suppressing mention of g, we shall refer to Σ itself as the analytic disc and to $\partial\Sigma$ as the boundary. (Note that $\partial\Sigma$ in general is not the set-theoretic boundary of the point set Σ.)

Theorem 8. Let $D^{\text{open}} \subset \mathbb{C}^n$. Then the following are equivalent:

i) Let $\{\Sigma_j\}$ be a sequence of analytic discs in D. If $\bigcup_{j=1}^{\infty} \partial\Sigma_j \subset\subset D$, then $\bigcup_{j=1}^{\infty} \Sigma_j \subset\subset D$. ("Kontinuitätssatz").

ii) $-\log \Delta(z)$ is plurisubharmonic in D, where $\Delta(z)$ is taken in any norm.

iii) For any analytic disc Σ in D, $\Delta(\Sigma) = \Delta(\partial\Sigma)$.

iv) There exists a pseudoconvex function ϕ in D such that, for every $N > 0$ there exists a $K \subset\subset D$ for which $\phi \geq N$ on D-K. (Informally, $\phi = +\infty$ on the boundary of D.)

Exercises.

a) In Euclidean space, i) has the following analog:

Let $D^{\text{open}} \subset\subset \mathbb{E}^n$. Let $\{\Sigma_j\}$ be a sequence of segments in D. If $\bigcup \partial\Sigma_j \subset\subset D$, then $\bigcup \Sigma_j \subset\subset D$. Show that this property holds if and only if every component of D is convex.

b) Find the analog of ii) in $\mathbb{E}^n$.

Definition 22. A region with any and hence all of the above properties is said to be pseudoconvex.

Corollary 1. Let $D_j^{\text{open}} \subset \mathbb{C}^n$, D_j pseudoconvex. If $\bigcap_{j=1}^{\infty} D_j$ is open, then $\bigcap_{j=1}^{\infty} D_j$ is pseudoconvex.

Corollary 2. Let $D^{\text{open}} \subset \mathbb{C}^n$. Then D is pseudoconvex if and only if each component is pseudoconvex.

Corollary 3. The holomorphic image of a pseudoconvex region is pseudoconvex.

Corollary 4. Let $\{A_j\}$ be a sequence of pseudoconvex domains such that $A_j \subset A_{j+1}$, then $\bigcup A_j$ is pseudoconvex.

B. Proof of Theorem 8.

iii) implies i). $\bigcup \partial \Sigma_j \subset\subset D$ and $\Delta(\Sigma_j) = \Delta(\partial\Sigma_j)$ imply $cl(\bigcup \Sigma_j) \subset D$. That $\bigcup \Sigma_j$ is bounded follows from the fact that $|g|$ assumes its maximum over E on $\{|\zeta| = 1\}$, and $\bigcup \partial\Sigma_j$ is bounded.

ii) implies iv). If D is bounded, we may choose $\phi(z) = -\log\Delta(z)$. If D is unbounded, choose $\phi(z) = \max(-\log\Delta(z), |z_1|^2 + \ldots + |z_n|^2)$; where $\Delta(z)$ is taken in the Euclidean norm.

ii) implies iii). Let Σ be given by:

$$g: \{|\zeta| \leq 1\} \to D.$$

Then $-\log\Delta(g(\zeta))$ is subharmonic and continuous; hence it has a maximum on $\{|\zeta| \leq 1\}$, which is assumed on $\{|\zeta| = 1\}$. Therefore,

$$-\log \Delta(\partial\Sigma) \geq -\log \Delta(\Sigma)$$

i. e.
$$\Delta(\partial\Sigma) \leq \Delta(\Sigma)$$

But clearly,
$$\Delta(\partial\Sigma) \geq \Delta(\Sigma).$$

iv) implies i). Let Σ_j be given by

$$g_j: \{|\zeta| \leq 1\} \to D.$$

Consider the subharmonic functions

$\phi(g_j(\zeta))$, $j = 1, 2, \ldots$; where ϕ is the function given by iv). Note that

$$\max_{|\zeta| \leq 1} \phi(g_j(\zeta)) = \max_{|\zeta| = 1} \phi(g_j(\zeta)).$$

i. e.
$$\max_{z \in \partial\Sigma_j} \phi(z) = \max_{z \in \Sigma_j} \phi(z).$$

Now, $$\bigcup \partial \Sigma_j \subset\subset D;$$

hence $$\sup \psi\left(\bigcup_{j=1}^{\infty} \partial \Sigma_j\right) < M < \infty .$$

But therefore $$\sup \psi\left(\bigcup_{j=1}^{\infty} \Sigma_j\right) < M < \infty ,$$

implying $$\bigcup_{j=1}^{\infty} \Sigma_j \subset\subset D .$$

i) implies ii). (Proof due to Hartogs). Since $\Delta(z)$ is continuous, $-\log \Delta(z)$ is continuous. In fact $-\log \Delta(z)$ satisfies a Lipschitz condition on compact subsets of D. To prove that $-\log \Delta(z)$ is plurisubharmonic in D, it is sufficient to show plurisubharmonicity at a point $z_o \in D$. Thus we must show that, for the set of points $z = z_o + \zeta z_1$ where z_o and z_1 are arbitrary points of D and $\mathbb{C}^n$ respectively, and $\zeta \in \mathbb{C}$ is sufficiently small, $-\log \Delta(z_o + \zeta z_1) = \psi(\zeta)$ is subharmonic as a function of ζ. For $||z_1||$ small enough in the norm used to measure $\Delta(z)$, $z \in D$ for $|\zeta| \leq 1$. Furthermore, $\psi(\zeta)$ is subharmonic if it satisfies the mean value property of subharmonic functions, and it is enough to show this for $\zeta = 0$; i. e. to show that

$$(!) \qquad \psi(0) \leq \frac{1}{2\pi} \int_{o}^{2\pi} \psi(e^{i\theta})\, d\theta .$$

Let $g(\zeta) = h(\zeta) + ih^*(\zeta)$, with h a real-valued function such that $h(e^{i\theta}) = \psi(e^{i\theta})$, g holomorphic for $|\zeta| < 1$, and continuous for $|\zeta| \leq 1$. We claim that such a function exists. Firstly, the continuity of ψ implies the existence of a harmonic function h, in $|\zeta| < 1$, equal to ψ on $|\zeta| = 1$. Thus h is defined and is continuous on the closed unit disc. Secondly, take h^* to be some conjugate function to h. Since g is now defined and holomorphic, and its real part satisfies a Lipschitz condition on $|\zeta| = 1$, its imaginary part satisfies a Hölder condition. Hence h^* is continuous on $|\zeta| = 1$.

Next, let b be any vector in $\mathbb{C}^n$ with $||b|| = 1$, and let λ_o satisfy $0 < \lambda_o < 1$. Consider the analytic disc in $\mathbb{C}^n$

$$\Sigma(\lambda): \zeta \to z_0 + \zeta z_1 + \lambda e^{-g(\zeta)} b, \text{ for } |\zeta| \leq 1 \text{ and } \lambda \text{ fixed } 0 \leq \lambda \leq \lambda_0.$$

(1) $\Sigma(0) \subset D$. This is obvious.

(2) If $p \in \Sigma(\lambda)$ and $\lambda_j \to \lambda$, then there exist $p_j \in \Sigma(\lambda_j)$ such that $p_j \to p$. (Namely, $p_j = z_0 + \zeta_0 z_1 + \lambda_j e^{-g(\zeta_0)} b$ where ζ_0 is the preimage of p.)

(3) $\bigcup\limits_{0 \leq \lambda \leq \lambda_0} \partial\Sigma(\lambda) \subset\subset D$.

(For if $z \in \partial\Sigma(\lambda)$ then

$\|z-(z_0 + e^{i\theta} z_1)\| \leq \lambda e^{-h(e^{i\theta})} \leq \lambda_0 e^{\log\Delta(z_0 + e^{i\theta} z_1)} = \lambda_0 \Delta(z_0 + e^{i\theta} z_1)$ and hence (3) holds.

(4) $S = \left\{\lambda \,\middle|\, 0 \leq \lambda \leq \lambda_0 \text{ and } \Sigma(\lambda) \subset D\right\}$ is open in the space $\Lambda = [0, \lambda_0]$.

(5) S is closed in Λ.

(This follows from (2), (3), and the Kontinuitätssatz.)

(6) S is the set $[0, \lambda_0]$.

(From (1), (4), and (5), $S = \Lambda$.)

Hence $\Sigma(\lambda) \subset D$ for $0 \leq \lambda < 1$. Consequently, $z_0 + \lambda e^{-g(0)} b \in D$ for $0 \leq \lambda < 1$, in fact for λ complex, $|\lambda| < 1$; $z_0 + \lambda e^{-i\alpha} e^{-g(0)} b \in D$ for $0 \leq \lambda < 1$ and α real, since we can incorporate $e^{-i\alpha}$ into b, as $\|e^{i\alpha} b\| = \|b\| = 1$. This means that a ball about z_0 of radius $|\lambda e^{-g(0)}|$ is contained in D, and therefore $\Delta(z_0) \geq |e^{-g(0)}| = e^{-h(0)}$. Hence $-\log \Delta(z_0) \leq h(0)$, but $-\log \Delta(z_0) = \psi(0)$, and since h is harmonic

$$(!) \qquad \psi(0) \leq h(0) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta})\, d\theta .$$

C. Lemma. Let Σ be an analytic disc in D, $D \subset \mathbb{C}^n$. Then $\Sigma \subset \widehat{\text{bdry}\,\Sigma}$, the hull of the boundary of Σ with respect to holomorphic functions.

Proof. Let Σ be given by the holomorphic mapping :

$$g: \{|\zeta| \leq 1\} \to D.$$

For every f holomorphic in D, $f(g(\zeta))$ is holomorphic and therefore $|f(g(\zeta))|$ assumes its maximum M on $\{|\zeta| = 1\}$. Therefore $|f(g(\zeta))| \leq M$ for $g(\zeta) \in \partial\Sigma$

implies $|f(g(\zeta))| \leq M$ for $g(\zeta) \in \sum$.

Theorem 9. If D is a region of holomorphy, then D is pseudoconvex.

Proof. Let $\{\sum_j\}$ be a sequence of analytic discs in D; such that $\bigcup_{j=1}^{\infty} \partial\sum_j \subset\subset D$.

But $\bigcup_{j=1}^{\infty} \partial\sum_j \subset \widehat{\bigcup_{j=1}^{\infty} \partial\sum_j} \subset\subset D$, as D is a region of holomorphy.

But by the above lemma, $\bigcup_{j=1}^{\infty} \sum_j \subset \bigcup_{j=1}^{\infty} \widehat{\partial\sum_j}$, hence $\bigcup_{j=1}^{\infty} \sum_j \subset\subset D$. Now apply theorem 8.

Theorem 10. Let $D^{open} \subset \mathbb{C}^n$. If, for every $\zeta \in \partial D$, there exists a ball N about ζ such that $N \cap D$ is pseudoconvex, then D is pseudoconvex.

Proof. Assume that D is bounded. Then ∂D is compact. By hypothesis, for $\zeta \in \partial D$, there is a ball N_ζ, about ζ, such that $N_\zeta \cap D$ is pseudoconvex. The set $\{N_\zeta\}$ is an open covering of ∂D. The compactness of ∂D implies that a finite number of the N_ζ cover ∂D; call them $N_1, \ldots, N_p$. If $N_i \cap N_j \cap \partial D \neq \phi$, set $Q_{ij} = N_i \cap N_j$, for each $i \neq j$; $i,j = 1,2,\ldots,p$. In each Q_{ij} choose any ball B_{ij} centered at any point $\zeta \in (Q_{ij} \cap \partial D)$ such that $B_{ij} \subset Q_{ij}$. Since there are only a finite number of sets Q_{ij}, there are only finitely many B_{ij}. Let r = min (radius of B_{ij}). At each point $\zeta \in \partial D$ the ball $S(\zeta,r)$ of radius r centered at ζ is contained in some N_i, and hence $S(\zeta,r) \cap D$ is pseudoconvex; because $S(\zeta,r) \cap D = S(\zeta,r) \cap (N_i \cap D)$ and $S(\zeta,r)$ is pseudoconvex by Cor. 5, p. 16 and Thm. 9.

Now, consider the function $\phi(z) = \max(-\log \frac{r}{2}, -\log \Delta(z))$. We claim that $\phi(z)$ is pseudoconvex. Clearly $\phi(z)$ is continuous. If $A = \{z \mid \Delta(z) \geq r/2\}$, then for $z \in A$, $\phi(z) = -\log \frac{r}{2}$ = constant and therefore is plurisubharmonic. If $B = \{z \mid \Delta(z) < r/2\}$, then for $z \in B$, $\phi(z) = -\log\Delta(z)$. But for $z \in B$, $z \in S(\zeta,r)$ and $\Delta(z) \equiv \Delta_D(z) = \Delta_K(z)$ where $K = S(\zeta,r) \cap D$. Since K is pseudoconvex, $-\log\Delta(z)$ is pseudoconvex. Thus $\phi(z)$ is pseudoconvex in D. As z approaches the ∂D, $\phi(z)$ becomes infinite. By part (4) of Theorem 8, D is pseudoconvex.

Now, consider the case when D is unbounded. Set $D_j = D \cap \{z \mid \|z\| < j\}$. Each D_j is a bounded set. If $\zeta \in \partial D$ then either (i) $\zeta \in \partial D$ and $\zeta \in S_j = \{z \mid \|z\| < j\}$ or (ii) $\zeta \in \partial S_j$ and $\zeta \notin \partial D$, or (iii) $\zeta \in \partial D$ and $\zeta \in \partial S_j$. For case (i), by hypothesis, there exists an N_ζ about ζ such that $N_\zeta \cap D$ is pseudoconvex. For case (ii), since S_j is convex, any ball about ζ, N_ζ lying in D, satisfies $N_\zeta \cap D_j$ is pseudoconvex. For case (iii), there exists a ball $N(\zeta,r)$ such that $N \cap D$ is pseudoconvex. But $N \cap S_j$ is also pseudoconvex. Therefore $N \cap D_j$ is pseudoconvex. Therefore each D_j satisfies the hypothesis of this theorem and is bounded. We have already shown that therefore D_j is pseudoconvex, $j = 1,2,\ldots$. Since $D_j \subset D_{j+1}$ and $\{D_j\} \to D$, D is pseudoconvex (by Cor. 4 of Thm. 8).

Establishing the converse of Theorem 9 is the Levi Problem.

§3. Solution of the Levi Problem for tube domains

Definition 23. Let $z_j = x_j + iy_j$. The set $D = \{(z_1,\ldots,z_n) \mid (x_1,\ldots,x_n) \in B \subset \mathbb{R}^n\}$ where B is some open subset of $\mathbb{R}^n$, is called a tube domain. B is called the base of the tube domain.

Example. $D = \{(z_1,z_2) \mid |x_1| < 1, |x_2| < 1\}$ is a tube domain. Here, the base B, of D, is the unit square in $\mathbb{R}^2$.

Theorem 11. Let D be a tube domain with base B. The following properties are equivalent.

(1) D is pseudoconvex.

(2) Every component of B is convex.

(3) D is a region of holomorphy.

Note. Assuming Theorem 11, (1) implies (3) is the solution of the Levi Problem for tube domains.

Proof of Theorem 11. (1) implies (2). Let the norm be the Euclidean norm, and let $\psi(z) = -\log \Delta(z))$. Then (1) implies that ψ is pseudoconvex in D, and $\psi(z) = -\log \Delta_D(x_1 + iy_1,\ldots,x_n + iy_n) = -\log \Delta_B(x_1,\ldots,x_n)$ since D is a tube domain.

Suppose $\psi \in C^2$, then the matrix $\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k}$ is positive definite. But

$\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)\psi$; and since ψ is independent of the y's, $\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\frac{\partial^2\psi}{\partial x_j \partial x_k}$. Thus every diagonal element is positive, i. e. $\frac{\partial^2\psi}{\partial x_j^2} \geq 0$. This means that ψ is convex on every straight line segment in B parallel to a coordinate axis, and since ψ is pseudoconvex this property is invariant under linear transformations. Hence ψ is convex on every straight line segment in B, and hence in every component of B. We claim that this implies that every component B_o of B is convex:

(a) Firstly, we show that if $\{\lambda_\nu\}$ is a sequence of straight line segments in B_o such that $\lim_{\nu\to\infty} \lambda_\nu = \lambda$ and $\lim_{\nu\to\infty} \partial\lambda_\nu = \mu$ then if $\mu \subset B_o$, $\lambda \subset B_o$.

Indeed, since ψ is convex on each straight line segment in B_o, on each λ_ν $\max_{x\in\lambda_\nu} \psi(x) = \max_{x\in\partial\lambda_\nu} \psi(x)$, i. e. $\max_{x\in\lambda_\nu}(-\log\Delta_B(x)) = \max_{x\in\partial\lambda_\nu}(-\log\Delta_B(x))$, or equivalently $\min_{x\in\lambda_\nu} \Delta_B(x) = \min_{x\in\partial\lambda_\nu} \Delta_B(x)$. Since $\Delta_B(x)$ is a continuous function, the equality holds in the limit $\min_{x\in\lambda} \Delta_B(x) = \min_{x\in\mu} \Delta_B(x)$. But μ is a closed set and $\mu \subset B_o$, therefore $\min_{x\in\mu} \Delta_B(x) > 0$; hence $\lambda \subset B_o$.

(b) Now, let $x, y \in B_o$. We must show that the line segment joining them belongs to B_o. Since B_o is connected, there exists a curve $\phi(t)$, $0 \leq t \leq 1$, lying in B_o, joining x and y; $\phi(0)=x, \phi(1)=y$. For t sufficiently small, the line segment $(x, \phi(t)) \subset B_o$. As $t \to 1$ there cannot exist a t_o such that for all $t < t_o$ the line segment $(x, \phi(t)) \subset B_o$ but the segment $(x, \phi(t_o)) \not\subset B_o$, because this would violate (a).

Hence B_o <u>is</u> convex.

In the case $\psi \notin C^2$, it suffices to show that if B_o is any component of B and if λ is a straight line segment in B_o, then $\max_{x \in \lambda} \psi(x) = \max_{x \in \partial\lambda} \psi(x)$, for then the proof follows as above. So, let λ be a straight line segment in B_o. Then $\lambda \subset \hat{B}^{\text{domain}} \subset\subset B_o$ and the tube domain $\hat{D}$ over $\hat{B}$ satisfies $\hat{D} \subset\subset D$. Hence for every $\epsilon > 0$ there is a pseudoconvex C^∞ function ψ_ϵ in $\hat{D}$ such that $|\psi - \psi_\epsilon| < \epsilon$ in $\hat{D}$, ψ_ϵ depending only on the x' s; as we can define smoothing functions K_ϵ depending only on the x' s in the proof of Proposition 2. As in the previous case, each ψ_ϵ is convex on $\hat{B}$, and therefore $\max_{x \in \lambda} \psi_\epsilon(x) = \max_{x \in \partial\lambda} \psi_\epsilon(x)$.

But, $\max_{x \in \lambda} \psi(x) < \max_{x \in \lambda} \psi_\epsilon(x) + \epsilon = \max_{x \in \partial\lambda} \psi_\epsilon(x) + \epsilon < \max_{x \in \partial\lambda} \psi(x) + 2\epsilon$, similarly

$\max_{x \in \lambda} \psi(x) > \max_{x \in \lambda} \psi_\epsilon(x) - \epsilon = \max_{x \in \partial\lambda} \psi_\epsilon(x) - \epsilon > \max_{x \in \partial\lambda} \psi(x) - 2\epsilon$. Letting $\epsilon \searrow 0$ gives the desired equality.

(2) implies (3). Since every component of B is convex, every component of D is convex. Hence D is a region of holomorphy.

(3) implies (1) has already been proved.

Chapter 4. Zeroes of Holomorphic Functions. Meromorphic Functions.

§1. Weierstrass Preparation Theorem

A. Definition 24. Let $\sum a_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n}$ be a formal series about the origin. (This series is not assumed to be convergent). Then the order of the series is the least integer K such that $a_{\nu_1 \nu_2 \ldots \nu_n} \neq 0$, $\nu_1+\nu_2+ \ldots + \nu_n = K$.

The series is said to be normalized with respect to z_1 (at the origin) if ord (series) = K and z_1^K occurs with non-zero coefficient $a_{K0\ldots0}$.

Note that the order of a series is invariant under holomorphic changes of variables, including non-singular linear changes, which leave the origin fixed

$$\zeta_j = \sum_{s=1}^{n} \alpha_{js} z_s , \qquad j = 1,\ldots,n,$$

$$\det (\alpha_{js}) \neq 0 ;$$

Corollary. If $\sum a_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n} = f(z)$ is a convergent series of order K, then the following are equivalent:

1) f is normalized with respect to z_1 at the origin

2) $\left(\dfrac{\partial^K f}{\partial z_1^K}\right) \neq 0$

3) $f_K(z_1,0,\ldots,0) \neq 0$, where f_K denotes the partial sum $\sum_{\nu_1+\ldots+\nu_n = K} a_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n}$ of the K th order homogeneous polynomials in the series f.

Note that 1) and 3) are also equivalent for formal power series. We shall write $f(z) = \sum a_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n}$ for the sake of brevity, for all formal power series. When convergence is assumed it will be mentioned explicitly.

Property 1. If $\sum a_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n} = f$ is any power series, it may be normalized with respect to z_1 (at the origin) by a linear change of variables: $z_j = \sum_{s=1}^{n} \alpha_{js} \zeta_s$, $j = 1,\ldots,n$; $\det (\alpha_{js}) \neq 0$.

Proof. Let f be of order K. Assume first that f is convergent. Then

$$f(\zeta) = f\left(\sum_{s=1}^{n} \alpha_{1s}\zeta_s, \ldots, \sum_{s=1}^{n} \alpha_{ns}\zeta_s\right),$$

and ord $f(\zeta)$ = ord $f(z)$.

$f(\zeta_1,0,\ldots,0) = f(\alpha_{11}\zeta_1,\ldots,\alpha_{n1}\zeta_1)$; hence $f_K(\zeta_1,0,\ldots,0)$ is non-zero for some choice of the α_{j1}, $j = 1,\ldots,n$. We may complete the matrix (α_{jk}) so that $\det(\alpha_{jk}) \neq 0$.

If $f(z)$ is nonconvergent

$$f = f_K + f_{K+1} + f_{K+2} + \ldots, \quad f_K \not\equiv 0 ;$$

where the f_j are homogeneous polynomials in the z_i of order j. Now, consider f_K as above.

Property 2. If $\{f^{(j)}(z)\}$ is a countable sequence of power series, they may be simultaneously normalized with respect to z_1 (at the origin) by one non-singular linear change of variables.

Proof. For each j,

$$f^{(j)} = f^{(j)}_{K_j} + f^{(j)}_{K_j+1} + \ldots ; \qquad \text{ord } f^{(j)} = K_j .$$

Consider the spherical hull

$$\Sigma = \left\{ (\alpha_{11},\alpha_{21},\ldots,\alpha_{n1}) \,\middle|\, \sum_{j=1}^{n} |\alpha_{j1}|^2 = 1 \right\}.$$

Now $f^{(j)}_{K_j}(z_1,\ldots,z_n)$ is a polynomial, and hence vanishes on a closed nowhere dense subset of Σ. But the union of countably many nowhere dense sets is nowhere dense, so there exists $(\alpha_{11},\ldots,\alpha_{n1}) \in \Sigma$ such that $f^{(j)}_{K_j}(\alpha_{11},\ldots,\alpha_{n1}) \neq 0$ for every j. But we may now complete $(\alpha_{11},\ldots,\alpha_{n1}) \neq 0$ to a nonsingular matrix.

We have shown that, if we consider countable collections of power series, and properties invariant under linear transformations, we may assume these series to be normalized with respect to z_1.

Note the following properties of order: (f, g denote formal power series)

$$\text{ord } fg = \text{ord } f + \text{ord } g$$

$$\text{ord } (f+g) \geq \min(\text{ord } f, \text{ord } g)$$

ord $f = 0$ if and only if f is a unit; i.e. if and only if there exists a g such thar $fg = 1$, where g is a formal power series.

We remark here that the set of formal power series at a point, as well as the subset of those power series which converge in some neighborhood of the origin, form commutative rings with unit. This ring is an integral domain, with units the series of order zero.

We remark also that the definitions and consequences stated above may easily be extended to series whose centers are any point $a \in \mathbb{C}^n$.

B. Theorem 12. (Weierstrass Preparation Theorem). Let f be a formal power series, normalized w.r.t. z_1, of order K. Then

$$f = h(z_1^K + a_1 z_1^{K-1} + \ldots + a_K)$$

where h is a unit, $a_1, \ldots, a_K$ power series in $z_2, \ldots, z_n$ and are non units; and this representation is unique.

If f is a convergent power series, then h and a_i are also.

Note. $z_1^K + a_1 z_1^{K-1} + \ldots + a_K$ as above is called a Weierstrass polynomial.

Proof. We first make a series of remarks:

For $K = 0$, the theorem is trivial.

For $z = z_1$, the theorem is also clear.

Furthermore, if $f = h(z_1^K + a_1 z_1^{K-1} + \ldots + a_K)$, the a_i must have no constant term, for

$$\text{ord } f = \text{ord } h + \text{ord } (z_1^K + a_1 z_1^{K-1} + \ldots + a_K) .$$

Therefore

$$K = \text{ord } (z_1^K + a_1 z_1^{K-1} + \ldots + a_K) .$$

If ord $a_i = 0$ for some a_i,

$$\text{ord } (z_1^K + a_1 z_1^{K-1} + \ldots + a_K) = K - 1 < K .$$

We now present a proof for the case of formal power series; this (constructive) proof will give uniqueness in both cases. However, if f is convergent, the convergence of h and the a_i is more easily shown by a second proof.

$f = f_K + f_{K+1} + \ldots$; f_j homogeneous polynomials of ord j where $f_K = z_1^K + \ldots$.

We wish to construct a power series

$$\frac{1}{h} = X_0 + X_1 + \ldots \quad \text{such that}$$

$$(1) \quad (f_K + f_{K+1} + \ldots)(X_0 + X_1 + \ldots) = y_K + y_{K+1} + \ldots$$

where $X_0 \neq 0$, $y_K = z_1^K + \ldots$

and all other y_j are of order at most $K-1$ in z_1.

From (1), we obtain

$$f_K X_0 = y_K$$

But $f_K = z_1^K + \ldots$, $y_K = z_1^K + \ldots$; hence $X_0 = 1$.

Similarly, $f_K X_1 + f_{K+1} X_0 = y_{K+1}$

i.e. $$f_{K+1} = - f_K X_1 + y_{K+1} .$$

We choose X_1 so that y_{K+1} has order at most $K-1$ in z_1, as follows: Let $f_{K+1} = \sum_{\nu_1+\ldots+\nu_n=K+1} a_{\nu_1\ldots\nu_n} z_1^{\nu_1}\ldots z_n^{\nu_n}$

$$f_K = z_1^K + \sum_{\substack{\mu_1+\ldots+\mu_n=K \\ \mu_1 \neq K}} b_{\mu_1\ldots\mu_n} z_1^{\mu_1}\ldots z_n^{\mu_n}$$

$$X_1 = c_1 z_1 + c_2 z_2 + \ldots + c_n z_n .$$

Take $c_1 = - a_{K+1,0,\ldots,0}$ and

$$c_j = - a_{K,0,\ldots,1,0,\ldots,0} - b_{K-1,0,\ldots,1,0,\ldots,0}\, a_{K+1,0,\ldots,0}$$

for $j \geq 2$; the subscripts 1 appearing in the j th places.

Choose X_2, etc., similarly.

Note that we have proven uniqueness for both convergent and nonconvergent series. We now proceed to the proof for the case: f convergent. It is due to Siegel:

As before, write

$$f = f_K + f_{K+1} + \dots ,$$
$$f_K = z_1^K + \dots .$$

For $|z_j| < |z_1|$, $j = 2,\dots,n$

$$|f_{K+1} + f_{K+2} + \dots| \leq c_1|z_1|^{K+1} + c_2|z_1|^{K+2} + \dots,$$

which is convergent for $|z_1| < \tilde{\rho}$ small.

Therefore $|f_{K+1} + f_{K+2} + \dots| \leq \frac{1}{2}|z_1|^K$; $|z_j| < |z_1| < \rho < \tilde{\rho}$,

as $\left|\frac{f_{K+1}+f_{K+2}+\dots}{z_1^K}\right| < \frac{1}{2}$ under the above conditions.

Consider f_K/z_1^K ; define $t_j = z_j/z_1$, $j = 2,\dots,n$.
Then f_K/z_1^K is a polynomial in the t_j ;

$$\frac{f_K}{z_1^K} = 1 + r ,$$

where r is a homogeneous polynomial in the t_j without a constant term. Therefore, for $|t_j| < \rho_1$, $|r| < 1/2$.
Under these conditions:

$$\begin{aligned}\frac{f}{f_K} &= \frac{f_K + (f_{K+1} + \dots)}{f_K}\\ &= 1 + \frac{f_{K+1}+\dots/z_1^K}{f_K/z_1^K}\\ &= 1 + \frac{f_{K+1}+\dots}{z_1^K}\,\frac{1}{1+r}\\ &= 1 + q ,\end{aligned}$$

where q is a power series in z_1 and the t_j. We restrict ourselves to the above inequalities; hence

$$|q| < 1 .$$

Choosing some determination of log, we obtain:

$$\begin{aligned}\log\frac{f}{f_K} &= \log(1+q)\\ &= q - q^2/2 + q^3/3 - q^4/4 + \dots ,\end{aligned}$$

a convergent power series. Substituting, we obtain a convergent power series in $z_1, t_2, \ldots, t_n$; replacing t_j by z_j/z_1 we obtain a power series in $z_2, \ldots, z_n$ and Laurent series in z_1.

Now assume also that $|z_1| > \varepsilon > 0$. Hence

$$\log \frac{f}{f_K} = \sum_{\nu=0}^{\infty} \alpha_\nu z_1^\nu + \sum_{\omega=1}^{\infty} \frac{\beta_\omega}{z_1^\omega} = v + w$$

where α_ν, β_ω are power series in $z_2, \ldots, z_n$. Therefore

$$\log f - \log f_K = v + w$$

i.e.

$$fe^{-v} = f_K e^w .$$

But fe^{-v} is an analytic function of z_1, and e^{-v} is a unit. Hence fe^{-v} converges in a neighborhood of the origin, by Abel's theorem. Therefore the series $f_K e^w$ cannot contain any negative powers of z_1; and also no power of $z_1 > K$, and we have obtained a convergent representation

$$f = e^v(f_K e^w)$$

as claimed, unique by the above.

Corollary 1. Let f be a convergent power series, and assume f vanishes at the origin. Then the set of zeroes of f in a neighborhood of the origin is of dimension $n-1$.

Proof. By a linear change of variables, we may assume f is normalized with respect to z_1,

$$f = h(z_1^K + a_1 z_1^{K-1} + \ldots + a_K)$$

For $z_2, \ldots, z_n$ small and fixed arbitrarily, $a_1, \ldots, a_K$ are small, and $z_1^K + a_1 z_1^{K-1} + \ldots + a_K$ is a complex polynomial, with K roots (counting multiplicity). Furthermore, these zeroes are located in a neighborhood of the origin as they depend continuously on the a_i.

Definition 25. Let $S^{\text{closed}} \subset D^{\text{open}} \subset \mathbb{C}^n$. Then S is said to be a (globally defined) analytic hypersurface or analytic variety of codimension 1, if S is the set of zeroes of a function $f \not\equiv 0$, analytic in D.

Corollary 2. S is locally arcwise connected.

Corollary 3. If $n > 1$ and f is holomorphic in $D \subset \mathbb{C}^n$, then the zeroes of f are not isolated.

§2. Rings of power series

A. As we have remarked previously, the set of convergent power series in $z_1,\ldots,z_n$ at a point forms a ring, as does the set of formal power series. We shall now state some algebraic results which will prove useful in the sequel. Refer to any standard algebra text, e.g. van der Waerden, Moderne Algebra, for proofs and details.

Definition 26. Let R be a commutative unitary ring. R is said to be an integral domain if: $a \in R$, $b \in R$, $ab = 0$ implies $a = 0$ or $b = 0$. An element $a \in R$ is called a unit if a has an inverse in R. Elements $a,b \in R$ will be called equivalent, written $a \equiv b$, if $a = eb$, where e is a unit. An element $a \in R$ is called reducible if $a = bc$, where b and c are non-units; a is otherwise called irreducible or prime. R is a unique factorization domain (U.F.D.) if every element may be written as a product of primes, unique up to order and equivalence. A subset I of R is an ideal if $a,b \in I$ implies $a-b \in I$, and $a \in I$, $r_t \in R$ implies $ar_t \in I$. I is a proper ideal if $I \neq R$, and maximal if it is proper and such that if $\mathcal{J}$ is any ideal satisfying $I \subset \mathcal{J} \subset R$, then $I = \mathcal{J}$ or $\mathcal{J} = R$. A ring R is called a local ring if there exists a unique maximal ideal. $R[t]$ denotes the ring of polynomials with coefficients in R and $\bar{R}[t]$ the ring of power series in t with coefficients in R. Let $\sum^{\infty} \alpha_j t^j \in \bar{R}[t]$, $\alpha_j \in R$, t an indeterminate. $\sum^{\infty} \alpha_j t^j$ is called primitive if the coefficients α_j of t have no common factor except units. (Note that $f \in \bar{R}[t]$ implies $f = \alpha g$ with $\alpha \in R$ and $g \in \bar{R}[t]$ and primitive.) Two polynomials will be called relatively prime or coprime if they have no common polynomial factor. We use the following results.

Lemma a. (Gauss' Lemma) If R is UFD, $R[t]$ is UFD.

Lemma b. Let $p,q \in R[t]$ $(\overline{R}[t])$. Then p and q primitive implies pq primitive.

Lemma c. Let $P_1, P_2 \in R[t]$, R an integral domain.

$$P_1 = t^K + a_1 t^{K-1} + \dots + a_K$$

$$P_2 = t^L + b_1 t^{L-1} + \dots + b_L \quad .$$

Then there exists a polynomial r in the coefficients a_i, b_j called the resultant of P_1 and P_2, which is zero if and only if P_1 and P_2 have a common factor; i.e. if and only if there exist $p,q,s \in R[t]$, $\deg q > 0$, such that $P_1 = pq$, $P_2 = sq$. Furthermore, there exist polynomials A and B such that $AP_1 + BP_2 = r$.

Lemma d. No proper ideal I of a unitary ring R contains a unit.

Lemma e. R is a local ring if the nonunits in R form an ideal.

B. Definition 27. Let $\mathcal{O}_n$ denote the ring of formal power series at the origin in n complex variables.

Property 1. $\mathcal{O}_n$ is an integral domain with unit.

Property 2. The nonunits of $\mathcal{O}_n$ form an ideal; hence $\mathcal{O}_n$ is a local ring.

Property 3. $\mathcal{O}_n$ is UFD.

Proof. We use induction on n. For $n = 1$, units and elements of order 1 are irreducible. All elements of order > 1 are reducible, for if $\operatorname{ord} g = K > 1$,

$$g(z) = z^K g_1(z) \quad ,$$

where $g_1(z)$ is a unit, and the decomposition is unique. Hence, assume $\mathcal{O}_{n-1}$ is UFD. We may assume $f \in \mathcal{O}_n$ is normalized at the origin. Then:

$$f = h(z_1^K + a_1 z_1^{K-1} + \dots + a_K) \equiv hp \quad ,$$

where p is a Weierstrass polynomial; i.e. $p \in \mathcal{O}_{n-1}[z_1]$ and is monic. Now f is prime in $\mathcal{O}_n$ if and only if p is prime in $\mathcal{O}_{n-1}[z_1]$, for:

Assume p is reducible; i.e. $p = p_1 p_2$. We may take p_1, p_2 to be Weierstrass polynomials. Hence $f = (hp_1)p_2$. Conversely, assume f is reducible:

$$\begin{aligned} f &= f_1 f_2 \\ &= (h_1 p_1)(h_2 p_2) \\ &= (h_1 h_2)(p_1 p_2) . \end{aligned}$$

But $p_1 p_2 = p$ by uniqueness, and $h_1 h_2 = h$.

But $\mathcal{O}_{n-1}[z_1]$ is UFD by assumption and Gauss' lemma. Now let $f \in \mathcal{O}_n$.

$$f = hp .$$

But
$$p = p_1 \cdots p_r$$

where the p_i are irreducible Weierstrass polynomials. Hence

$$f = hp_1 \cdots p_r ,$$

and this decomposition is unique up to order and equivalence, for if

$$f = t_1 \cdots t_\ell ,$$

$$t_i = h_i r_i ,$$

by the Weierstrass Theorem and then

$$r_1 \cdots r_\ell = p_1 \cdots p_r$$

by uniqueness. Hence $\{r_1, \ldots, r_\ell\} \equiv \{p_1, \ldots, p_r\}$ as $\mathcal{O}_{n-1}[z_1]$ is UFD.

Definition 28. Two holomorphic functions are said to be relatively prime or coprime at a point if their power series expansions at that point have no common irreducible factor other than a unit.

Lemma f. Let f, g be holomorphic functions in D, $0 \in D \subset \mathbb{C}^n$, $n > 1$ coprime at the origin, such that

$f(0) = g(0) = 0$. Then in any neighborhood of the origin, there exist points at which f vanishes and g does not, and points at which g vanishes and f does not.

Proof. We may assume f and g are Weierstrass polynomials

$$f = z_1^K + a_1 z_1^{K-1} + \dots + a_K$$

$$g = z_1^L + b_1 z_1^{L-1} + \dots + b_L .$$

Suppose there exist no such points. Let $r(a_i, b_j)$ be the resultant of f and g.

$$r(a_i, b_j) \quad \varepsilon \quad \mathcal{O}_{n-1} .$$

For each $z_2, \dots, z_n$ in a neighborhood of the origin, there exist z_1 such that f and g vanish simultaneously. But if f and g have a common zero viewed as polynomials in one variable, then f,g have a common factor. Hence $r(a_i, b_j)$ is zero for each $z_2, \dots, z_n$. But r is analytic, and hence $r \equiv 0$ near the origin, implies f,g have a common factor as polynomials in $\mathcal{O}_n$.

Lemma g. Let f,g be holomorphic functions coprime at the origin. They they are also coprime in some neighborhood of the origin.

Proof. Let
$$f = up$$
$$g = vq ,$$

where p,q are Weierstrass polynomials, and let

$$r = Ap + Bq$$

be the resultant of p and q. Let N be a neighborhood of the origin so small that u,p,v,q,A and B are convergent. Let $a = (a_1, \dots, a_n) \ \varepsilon \ N$. To show f,g are coprime at a, it suffices to show p,q are coprime. The equation

$$r = Ap + Bq$$

persists where r,A,p,B and q are viewed as series about a, i.e. as series in ζ_1, where

$$\zeta_1 = z_1 - a_1 .$$

Then
$$p = \zeta_1^K + \ldots$$
$$q = \zeta_1^L + \ldots .$$

Now assume p and q have a common factor h "at a", $h \in \mathcal{O}_{n-1}[\zeta_1]$. But p,q are primitive in ζ_1; hence h is also.

$$h = h_0 + h_1\zeta_1 + \ldots$$

where $h_i \in \mathcal{O}_{n-1}$. But h divides r. Therefore

$$r = hk\lambda ,$$

$\lambda \in \mathcal{O}_{n-1}$ and k a primitive power series in ζ_1, $k \in \mathcal{O}_{n-1}[\zeta_1]$.

$$k = k_0 + k_1\zeta_1 + \ldots$$

By comparing coefficients

$$r = h_0k_0\lambda , \qquad \text{a relation in } \mathcal{O}_{n-1} .$$

Hence λ divides r, and

$$\frac{r}{\lambda} = hk.$$

But h and k are primitive, hence hk is primitive, hence h_0k_0 must be a unit, i.e. hk is a unit, implying that h is a unit. Thus p and q are coprime at a.

§3. Meromorphic functions

Let $x \in D^{open} \subset \mathbb{C}^n$, and consider functions which are each defined in some neighborhood of x in D. Call two such functions equivalent if they coincide on a neighborhood of x. This defines an equivalence relation, and the equivalence class of a function f at x, denoted by $[f]_x$, is called the germ of f at x.

If f is a holomorphic function, then $[f]_x$ amounts to a convergent power series.

Germs at x form a ring with the obvious definition of addition and multiplication.

The ring $\mathcal{O}_x$ of germs of holomorphic functions is a commutative integral domain with identity and unique factorization. Topologize the space of germs $\mathcal{O} = \bigcup_{x \in D} \mathcal{O}_x$ by defining the following basis for the open sets: Let $k \in \mathcal{O}$, then $k \in \mathcal{O}_{x_0}$ and so $k = [f]_{x_0}$, where f is defined in an ε-neighborhood N_{x_0} of x_0 in D. At each $y \in N_{x_0}$, take that class in $\mathcal{O}_y$ containing the direct analytic continuation of f, i.e. take $[f]_y$. Then define $\bigcup_{y \in N_{x_0}} [f]_y$ to be an open set and the collection of such sets to be the basis of open sets.

A holomorphic function f in D amounts to a continuous mapping, $f : D \to \mathcal{O}$ which assigns to each point in D a holomorphic germ over that point.

Now form the quotient field M_x of $\mathcal{O}_x$ for each $x \in D$. Topologize $M = \bigcup_{x \in D} M_x$ as follows: Let $\ell \in M$, then $\ell \in M_{x_0}$, $\ell = [\frac{\alpha}{\beta}]_{x_0}$ and is represented by $[f_1]_{x_0}/[f_2]_{x_0}$ where f_1 and f_2 are holomorphic functions at x_0, and because of unique factorization we may take f_1 and f_2 to be coprime at x_0. Let N_{x_0} be a neighborhood of x_0 in which f_1 and f_2 are defined and are still coprime. At each $y \in N_{x_0}$, take that class in M_y represented by $[f_1]_y/[f_2]_y$. The union over N_{x_0} of these classes we define as an open set and the collection of all such sets we take as the basis for the topology.

The elements of M_x are called germs of meromorphic functions over x.

Definition 29. A meromorphic function in D is a continuous mapping which assigns to each point of D a meromorphic germ over that point.

Meromorphic functions form a field.

At a point $z_0 \in D$, a meromorphic function g is efined by the quotient of two functions f_1, f_2 coprime and holomorphic at z_o.

a) If $f_2(z_o) \neq 0$, i.e. f_2 is a unit, then f_1/f_2 is holomorphic there and hence g is holomorphic at z_o and therefore in a neighborhood of z_o. z_o is called a regular point of g.

b) If $f_2(z_o) = 0$ and $f_1(z_o) \neq 0$, then g is said to have a pole at z_o.

c) If $f_2(z_o) = 0$ and $f_1(z_o) = 0$, then z_o is called a point of indeterminacy of g. The set of such points has topological dimension $2n-4$.

Corollary 1. The set of regular points of g is open, and $g|_{\text{regular pts}}$ is holomorphic.

Corollary 2. If z_o is a pole of g, then there exists a neighborhood N of z_o in which every point is a pole or a regular point. Furthermore, g has no isolated poles $(n > 1)$, and, for each number $M > 0$ there exists a neighborhood N_M of z_o in which $|g| > M$.

Corollary 3. A point of indeterminacy is a limit point of zeroes and poles of g.

Exercise. A point of indeterminacy of g is a limit point of zeroes of $g - \alpha$, where α is any complex number.

Poincaré's Problem

1) Weak form: Given a domain D, is every function meromorphic in D a quotient of two functions holomorphic in D?

2) Strong form: Given a domain D, g meromorphic in D, is g the quotient of two functions holomorphic in D and coprime at every point?

§4. Removable singularities

In this section we shall state three theorems. The first, Radó's theorem, facilitates the proof of the first theorem on removable singularities which is a direct generalization of the Riemann theorem in one complex variable.

A second theorem on removable singularities will be stated but not proven.

Theorem 13 (Radó). Let $f(z_1,\ldots,z_n)$ be continuous in $D^{\text{open}} \subset \mathbb{C}^n$ and holomorphic in $(D- \{z|f(z) = 0\})$. Then f is holomorphic in D.

Proof (Heinz). A function of n variables is holomorphic if and only if it is holomorphic in each variable separately. It is sufficient to prove the theorem for functions of one variable.

If D' is any open disc whose closure is contained in $D \subset \mathbb{C}^n$, we must prove that $f(z)$ is holomorphic in D'. Without loss of generality, assume that D' is the unit disc $(|z|<1)$. Let $\Gamma = (|z|=1)$ and $\Delta = (\{z|f(z)=0\} \cap D')$. Since $f(z)$ is continuous in $(D' \cup \Gamma)$, it is bounded there, and we may assume $|f(z)| < 1$ in $(D' \cup \Gamma)$. By hypothesis, $f(z)$ is holomorphic in $(D'-\Delta)$.

Construct a complex-valued harmonic function $g(z)$ in D' such that $g(z) = f(z)$ on Γ. Then, consider the following functions for $z \in (D'-\Delta)$ and $\alpha > 0$:

$$\phi_1(z) = \operatorname{Re}\,[f(z)-g(z)] + \alpha \log |f(z)|$$

$$\phi_2(z) = \operatorname{Re}\,[f(z)-g(z)] - \alpha \log |f(z)|$$

$$\phi_3(z) = \operatorname{Im}\,[f(z)-g(z)] + \alpha \log |f(z)|$$

$$\phi_4(z) = \operatorname{Im}\,[f(z)-g(z)] - \alpha \log |f(z)| .$$

Note that for $z \in (D'-\Delta)$, $\alpha \log |f(z)| < 0$. Now, as $z \to \partial(D'-\Delta)$, which consists of Γ and points where $f = 0$, either

(i) $z \to z_0 \in (\partial(D'-\Delta) \cap \Gamma)$ and then $[f(z)-g(z)] \to 0$ or (ii) $z \to z_0 \in (\partial(D'-\Delta) \cap \Delta)$ and then $[f(z)-g(z)]$ remains bounded and $\log |f(z)| \to -\infty$. In either event, $\phi_1(z)$ and $\phi_3(z) \to$ negative numbers while $\phi_2(z)$ and $\phi_4(z) \to$ positive numbers. But, since the ϕ_i are harmonic in $(D'-\Delta)$, they assume both their maximum and their minimum on $\partial(D'-\Delta)$. Hence, $\phi_1(z)$ and $\phi_3(z)$ are negative, and $\phi_2(z)$ and $\phi_4(z)$ are positive for all $z \in cl(D'-\Delta)$. Let $\alpha \to 0$, then

$$\phi_1(z) \to \text{Re}\,[f(z)-g(z)] \quad \text{which implies} \quad \text{Re}\,[f(z)-g(z)] \leq 0$$

$$\phi_2(z) \to \text{Re}\,[f(z)-g(z)] \quad \text{which implies} \quad \text{Re}\,[f(z)-g(z)] \geq 0$$

$$\phi_3(z) \to \text{Im}\,[f(z)-g(z)] \quad \text{which implies} \quad \text{Im}\,[f(z)-g(z)] \leq 0$$

$$\phi_4(z) \to \text{Im}\,[f(z)-g(z)] \quad \text{which implies} \quad \text{Im}\,[f(z)-g(z)] \geq 0$$

for all $z \in cl(D'-\Delta)$. Therefore $f(z) = g(z)$ for all $z \in (cl(D'-\Delta) \cup \Gamma)$. Since $\partial\Delta$ consists of points of Γ and points of $cl(D'-\Delta)$, $f(z) = g(z)$ for $z \in \partial\Delta$. In any component of the interior of Δ, $f \equiv 0$ and thus $g \equiv 0$. Therefore $f(z) = g(z)$ in $(D' \cup \Gamma)$, which means that f is harmonic in D'. Thus f has continuous first order partial derivatives. f satisfies the Cauchy-Riemann equations in $D-\Delta$, hence by continuity, on $\partial\Delta$ and on Γ. In the interior of Δ, $f \equiv 0$ and therefore satisfies the equations in Δ. Hence f satisfies the Cauchy-Riemann equations in D' and is therefore holomorphic in D'.

<u>Theorem 14</u>. Let $D^{open} \subset \mathbb{C}^n$, and let $g \not\equiv 0$ be holomorphic in D. Let f be holomorphic and bounded in $D - \{z | g(z) = 0\}$. Then f is holomorphic in D.

<u>Proof</u>. Consider the function

$$h = \begin{cases} gf & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

Then h is continuous as f is bounded, and holomorphic where it is not zero. By Radó's theorem, h is holomorphic. But g is holomorphic by assumption; hence

$$f = \frac{h}{g}$$

is meromorphic. But f is bounded, thus without poles, thus holomorphic.

Theorem 15. Let $D^{\text{open}} \subset \mathbb{C}^n$; g, h holomorphic in D, not identically zero and relatively prime at every point of D. Let f be holomorphic in the set $D - \{z \mid g(z) = h(z) = 0\}$. Then f is holomorphic in D.

Theorem 16. Let $D^{\text{domain}} \subset \mathbb{C}^n$ and let $g \not\equiv 0$ be holomorphic in D. Then $(D - \{z \mid g(z) = 0\})$ is connected.

Proof. Let $S = (D - \{z \mid g(z) = 0\})$. Suppose that S is not connected, then $S = U \cup V$ where U and V are open, disjoint sets. Define a function $h = \begin{cases} 1 \text{ in } U \\ 0 \text{ in } V \end{cases}$. h is holomorphic where $g \neq 0$, and is bounded; therefore h is holomorphic in D. This is impossible since it implies that h is identically 1 in D.

§5. Complex manifolds

Remark. From now on, "differentiable" means "C^∞".

Definition 30. X is a $\binom{\text{differentiable}}{\text{complex}}$ manifold of $\binom{\text{real}}{\text{complex}}$ dimension r, if the following conditions are satisfied:

(1) X is a Hausdorff space.

(2) Given an open set in X and a function defined in it, it is possible to say whether or not this function is $\binom{\text{differentiable}}{\text{holomorphic}}$.

(3) There exist coordinates: every point in X has a neighborhood where r $\binom{\text{real, differentiable}}{\text{complex, holomorphic}}$ functions are defined such that they give a homeomorphism of this neighborhood onto a domain in $\binom{R^r}{\mathbb{C}^r}$, and every function defined in this neighborhood is $\binom{\text{differentiable}}{\text{holomorphic}}$ if and only if it is $\binom{\text{differentiable as a function of } x_1,\dots,x_r}{\text{holomorphic in each variable of } z_1,\dots,z_r}$. The coordinates are called local coordinates and such a neighborhood is called a coordinate patch.

(4) There is a countable basis for the open sets of X, i.e. X is second countable.

Remarks. On a complex manifold we may talk about holomorphic and meromorphic functions; on a differentiable manifold, about differentiable functions.

A connected 1-dimensional complex manifold is called a Riemann surface.

Every complex manifold is a differentiable manifold. Therefore it is natural to ask on which differentiable manifolds we can introduce the concept of holomorphic functions: that is, which differentiable manifolds can be given a complex structure. Necessary conditions are that the differentiable manifold be orientable and of even dimension, $r = 2n$. If $n = 1$, these conditions are also sufficient; however, if $n > 1$, they are not. In fact, in the latter case, necessary and sufficient conditions are not known.

There are other differences between the cases $n = 1$ and $n > 1$:

(i) When $n = 1$, axiom (4) in definition 30 is unnecessary as it follows from axioms (1), (2), and (3). When $n > 1$, axiom (4) is essential.

(ii) If $n = 1$ and X is compact, there exist non-constant meromorphic functions on X (i.e. on every closed Riemann surface there exist non-constant meromorphic functions). However, when $n > 1$, there are compact complex manifolds having no non-constant meromorphic functions.

(iii) If $n = 1$ and X is not compact, there exist non-constant holomorphic functions (i.e. on every open Riemann surface), while if $n > 1$ this is not necessarily so.

For example, let Y be a compact connected complex manifold of dimension $n > 1$, then $X = (Y-\{p\})$ is not compact and if there existed a non-constant holomorphic function on X, it would be holomorphic also at p (by Hartogs' Theorem), and it would be constant (by the maximum modulus theorem).

Examples of Complex Manifolds.

1) $\mathbb{C}^n$

2) Any open subset of a complex manifold.

3) If X and Y are complex manifolds then $X \times Y$ is

a complex manifold, where a function f is holomorphic in $X \times Y$ if it is holomorphic in X and holomorphic in Y.

4) The complex projective space

$$P_n(\mathbb{C}) = \{[(z_0,z_1,\ldots,z_n)] \mid z_i \text{ are not all zero}\},$$

where $[(z_0,\ldots,z_n)]$ denotes the equivalence class of points $(z_0,\ldots,z_n) \in \mathbb{C}^{n+1}$, where two points $(\zeta_0,\ldots,\zeta_n)$ and $(\zeta'_0,\ldots,\zeta'_n)$ are equivalent if and only if there is a $t \neq 0$ such that $\zeta_j = t\zeta'_j$, $j = 0,1,\ldots,n$, with local coordinates in a neighborhood of $(z_0,\ldots,z_n)$ where $z_i \neq 0$ for some i, $0 \leq i \leq n$, being $z_0/z_i,\ldots,z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_i$, is a complex manifold. On this manifold there exist meromorphic functions; the ratio of two homogeneous polynomials of the same degree is such.

5) The special case of 4, $P_1(\mathbb{C})$ = Riemann sphere = $\{\mathbb{C} \cup \{\infty\}\}$.

6) Starting with a complex manifold of dimension $n > 1$, omitting a single point, and imbedding $P_1(\mathbb{C})$, we will obtain a new complex manifold. This procedure is known as the σ-process. We do this for $n = 2$, starting with $\mathbb{C}^2$.

First we define the space X to consist of two types of points:

$$I = \{(z_1,z_2) \mid (z_1,z_2) \in \mathbb{C}^2 \text{ and } (z_1,z_2) \neq (0,0)\}$$

$$II = \{[(\zeta_1,\zeta_2)] \mid (\zeta_1,\zeta_2) \in \mathbb{C}^2 \text{ and } (\zeta_1,\zeta_2) \neq (0,0)\}.$$

$$X = I \cup II.$$

Secondly, we make X into a Hausdorff space by defining the following basis of open sets:

A neighborhood of a point $p \in I$ shall be a neighborhood in the ordinary topology of $\mathbb{C}^2$ such that its closure does not contain $(0,0)$.

A neighborhood of a point $p \in II$, $p = [(\zeta_1,\zeta_2)]$, and say $\zeta_1 \neq 0$, shall be the set of points; $[(1,\zeta)] \in II$ satisfying $|\zeta - \zeta_2/\zeta_1| < \varepsilon$, and $(z_1,z_2) \in I$ satisfying

$z_1 \neq 0$, $|z_1| < \varepsilon$, $|z_2| < \varepsilon$ and $|z_2/z_1 - \zeta_2/\zeta_1| < \varepsilon$, for some $\varepsilon > 0$.

Thirdly, we define local coordinates.. Near a point of I, take z_1 and z_2 as coordinates. Near a point $[(\zeta_1,\zeta_2)]$ of II, where say $\zeta_2 \neq 0$ and hence $[(\zeta_1,\zeta_2)] = [(\zeta_1/\zeta_2,1)] = [(\zeta_0,1)]$ say, take as local coordinates t and τ :

$$\begin{aligned} z_1 &= (1+t)\tau\, \zeta_0 \\ z_2 &= \tau \\ \zeta_1 &= (1+t)\zeta_0 \\ \zeta_2 &= 1 \ . \end{aligned}$$

On this new manifold X, the following holds: every holomorphic function on X is constant on II. For if f is holomorphic on X, it is holomorphic on $X-P_1(\mathbb{C}) = \mathbb{C}^2-\{0\}$. By Hartogs' Theorem, f is also holomorphic at the origin $(0,0)$. Therefore f approaches some complex number, a, as its argument goes to the origin. Hence near every point of $P_1(\mathbb{C})$, the value of f is close to a. Thus $f = a$ on every point of $P_1(\mathbb{C})$, but then f must be identically a on II.

7) A globally presented, regularly imbedded analytic subvariety Y, of codimension r in an n-dimensional complex manifold X is defined as follows:

Let $f_1,\ldots,f_r$ be holomorphic functions defined on X, such that, for every $x \in X$ at which $f_1(x) = \ldots = f_r(x) = 0$, the rank of the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial z_j}\right)$$

is r, i.e. J is of maximal rank. The derivatives $\partial f_i/\partial z_j$ are to be understood in the following way: Let N be a coordinate patch containing x, and let $\Phi : N \to \mathbb{C}^n$ define the local coordinates

$$\Phi = (z_1, \ldots, z_n) .$$

Then
$$\frac{\partial f_i}{\partial z_j} \equiv \frac{\partial f_i \cdot \Phi^{-1}(z_1, \ldots, z_n)}{\partial z_j}$$

Then Y, as a subset of X, is defined as:

$$Y = \{ x \mid f_1(x) = \ldots = f_r(x) = 0 \} .$$

Note that Y is closed in X, and that axioms 1, 2 and 4 are clearly satisfied. We now define the local coordinates in Y.

Let $y \in Y$, with local coordinates $\Phi(y) = (z_1(y), \ldots, z_n(y))$ defined in a neighborhood N of y, where $N \subset X$, Assume that, at y, $\det(\partial f_i/\partial z_j) = \det A \neq 0$, $j = 1, \ldots, r$ by relabeling the z_j if necessary. Define new coordinates

$$\begin{aligned}
\zeta_1 &= f_1(z_1, \ldots, z_n) \\
&\vdots \\
\zeta_r &= f_r(z_1, \ldots, z_n) \\
\zeta_{r+1} &= z_{r+1} \\
&\vdots \\
\zeta_n &= z_n .
\end{aligned}$$

Then the transformation taking z to ζ is given by the square matrix;

$$\left(\frac{\partial \zeta}{\partial z}\right) = \left(\begin{array}{c|c} A & B \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right)$$

which is nonsingular as $\det A \neq 0$. Then the local coordinates in Y are $(\zeta_{r+1},\ldots,\zeta_n)$, as $\zeta_1 = \ldots = \zeta_r = 0$ on Y.

Note. This example exhibits the technique by which statements in $\mathbb{C}^n$ that are local, i.e. refer to some neighborhood of a point, are transformed into statements about analytic manifolds X of dimension n. Hence, when proving results about manifolds, we shall sometimes assume $X \subset \mathbb{C}^n$. The modification of notation needed for arbitrary manifolds will be left to the reader.

Note. A regularly imbedded globally presented analytic subvariety of codimension 1 is called a hypersurface.

8) A regularly imbedded analytic subvariety Y of codimension r is a closed subset of X such that every point of Y has a neighborhood N in X such that $Y \cap N$ is a globally presented regularly imbedded analytic subvariety of codimension r in N.

Chapter 5. The Additive Cousin Problem

§1. The Additive Problem, formulated

A. This "first" Cousin problem (there is a second one) is a direct generalization of the Mittag-Leffler problem in one complex variable:

> Given a domain D, a discrete set of points $a_\nu \in D$, and polynomials $P_\nu(\frac{1}{z-a_\nu})$, without constant term, find a function f, meromorphic in D, with singular part P_ν at a_ν.

As we know, this problem is solvable for all domains $D \subset \mathbb{C}$. The Additive Cousin Problem is as follows (we shall denote it by "Cousin I", or simply C.I in the sequel):

C.I Let X be a complex manifold, and $U = \{u_i\}$, $i \in I$ be a given open covering of X, I some index set. Let meromorphic functions F_i defined in u_i, be given, such that $F_i - F_j$ is holomorphic in $u_i \cap u_j$. Find a function F, meromorphic and defined on X, such that $F - F_i$ is holomorphic in u_i.

This problem is not always solvable, as seen from the following theorems:

Theorem 17. Extension Theorem (Oka). Let X be a complex manifold such that the Cousin problem is always solvable. Let Y be a globally presented regularly imbedded analytic hypersurface of codimension 1. Then extension from Y is always possible, i.e. given ϕ, holomorphic on Y, there exists Φ, holomorphic on X, such that $\phi = \Phi$ on Y.

Assuming this theorem for the moment, we exhibit the following application:

Theorem 18 (Cartan). Let $X \subset \mathbb{C}^2$, open, such that the Cousin problem is always solvable in X. Then X is a region of holomorphy.

Proof. We may assume that X is a domain, that $0 \in X$, and that $X \neq \mathbb{C}^2$, as we already know that every $\mathbb{C}^n$ is a domain of holomorphy. Hence, assume $b \in \operatorname{bdry} X$; $b = (b_1, b_2)$

where b_1, b_2 are fixed complex numbers. Then if $f(z) = b_2 z_1 - b_1 z_2$. $f(z) = 0$ is an analytic plane through 0 and b. Let $Y = \{z \mid f(z) = 0\}$. Now $Y_1 = Y \cap X$ is an open set in C; hence Y_1 is a region of holomorphy. Therefore there exists a function ϕ, holomorphic in Y_1 and singular at b. But by the extension theorem, there exists a Φ, holomorphic in X such that $\Phi = \phi$ on Y_1, and Φ is singular at b.

B. Proof of the Extension Theorem

Let $Y = \{f = 0\}$; and let ϕ be holomorphic in Y. There exists a covering $U = \{u_i\} \varepsilon X$ of Y such that $Y \subset \bigcup_i u_i$; and in each u_i there exists a Φ_i such that Φ_i is holomorphic in u_i and equal to ϕ on $Y \cap u_i$, by definition of holomorphicity on closed sets (see proof of Lemma p. 74). Let $u_0 = X - Y$, an open set in X.

Define:
$$F_i = \Phi_i / f, \qquad i \neq 0$$
$$F_0 = 1 .$$

This covering and set of associated functions defines a Cousin problem, for, on $u_0 \cap u_i$, $F_0 - F_i$ is holomorphic. Indeed, on $u_i \cap u_j$, $F_i - F_j$ is holomorphic except possibly for points on Y. Hence, assume $f(z_0) = 0$, $z_0 \varepsilon u_i \cap u_j$. Introduce local coordinates $(\zeta_1, \ldots, \zeta_n)$ such that $f = \zeta_1$. But now,

$$\Phi_i(0, \zeta_2, \ldots, \zeta_n) - \Phi_j(0, \zeta_2, \ldots, \zeta_n) = 0 ,$$

as Φ_i and Φ_j agree on Y, and

$$F_i - F_j = \frac{\Phi_i(\zeta_1, \ldots, \zeta_n) - \Phi_j(\zeta_1, \ldots, \zeta_n)}{\zeta_1} .$$

But in the power series expansion of $\Phi_i - \Phi_j$, only terms containing powers of ζ_1 appear; hence $F_i - F_j$ is holomorphic in $u_i \cap u_j$.

By hypothesis, there exists F, meromorphic in X, such that $g_i = F - F_i$ is holomorphic in u_i. We claim that $\Phi = fF$ is holomorphic in X and equal to ϕ on Y.

Clearly, Φ is holomorphic on $u_0 = X - Y$. Consider F on u_i, $i \neq 0$;

$$F = g_i + \frac{\Phi_i}{f}, \text{ and } g_i \text{ is holomorphic.}$$

Hence, $\Phi = fg_i + \Phi_i$ in u_i, and on $Y \cap u_i$, $\Phi = \phi$.

§ 2. Reformulation of the Cousin Problem

C.I'. (Cousin problem belonging to the covering U). Given X, an analytic manifold of dimension n, covering $U = \{u_i\}$ and holomorphic functions f_{ij} defined in $u_i \cap u_j$ satisfying

$$f_{ji} = - f_{ij} \qquad \text{(antisymmetry)}$$

$$f_{ij} + f_{jk} + f_{ki} = 0 \qquad \text{(compatibility)},$$

find holomorphic f_i, defined in u_i, such that $f_{ij} = f_i - f_j$.

Claim. C.I' implies C.I.

Proof. Assume C.I is given, and let $f_{ij} = F_i - F_j$, where the F_i are meromorphic functions defined in u_i. Then the f_{ij} are holomorphic and satisfy the symmetry and compatibility conditions. Let f_i be the solution functions of C.I', and define $F = F_i - f_i$ in u_i. Then F is globally well-defined, for in $u_i \cap u_j$

$$F_i - F_j = f_{ij} = f_i - f_j,$$

hence
$$F_i - f_i = F_j - f_j,$$

and F solves C.I.

Induced Cousin Problem. Let $U = \{u_j\}$, $j \in J$; $V = \{v_i\}$, $i \in I$ be two coverings of X, and assume that V is a refinement of U (i.e. every v_i is contained in some u_j), and let a Cousin problem belonging to the covering U be given. We induce a Cousin problem belonging to V as follows:

Let σ be an "affinity" function from I to J such that $v_i \subset u_{\sigma(i)}$. In $v_i \cap v_j$, assign $\hat{f}_{ij} = f_{\sigma(i)\sigma(j)}$. Clearly, the $\hat{f}_{ij}$ are antisymmetric and satisfy the compatibility condition.

We now reformulate the Cousin problem a second time:

C.I". Let a Cousin problem belonging to the covering U be given. Find a refinement V of U such that the induced problem is solvable with respect to V. (We shall show later that the choice of the affinity function is immaterial.)

Claim. C.I" implies C.I.

Let $U = \{u_i\}$ be a given covering, with associated meromorphic functions F_i. As before, define $f_{ij} = F_i - F_j$. Let $V = \{v_i\}$, with associated g_{ij} be the solvable induced Cousin problem, with affinity function σ:

$$v_i \subset u_{\sigma(i)} ,$$

$$g_{ij} = f_{\sigma(i)\sigma(j)} .$$

Let g_i be the solution functions. Define

$$F = F_{\sigma(i)} - g_i \quad \text{in } v_i .$$

Then F is globally well defined and solves C.I, for

$$F_{\sigma(i)} - F_{\sigma(j)} = f_{\sigma(i)\sigma(j)} = g_{ij} = g_i - g_j .$$

Furthermore, $F - F_i$ is holomorphic in u_i, for, let $x \in u_i$. Then there exists a v_j containing x, and consider:

$$\begin{aligned} F - F_i &= F - F_{\sigma(j)} + F_{\sigma(j)} - F_i \\ &= -g_j + f_{\sigma(j)i} , \end{aligned}$$

defined and holomorphic in $v_j \cap u_{\sigma(j)} \cap u_i = v_j \cap u_i$, and $x \in v_j \cap u_i$.

Remark. We may now consider the Cousin problem for locally finite coverings only (i.e., for coverings such that every point of X has a neighborhood which intersects only a finite subcollection of the cover), by virtue of the following observations:

i) (Paracompactness). In any manifold every covering has a locally finite refinement by open sets each relatively compact in some open set of the origin covering and some coordinate patch (see de Rham, Variétés Différentiables).

ii) C.I" implies C.I.

§3. Reduction of the Cousin Problem to non-homogeneous Cauchy-Riemann equations

A. Intermediate Problem. Given a complex manifold X, locally finite open covering $U = \{u_i\}$, and holomorphic functions f_{ij} defined in $u_i \cap u_j$, antisymmetric and satisfying the compatibility condition, find functions g_i, defined in u_i, such that $f_{ij} = g_i - g_j$ where the $g_i \in C^\infty$.

Proposition 1. The Intermediate Problem is always solvable. The proof of this proposition will be presented subsequently (p. 64).

Let the functions a_ν, $\nu = 1,\ldots,n$ be defined on X as follows:

$$a_\nu = \frac{\partial g_j}{\partial \bar{z}_\nu} \text{ in } u_j \,; \quad \nu = 1,\ldots,n.$$

We claim the a_ν are globally well defined, i.e.,

$$\frac{\partial g_j}{\partial \bar{z}_\nu} - \frac{\partial g_i}{\partial \bar{z}_\nu} = 0 \text{ in } u_i \cap u_j.$$

But this is clear, for $g_j - g_i = f_{ji}$ and f_{ji} is holomorphic there. Furthermore, the a_ν satisfy the following compatibility condition:

$$\frac{\partial a_\nu}{\partial \bar{z}_\mu} = \frac{\partial a_\mu}{\partial \bar{z}_\nu}$$

and this is clear. Now we can state the final form of C.I:

Final Problem. Find a function $A \varepsilon C^{\infty}$, defined on X, such that $\partial A/\partial \bar{z}_\nu = a_\nu$, $\nu = 1,\ldots,n$; where the a_ν are as above.

More precisely, given a complex manifold X of dimension n and C^{∞} functions a_ν, $\nu = 1,\ldots,n$; defined on X and satisfying the compatibility conditions

$$\frac{\partial a_\nu}{\partial \bar{z}_\mu} = \frac{\partial a_\mu}{\partial \bar{z}_\nu} \; ; \qquad \mu,\nu = 1,\ldots,n$$

find a function A, defined on X, such that $\partial A/\partial \bar{z}_\nu = a_\nu$, $\nu = 1,\ldots,n$.

Note. These differential equations are known as non-homogeneous Cauchy-Riemann equations.

Proposition 2. The final problem implies C.I.

Proof. Assume there exists a function A, defined as above. Set $f_i = g_i - A$, defined in u_i, where the $g_i \varepsilon C^{\infty}$ are given by Proposition 1. Note that

$$\frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial g_i}{\partial \bar{z}_\nu} - \frac{\partial A}{\partial \bar{z}_\nu} = 0 \; ;$$

i.e. the f_i are holomorphic. But $f_i - f_j = g_i - g_j = f_{ij}$.

Definition 31. Let X be a differentiable manifold, $U = \{u_i\}$ a locally finite open covering. Then a partition of unity subordinated to the covering U is a system of functions ω_j, defined on X, positive and C^{∞}, such that

$$\omega_j \equiv 0 \quad \text{on} \quad X - u_j$$

$$\sum \omega_j = 1 \quad \text{at each point of } X .$$

Proposition 3. Given any manifold X and locally finite open covering U, there exists a partition of unity subordinated to the covering U.

Proof. Note that, if V is a refinement of U, and if there exists a partition w_j for V, we may define a partition ω_j of U as follows: Let σ be an affinity function, as before $(v_i \subset u_{\sigma(i)})$. Now set

$$\omega_{\sigma(i)} = \begin{cases} w_i & \text{on } v_i \\ 0 & \text{otherwise} \end{cases} .$$

For those u_j not as yet included, define $\omega_j = 0$.

Let U be given. By Remark i), p. 63, U has a locally finite refinement $V = \{v_i\}$ such that $v_i \subset\subset$ some coordinate patch P_i. Since X is paracompact and Hausdorff it is normal, so that there is a locally finite open covering $V' = \{v_i'\}$ such that $v_i' \subset\subset v_i$. For each i, let f_i be a diffeomorphism of P_i into $\mathbb{R}^n$. Let $s_i' = f(v_i')$ and $s_i = f(v_i)$; then $s_i'^{\text{open}} \subset\subset s_i^{\text{open}}$. In $\mathbb{R}^n$ there are C^∞ functions $\psi_i : \mathbb{R}^n \to [0,1]$ satisfying $\psi_i \equiv 1$ on $\mathrm{cl}\, s_i'$ and $\equiv 0$ outside s_i. Let $\psi_i \cdot f = \theta_i$, and set $\omega_i = \theta_i / \sum_i \theta_i$.

B. <u>Proof of Proposition 1.</u>

Let ω_j be a fixed partition of unity subordinate to $U = \{u_i\}$. Set $g_j = \sum_i \omega_i f_{ij}$ in u_j, where this sum is understood as follows: for $x \in u_j$, $\omega_i(x) = 0$ unless $x \in u_i \cap u_j$. When $\omega_i(x) = 0$, set $\omega_i f_{ij} = 0$. When $\omega_i(x) \neq 0$, f_{ij} is defined. Note that $\omega_i(x) = 0$ for all but a finite number of indices i.

The g_j solve the intermediate problem, for

$$\begin{aligned} g_i - g_j &= \sum_k \omega_k f_{ik} - \sum_k \omega_k f_{jk} \\ &= \sum_k \omega_k (f_{ik} + f_{kj}) \\ &= \sum_k \omega_k (f_{ij}) = f_{ij} . \end{aligned}$$

Hence, we have reduced the Additive Cousin problem to an existence theorem for the nonhomogeneous Cauchy-Riemann equations. We shall exhibit a solution for a polydisc shortly.

<u>Example</u>. Let X be a simply connected differentiable manifold, $U = \{u_i\}$ a locally finite open covering. We pose a "Cousin Problem" as follows:

To each $u_i \cap u_j$ let there be assigned a <u>complex</u>

<u>number</u> f_{ij} such that $f_{ij} = - f_{ji}$ and $f_{ij} + f_{jk} + f_{ki} = 0$. Then, find constants f_i such that $f_{ij} = f_i - f_j$.

Let g_i be the solutions of the intermediate problem; $f_{ij} = g_i - g_j$; where g_i is defined in u_i and $g_i \in C^\infty$. Define:

$$a_\nu = \frac{\partial g_i}{\partial x_\nu}, \text{ in } u_i; \qquad \nu = 1,\ldots,n.$$

Then the a_ν are defined globally, as before, and $\partial a_\nu/\partial x_\mu = \partial a_\mu/\partial x_\nu$. The final problem becomes: Find a function A such that

$$\frac{\partial A}{\partial x_\nu} = a_\nu , \qquad \nu = 1,\ldots,n.$$

Such a function exists, by Stokes' theorem. Set

$$f_i = g_i - A .$$

The f_i are constants, as $\partial f_i/\partial x_\nu = 0$, $\nu = 1,\ldots,n$.

<u>Exercise</u>. Prove the converse of the above example, in the following form:

Let X be a domain in $\mathbb{R}^n$. Then C.I solvable implies that every curl-free vector field is a gradient.

<u>Theorem 19</u>. Let $D = \{(z_1,\ldots,z_n) \mid |z_j| < R_j \leq \infty\}$. If $a_j(z_1,\ldots,z_n)$ are defined and C^∞ in D, $j = 1,\ldots,n$, and satisfy $\partial a_j/\partial \bar{z}_k = \partial a_k/\partial \bar{z}_j$ then there is a C^∞ function ϕ in D such that $\partial\phi/\partial\bar{z}_j = a_j$ for $j = 1,\ldots,n$.

<u>Proof</u>. 1. Let $D_0 = \{(z_1,\ldots,z_n) \mid |z_j| < r_j < \infty\} \subset\subset D$. Then the a_j are defined and C^∞ in a neighborhood of $\bar{D}_0$. We claim that there exists a ϕ defined and C^∞ in perhaps a smaller neighborhood of $\bar{D}_0$ satisfying $\partial\phi/\partial\bar{z}_j = a_j$, $j = 1,\ldots,n$. The proof is by induction on k, where we assume $a_j \equiv 0$ for $j > k$. For $k = 0$, the problem is reduced to solving the system of homogeneous Cauchy-Riemann equations, $\partial\phi/\partial\bar{z}_j = 0$, of which $\phi = 0$ is a solution. Assume that the problem can be solved for $k = \ell-1$, and consider the case $k = \ell$. Choose $\varepsilon > 0$ sufficiently small, and for $\zeta = \xi + i\eta$ and $(z_1,\ldots,z_n)$ in an ε_1 neighborhood of $\bar{D}_0$, $N(\varepsilon_1,\bar{D}_0)$, $\varepsilon_1 < \varepsilon$, define

$$\omega(z_1,\dots,z_n) = -\frac{1}{\pi} \iint\limits_{|\zeta|<r_\ell+\varepsilon} \frac{a_\ell(z_1,\dots,z_{\ell-1},\zeta,z_{\ell+1},\dots,z_n)}{\zeta-z_\ell}\, d\xi\, d\eta$$

$\omega(z_1,\dots,z_n)$ is then defined and C^∞ in all variables in $N(\varepsilon_1,\bar{D}_0)$ and satisfies $\partial\omega/\partial\bar{z}_\ell = a_\ell$ and $\partial\omega/\partial\bar{z}_j = 0$ for $j > \ell$, by the compatibility conditions $\partial a_\ell/\partial\bar{z}_j = \partial a_j/\partial\bar{z}_\ell$. The system

(1) $\dfrac{\partial\phi}{\partial\bar{z}_j} = a_j,\quad j = 1,\dots,\ell;\qquad \dfrac{\partial\phi}{\partial\bar{z}_j} = 0,\quad j = \ell+1,\dots,n;$

is then equivalent to the system

(2) $\dfrac{\partial(\phi-\omega)}{\partial\bar{z}_j} = a_j - \dfrac{\partial\omega}{\partial\bar{z}_j},\quad j = 1,\dots,\ell;\qquad \dfrac{\partial(\phi-\omega)}{\partial\bar{z}_j} = 0,\quad j = \ell+1,\dots,n;$

but since $\partial\omega/\partial\bar{z}_\ell = a_\ell$, (2) is actually

(2') $\dfrac{\partial(\phi-\omega)}{\partial\bar{z}_j} = a_j - \dfrac{\partial\omega}{\partial\bar{z}_j},\quad j=1,\dots,\ell-1;\qquad \dfrac{\partial(\phi-\omega)}{\partial\bar{z}_j} = 0,\ j=\ell,\dots,n.$

Now $(a_j - \partial\omega/\partial\bar{z}_j) \in C^\infty$ in $N(\varepsilon_1,\bar{D}_0)$ and satisfies the compatibility conditions since $\partial/\partial\bar{z}_k(a_j - \partial\omega/\partial\bar{z}_j) = \partial a_j/\partial\bar{z}_k - \partial/\partial\bar{z}_k(\partial\omega/\partial\bar{z}_j) = \partial a_k/\partial\bar{z}_j - \partial/\partial\bar{z}_j(\partial\omega/\partial\bar{z}_k) = \partial/\partial\bar{z}_j(a_k - \partial\omega/\partial\bar{z}_k)$. Hence, by our induction hypothesis, (2') has a C^∞ solution and therefore (1) has a C^∞ solution in a neighborhood of $\bar{D}_0$.

2. Construct open polydiscs $D_j \subset\subset D_{j+1}$ whose union is D. By 1., there exist functions $\phi_1,\phi_2,\dots$ such that $\phi_j \in C^\infty$ on $\bar{D}_j$ and $\partial\phi_j/\partial\bar{z}_k = a_k$ for $k = 1,\dots,n$ and $(z_1,\dots,z_n) \in \bar{D}_j$. Choose $\varepsilon_j > 0$ such that $\sum \varepsilon_j < \infty$. If the ϕ_j satisfied $|\phi_{j+1}-\phi_j| < \varepsilon_j$ in D_j then $\lim_{j\to\infty} \phi_j = \phi$ would exist uniformly on compact subsets of D, since $|\phi_{j+p}-\phi_j| < \varepsilon_j + \varepsilon_{j+1} + \dots + \varepsilon_{j+p} < \sum_{k=j}^{\infty} \varepsilon_k$ in D_j and for j sufficiently large $\sum_j^\infty \varepsilon_k$ is the tail end of a convergent series. Moreover, for k fixed, $(\phi_j-\phi_k)$ is holomorphic on D_k for $j > k$ since $\partial/\partial\bar{z}_i(\phi_j-\phi_k) = 0$ on D_k, $i = 1,\dots,n$. Therefore since the sequence $\{(\phi_j-\phi_k)\}$,

$j > k$ would converge uniformly to $(\phi-\phi_k)$, all derivatives would also converge, so that ϕ would be C^∞ and would satisfy the differential equations $\partial\phi/\partial\bar{z}_k = a_k$ in D. So the next step is to construction new ϕ_j's, $\hat{\phi}_j$, which are also solutions of the differential equations and satisfy $|\hat{\phi}_{j+1}-\hat{\phi}_j| < \varepsilon_j$ on D_j. Let $\hat{\phi}_1 = \phi_1$. Let $\hat{\phi}_2 = \phi_2 - h_1$, where h_1 is a polynomial. h_1 must satisfy $|(\phi_2-\phi_1)-h_1|<\varepsilon_1$ on D_1. But on $\bar{D}_1$, $\phi_2-\phi_1$ is holomorphic and hence has a power series representation which can be approximated as closely as desired by a polynomial. Let $\hat{\phi}_3 = \phi_3-h_2$, etc. Since the polynomials are holomorphic, $\partial\hat{\phi}_j/\partial\bar{z}_k = \partial\phi_j/\partial\bar{z}_k = a_k$, $k = 1,\ldots,n$, on $\bar{D}_j$, and $|\hat{\phi}_{j+1}-\hat{\phi}_j| < \varepsilon_j$ on D_j by construction. Hence $\lim\limits_{j\to\infty} \hat{\phi}_j = \hat{\phi}$ is a C^∞ solution of $\partial\phi/\partial\bar{z}_k = a_k$ in D.

<u>Note</u>. If X and Y are homeomorphic manifolds and the Cousin Problem is solvable in X then it is solvable in Y. Thus by the previous theorem, the Cousin Problem is solvable in any domain which is the product of simply connected domains in $\mathbb{C}$.

Chapter 6. Cohomology

§1. Cohomology of a complex manifold with holomorphic functions as coefficients

Let X be a complex manifold and $U = \{u_i\}$, $i \in I$, be a fixed open covering of X. (We always assume that $u_i \neq u_j$ for $i \neq j$.)

A. Definition 32. An r-cochain f on U is a rule which assigns to every ordered intersection of $(r+1)$ sets, $u_{i_0} \cap \dots \cap u_{i_r}$, a holomorphic function $f_{i_0 \dots i_r}(z)$ defined in this intersection such that

1. (a) when the i_j are distinct and $\bigcap u_{i_j} \neq \phi$, $f_{i_0 \dots i_r}(z)$ is a function holomorphic in $\bigcap u_{i_j}$.

   (b) when the i_j are either nondistinct or $\bigcap u_{i_j} = \phi$, $f_{i_0 \dots i_r}(z) \equiv 0$.

2. f (odd permutation of $(i_0 \dots i_r)) = - f(i_0 \dots i_r)$

   f (even permutation of $(i_0 \dots i_r)) = f(i_0 \dots i_r)$.

Examples.

1. A 0-cochain is a rule f which assigns to every $u_i \in U$ a holomorphic function $f_i(z)$ defined in u_i.

2. A 1-cochain is a rule f which assigns to every ordered intersection $u_i \cap u_j$, a holomorphic function $f_{ij}(z)$ defined in $u_i \cap u_j$ such that $f_{ij}(z) \equiv 0$ if $i = j$ or $u_i \cap u_j = \phi$, $f_{ij} = - f_{ji}$.

Cochains of the same dimension form an Abelian group under addition, $C^r = C^r(X,U,\mathcal{O})$, where $\mathcal{O}$ refers to holomorphic functions. This group is also a vector space over $\mathbb{C}$ and a module over holomorphic functions on X.

Definition 33. The coboundary operator, δ, is a linear mapping of C^r into C^{r+1} (and therefore a homomorphism of the group C^r) given by

$$(\delta f)(i_0 \dots i_{r+1}) = \sum_{j=0}^{r+1} (-1)^j f(i_0 \dots \hat{i}_j \dots i_{r+1}) \quad \text{for } f \in C^r,$$

where $\hat{i}_j$ denotes the deletion of i_j.

δf is called an (r+1)-coboundary. There are no 0-dimensional coboundaries. The coboundaries form an Abelian group under addition, $B^r = B^r(X,U,\mathcal{O})$. This group is also a vector space and a module, as above.

Definition 34. A cochain is called a cocycle when its coboundary is 0, i.e. $\delta f = 0$.

Since δ is a linear map, the sum and difference of two cocycles is a cocycle, and the cocycles form an Abelian group $Z^r = Z^r(X,U,\mathcal{O})$, a vector space, and a module.

Examples.

1. $Z^0 = \{$holomorphic functions on $X\}$ For $0 = (\delta f)(ij) = f(j) - f(i)$ implies that $f_i(z) = f_j(z)$ on $u_i \cap u_j$, and hence that the f_k are restrictions of holomorphic functions on X.

2. $Z^1 = \{$Cousin data$\}$. For $(\delta f)(ijk) = f(jk) - f(ik) + f(ij) = 0$ implies, since $f(ij) = -f(ji)$, that $f(ij) + f(jk) + f(ki) = 0$. Therefore we have $f_{ij}(z)$ defined in $u_i \cap u_j$ satisfying $f_{ij}(z) + f_{jk}(z) + f_{ki}(z) = 0$ and $f_{ij}(z) = -f_{ji}(z)$.

Corollary. $\delta^2 = 0$.

Proof. A typical term of $(\delta\delta f)(i_0 \ldots i_{r+1})$ is $\alpha = f(i_0 \ldots \hat{i}_j \ldots \hat{i}_k \ldots i_{r+1})$. We must show that this term appears with zero coefficient. If we first delete k and then j, the coefficient of α is $(-1)^k(-1)^j$. But this term is also obtained by first deleting j and then k, in which case the coefficient of α is $(-1)^j(-1)^{k-1}$ (since $j < k$). Hence the coefficient of α in $\delta\delta f$ is $(-1)^k(-1)^j + (-1)^j(-1)^{k-1} = 0$.

By the corollary, every coboundary is a cocycle. Hence $B^r \subset Z^r \subset C^r$. Z^r/B^r is called the r th cohomology group $= H^r(X,U,\mathcal{O})$. H^r is also a vector space over $\mathbb{C}$ and a module over holomorphic functions on X.

$H^0 = Z^0 = \{$holomorphic functions on $X\}$.

If $H^1 = 0$, then every cocycle is a coboundary which means that for every $f_{ij}(z)$ holomorphic in $u_i \cap u_j$,

$f_{ij}(z) = - f_{ji}(z)$ and $f_{ij}(z) + f_{jk}(z) + f_{ki}(z) = 0$, there exist functions $f_i(z)$ and $f_j(z)$, holomorphic in u_i and u_j respectively, such that $f_{ij}(z) = f_j(z) - f_i(z)$. In other words, $H^1 = 0$ implies that every Cousin Problem is solvable (for the covering U).

B. Let $V = \{v_j\}$, $j \varepsilon J$, be a refinement of the covering U. Then every v_j is contained in some $u_i \varepsilon U$, $i \varepsilon I$. Let $\sigma(j) = \bar{j}$ be an affinity function which assigns to each v_j one u_i which we call $u_{\bar{j}}$ such that $v_j \subset u_{\bar{j}}$.

We assign to every r-cochain $\bar{f}$ in U an r-cochain f in V as follows: (Call this mapping σ^*).

Corresponding to every non-empty ordered intersection of (r+1) sets, $\bigcap_{i=0}^{r} v_{j_i}$, there is an ordered intersection of sets, $\bigcap_{i=0}^{r} u_{\bar{j}_i}$, and hence an r-cochain $\bar{f}$, defined on $\bigcap u_{\bar{j}_i}$. Assign to $\bar{f}$ its restriction on $\bigcap v_{j_i}$, and call it f. Then the holomorphic function which f assigns to $\bigcap v_{j_i}$ is the holomorphic function which $\bar{f}$ assigns to $\bigcap u_{\bar{j}_i}$ restricted to $\bigcap v_{j_i}$.

$$\sigma^* : C^r(X,U,\mathcal{O}) \to C^r(X,V,\mathcal{O}) .$$

Properties.

1. σ^* is a homomorphism of the group $C^r(X,U,\mathcal{O})$.

2. $\delta\sigma^* = \sigma^*\delta$. For if $\bar{f} \varepsilon C^r(X,U,\mathcal{O})$ then

$$(\delta\sigma^*\bar{f})(j_0\ldots j_{r+1}) = (\delta f)(j_0\ldots j_{r+1}) = \sum_{k=0}^{r+1} (-1)^k f(j_0\ldots\hat{j}_k\ldots j_{r+1})$$

and

$$(\sigma^*\delta\bar{f})\ (j_0\ldots j_{r+1}) = \sigma^*\Big(\sum_{k=0}^{r+1} (-1)^k \bar{f}(j_0\ldots\hat{j}_k\ldots j_{r+1})\Big)$$

$$= \sum_{k=0}^{r+1} (-1)^k \sigma^*\bar{f}(j_0\ldots\hat{j}_k\ldots j_{r+1}) \quad \text{(by 1.)}$$

$$= \sum_{k=0}^{r+1} (-1)^k f(j_0\ldots\hat{j}_k\ldots j_{r+1}) .$$

3. (a) $\sigma^* : Z^r(X,U,\mathcal{O}) \to Z^r(X,V,\mathcal{O})$. For if $\bar{f} \varepsilon Z^r(X,U,\mathcal{O})$ then $\delta\bar{f} = 0$ implies that $0 = \sigma^*\delta\bar{f} = \delta\sigma^*\bar{f} = \delta f$.

(b) $\sigma^* : B^r(X,U,\mathcal{O}) \to B^r(X,V,\mathcal{O})$. For if $\bar{f} \in B^r(X,U,\mathcal{O})$ then $\bar{f} = \delta\bar{F}$ implies that $\sigma^*\bar{f} = \sigma^*\delta\bar{F} = \delta F$.

<u>Note</u>. σ^* depends on the choice of the affinity function σ.

<u>Lemma</u>. If σ and τ are two affinity functions and σ^* and τ^* are their corresponding mappings of $C^r(X,U,\mathcal{O}) \to C^r(X,V,\mathcal{O})$, respectively, then there exists a linear mapping, $\theta : C^r(X,U,\mathcal{O}) \to C^{r-1}(X,V,\mathcal{O})$, such that for $\bar{f} \in C^r(X,U,\mathcal{O})$, $\tau^*\bar{f} - \sigma^*\bar{f} = \theta\delta\bar{f} + \delta\theta\bar{f}$.

<u>Proof</u>. Define θ as follows:

$$(\theta f)(i_0 \ldots i_{r-1}) = \sum_{k=0}^{r-1} (-1)^k \bar{f}(\sigma(i_0)\ldots\sigma(i_{k-1})\sigma(i_k)\tau(i_k)\ldots\tau(i_{r-1}))$$

Now

$$\delta(\theta f)(i_0\ldots i_r) = \sum_{k=0}^{r} (-1)^k (\theta\bar{f})(i_0\ldots\hat{i}_k\ldots i_r)$$

$$= \sum_{k=0}^{r} (-1)^k \Big\{ \sum_{\ell=0}^{k-1} (-1)^\ell \bar{f}(\sigma(i_0)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\hat{\tau}(i_k)\ldots\tau(i_r))$$

$$+ \sum_{\ell=k}^{r-1} (-1)^\ell \bar{f}(\sigma(i_0)\ldots\hat{\sigma}(i_k)\ldots\sigma(i_{\ell+1})\tau(i_{\ell+1})\ldots\tau(i_r))\Big\}$$

$$= \sum_{k=0}^{r}\sum_{\ell=0}^{k-1} (-1)^{k+\ell} \bar{f}(\sigma(i_0)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\hat{\tau}(i_k)\ldots\tau(i_r))$$

$$- \sum_{k=0}^{r}\sum_{\ell'=k+1}^{r} (-1)^{k+\ell'} \bar{f}(\sigma(i_0)\ldots\hat{\sigma}(i_k)\ldots\sigma(i_{\ell'})\tau(i_{\ell'})\ldots\tau(i_r))$$

and

$$\theta(\delta f)(i_0\ldots i_r) = \sum_{\ell=0}^{r} (-1)^\ell \delta\bar{f}(\sigma(i_0)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\tau(i_r))$$

$$= \sum_{\ell=0}^{r} (-1)^\ell \Big\{ \sum_{k=0}^{\ell} (-1)^k \bar{f}(\sigma(i_0)\ldots\hat{\sigma}(i_k)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\tau(i_r))$$

$$+ \sum_{k=\ell+1}^{r+1} (-1)^k \bar{f}(\sigma(i_0)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\hat{\tau}(i_{k-1})\ldots\tau(i_r))\Big\}$$

$$= \sum_{\ell=0}^{r}\sum_{k=0}^{\ell} (-1)^{k+\ell} \bar{f}(\sigma(i_0)\ldots\hat{\sigma}(i_k)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\tau(i_r))$$

$$- \sum_{\ell=0}^{r}\sum_{k'=\ell}^{r} (-1)^{k'+\ell} \bar{f}(\sigma(i_0)\ldots\sigma(i_\ell)\tau(i_\ell)\ldots\hat{\tau}(i_{k'})\ldots\tau(i_r))$$

$$= \sum_{\ell=0}^{r} \sum_{k=0}^{\ell-1} (-1)^{k+\ell} \bar{f}(\sigma(i_0)\dots\hat{\sigma}(i_k)\dots\sigma(i_\ell)\tau(i_\ell)\dots\tau(i_r))$$

$$- \sum_{\ell=0}^{r} \sum_{k'=\ell+1}^{r} (-1)^{k'+\ell} \bar{f}(\sigma(i_0)\dots\sigma(i_\ell)\tau(i_\ell)\dots\hat{\tau}(i_{k'})\dots\tau(i_r)) + A$$

where

$$\begin{aligned} A &= \sum_{\ell=0}^{r} \bar{f}(\sigma(i_0)\dots\sigma(i_{\ell-1})\tau(i_\ell)\dots\tau(i_r)) \\ &\quad - \sum_{\ell=0}^{r} \bar{f}(\sigma(i_0)\dots\sigma(i_\ell)\tau(i_{\ell+1})\dots\tau(i_r)) \\ &= \bar{f}(\tau(i_0)\dots\tau(i_r)) - \bar{f}(\sigma(i_0)\dots\sigma(i_r)) \\ &= \tau^* \bar{f}(i_0\dots i_r) - \sigma^* \bar{f}(i_0\dots i_r) \quad , \end{aligned}$$

and it is clear that the remaining terms cancel, when one writes the sums $\sum_{k=0}^{r} \sum_{\ell=0}^{k-1}$, for example, as $\sum_{k,\ell=0;\ \ell<k}^{r}$.

Hence, $\tau^* - \sigma^* = \theta\delta + \delta\theta$, as claimed.

Corollary. If $\delta\bar{f} = 0$, $\bar{f} \in C^r(X,U,\mathcal{O})$, then $\tau^*\bar{f} - \sigma^*\bar{f} = \delta\theta\bar{f}$. Hence if $\bar{f}$ is a cocycle on U then $\tau^*\bar{f}$ and $\sigma^*\bar{f}$ are cohomologous cocycles on V.

Definition 35. If U^1 and U^2 are open coverings of X, and if $f^1 \in H^r(X,U^1,\mathcal{O})$, (i.e. f^1 is a cohomology class of r-cocycles) and $f^2 \in H^r(X,U^2,\mathcal{O})$ then f^1 and f^2 are equivalent if they induce the same cohomology class of r-cocycles on some common refinement of U^1 and U^2.

Let $[f^1]$ denote the equivalence class of f^1. f^1 is a representative of $[f^1]$. Define addition of equivalence classes as follows:

$$[f^1] + [f^2] = [g_1 + g_2]$$

where f^1 induces g_1 and f^2 induces g_2 in some common refinement of U^1 and U^2. It must be checked that this definition is independent of the choice of representatives chosen. (Exercise for the reader). Then these equivalence classes of cohomology classes of r-cocycles from an Abelian group under addition, $H^r(X,\mathcal{O})$, called the r th cohomology group of the manifold X. ($H^r(X,\mathcal{O})$ is independent of the choice of covering of X). $H^r(X,\mathcal{O}) = \varinjlim H^r(X,U,\mathcal{O})$. An element of $H^r(X,\mathcal{O})$ can be represented by an r-cocycle on a covering of X. $H^r(X,\mathcal{O}) = 0$ means that any r-cocycle on a covering U of X induces a coboundary on some refinement of U. Note that $f \to [f]$ is a homomorphism of $H^r(X,U,\mathcal{O})$ into $H^r(X,\mathcal{O})$.

$$H^0(X,\mathcal{O}) = \{\text{holomorphic functions on } X\}.$$

If $H^1(X,\mathcal{O}) = 0$ then every Cousin Problem is solvable (cf. 2nd reformulation of Cousin Problem).

Holomorphically equivalent manifolds have the same cohomology groups.

§2. Applications

Theorem 20. Let X be a complex manifold and Y a globally presented, regularly imbedded hypersurface, $Y = \{(z_1,\ldots,z_n) \mid \phi(z_1,\ldots,z_n) = 0\}$ where ϕ is a holomorphic function on X. If for some $r > 0$, $H^{r+1}(X,\mathcal{O}) = H^r(X,\mathcal{O}) = 0$, then $H^r(Y,\mathcal{O}) = 0$.

Proof. We say that a covering $U' = \{u_i'\}$ of X is sufficiently fine if for every $(\bigcap_{\text{finite}} u_i') \cap Y$ a function holomorphic in this intersection can be continued to the $\bigcap_{\text{finite}} u_i'$.

Lemma. There exists a covering U_0 of X such that every refinement of U_0 is sufficiently fine.

Proof. Let $y \in Y$, then in some neighborhood of y, $N(y)$, there are local coordinates $\zeta_1,\ldots,\zeta_n$ such that Y is given by $\zeta_n = 0$. Any function holomorphic in $N(y) \cap Y$ is a function of $\zeta_1,\ldots,\zeta_{n-1}$ and hence is holomorphic as a function of $\zeta_1,\ldots,\zeta_n$, i.e. in $N(y)$. Clearly if N_1 is any open set contained in $N(y)$ any function holomorphic in $N_1 \cap Y$ can be holomorphically continued to N_1. Take this neighborhood system and add open sets not intersecting Y

to form an open covering of X. This covering has the desired property, i.e. is U_0.

From now on we consider only such coverings. Now take such a covering $U = \{u_i\}$ of X. We must show that every r-cocycle on Y induces an r-coboundary on some refinement of U. Let $\mathfrak{f}^y$ be an r-cocycle on Y, then $\mathfrak{f}^y$ assigns to every $(\bigcap_{j=0}^{r} u_{i_j}) \cap Y$ a holomorphic function $f_{i_0 \cdots i_r}(z)$ defined in this intersection. Since U is sufficiently fine, $f_{i_0 \cdots i_r}(z)$ can be continued to $\bigcap_{j=0}^{r} u_{i_j}$. Define $\mathfrak{f} \equiv 0$ in $\bigcap u_{i_j}$ when $(\bigcap u_{i_j}) \cap Y = \phi$. Therefore, there is an r-cochain $\mathfrak{f}^x$ on X such that $\mathfrak{f}^x = \mathfrak{f}^y$ on Y. Then $\delta\mathfrak{f}^x = 0$ on Y which means that $\delta\mathfrak{f}^x = \phi g$ where g is an (r+1)-cochain on X. But $0 = \delta\delta\mathfrak{f}^x = \phi\delta g$ and hence $\delta g = 0$; g is a cocycle on X. Since $H^{r+1}(X,\mathcal{O}) = 0$, g must induce an (r+1)-coboundary on some refinement U^1 of U. So consider everything above in this refinement U^1. Then $g = \delta h$ where h is an r-cochain on X. Therefore $\delta(\mathfrak{f}^x - \phi h) = \delta\mathfrak{f}^x - \phi\delta h = \delta\mathfrak{f}^x - \phi g = 0$, and hence $\mathfrak{f}^x - \phi h$ is an r-cocycle. Since $H^r(X,\mathcal{O}) = 0$, $\mathfrak{f}^x - \phi h$ induces an r-coboundary on some refinement U^2 of U^1. Considering the above in this refinement U^2, $\mathfrak{f}^x - \phi h = \delta F$ where F is an (r-1)-cochain on X. Hence on Y, $\mathfrak{f}^x - \phi h = \mathfrak{f}^x = \mathfrak{f}^y = \delta F$, and since F is an (r-1)-cochain on X, it is an (r-1)-cochain on Y.

Note. The following theorem gives sufficient conditions for a domain to be a domain of holomorphy. Later on we will prove that these conditions are also necessary.

Theorem 21. Let $D^{\text{open}} \subset \mathbb{C}^n$, $n \geq 2$. If $H^r(D,\mathcal{O}) = 0$ for $1 \leq r \leq n-1$, then D is a region of holomorphy.

Proof. Use induction on n. The case $n = 2$ follows from Theorem 18. Assume that every open set in $\mathbb{C}^{n-1}$ for which $H^r = 0$, $1 \leq r \leq n-2$, is a region of holomorphy. Rather than working with a component of D, assume that D is a domain. Then let $D^{\text{dom}} \subset \mathbb{C}^n$ and let $b \in$ boundary of D. Pass a hyperplane P through b and an interior point of D and set $P \cap D = Y$. By Theorem 20, all the cohomology groups of Y from H^1 to H^{n-2} are 0. By our induction hypothesis then, Y is a domain of holomorphy. Hence there is a function g holomorphic on Y and singular at b. Since $H^1(D,\mathcal{O}) = 0$, the Cousin Problem is solvable in D, and hence g can be continued holomorphically to D, and the extended function will be singular at b.

§3. Other Cohomologies

A. 1. Had we defined an r-cochain to assign a C^∞ function, instead of a holomorphic function, to intersections, then we would have gotten $H^r(X,C^\infty)$, where X could be a real manifold.

2. Had we defined an r-cochain to assign a constant function, i.e. complex number, to intersections, then we would have gotten $H^r(X,\mathbb{C})$. Here X is any topological space.

3. Had we defined an r-cochain to assign an integer to intersections, we would have gotten $H^r(X,\mathbb{Z})$, the Integral Cohomology group of the topological space X.

4. In fact, given any Abelian group Γ, we could have defined an r-cochain to assign an element of Γ, in which case we would get $H^r(X,\Gamma)$.

B. Definition 36. Let S be a topological space, X a Hausdorff space, and p a mapping of S onto X (called the projection mapping). Denote $p^{-1}(x)$ by S_x, called a stalk, and note that $S_x \subset S$, $\bigcup_{x\varepsilon X} S_x = S$. The triple (p,S,X) is called a sheaf of Abelian groups over X if

a) p is continuous, and for each $x \varepsilon X$ and each $s \varepsilon S_x$, p is a homeomorphism of a neighborhood of s in S onto a neighborhood of x.

b) Each stalk S_x is an Abelian group such that: $s \to -s$ is a continuous mapping of S into S; and $(s,t) \to s+t$, defined on the set R of pairs (s,t) such that s,t belong to the same stalk, is a continuous mapping of the subset R of $S \times S$ into S.

For simplicity, we call S the sheaf.

Definition 37. Let $U^{open} \subset X$. A section (or cross section) of S over U is a continuous map $\mathfrak{f}$ of U into S such that $p \cdot \mathfrak{f}$ is the identity mapping. Denote by $\Gamma(U)$ the additive group of all sections of S over U.

Examples.

1. Let X be arbitrary, and let $S_x = \mathbb{Z}$ for all $x \in X$. Let $S = \bigcup_x S_x$ have the discrete topology. Then a section of S over $U^{open} \subset X$ assigns some integer to U.

2. Let X be arbitrary, and let $S_x = G$, any Abelian group, for all $x \in X$. Let $S = \bigcup S_x$ have the discrete topology. A section of $U^{open} \subset X$ is an assignment of an element of G to U.

As in example 1, all stalks are isomorphic. Such sheaves are called constant sheaves.

3. Let X be a complex manifold, and $S_x = \mathcal{O}_x$, the set of germs of holomorphic functions.

Definition 38. Let $x \in X$, a complex manifold, and consider holomorphic functions at x, each defined in some neighborhood of x. We say that two such functions are equivalent if they coincide on some sufficiently small neighborhood of x. This is clearly an equivalence relation. The set of all holomorphic functions as above, modulo this equivalence relation, is called the set of germs of holomorphic functions at x; which we have denoted by $\mathcal{O}_x$. The set of all germs form a group over each point x.

Introduce a topology in the set of all germs $S = \bigcup S_x$ as follows: Take any element $\mathfrak{f} \in S$; then $\mathfrak{f} \in S_{x_0}$, and is an equivalence class of functions defined in a neighborhood of $x_0 \in X$. Take a representative $g \in S_{x_0}$; g is defined in N_{x_0}, a neighborhood of $x_0 \in X$. Then for each $y \in N_{x_0}$, assign that class in S_y containing the direct analytic continuation of g, say $\{g_y\}$. Then $\bigcup_{y \in N_{x_0}} \{g_y\}$ is, by definition, an open set; and these are to form a basis for the topology of S. Each section over U is a holomorphic function defined in U. This sheaf is called the <u>sheaf of germs of holomorphic functions.</u>

4. Let X be a differentiable manifold and $S_x = C_x^\infty$, where C_x^∞ denotes the set of germs of C^∞ functions at x, defined similarly to $\mathcal{O}_x$. Then $S = \bigcup C_x^\infty$ is made into a sheaf as in example 3 above.

With the aid of the concept of a sheaf, we may now define the cohomology groups in a more general setting.

Let X be a paracompact space with a sheaf S over it, and $U = \{u_i\}$, $i \in I$, an open covering of X. Define the cochains $C^r(X,U,S)$ on X associated with the covering U, with coefficients in S, as follows: $f \in C^r(X,U,S)$ assigns to each ordered intersection, $u_{i_0} \cap \ldots \cap u_{i_r}$, a section of S over this intersection so that f is antisymmetric in the indices. Note that we can add cochains, and talk of antisymmetry conditions, for we can add their values using the group structure of $\Gamma(u_0 \cap u_1 \cap \ldots \cap u_r)$.

Continuing the construction as before, we obtain $H^r(X,U,S)$, and then form the projective (direct) limit, $H^r(X,S)$.

<u>Theorem 22</u>. $H^r(X,C^\infty) = 0$ for all $r > 0$ and any differentiable manifold X. In fact $H^r(X,U,C^\infty)$, $r > 0$ is already trivial, for every locally finite covering U.

<u>Proof</u>. We define a homomorphism $\theta : C^r(X,U,C^\infty) \to C^{r-1}(X,U,C^\infty)$ $r > 0$, so that $f = \theta\delta f + \delta\theta f$, and this is sufficient.

Let $\{\omega_i\}$ be a partition of unity subordinate to U (see Definition 31 and Proposition 3, 3, Chapter V). Define

$$\theta f(i_0 \ldots i_{r-1}) = \sum \omega_i f(i i_0 \ldots i_{r-1}) ,$$

where this sum is understood as follows: $\omega_i f = 0$ if $f = 0$ or if $\omega_i = 0$. Note that the local finiteness of U insures that almost all terms of this sum vanish at any point of X. Now

$$(\theta\delta f + \delta\theta f)(i_0 \ldots i_r) = \sum_i \omega_i \; \delta f(i i_0 \ldots i_r) + \sum_{k=0}^{r} (-1)^k \theta f(i_0 \ldots \hat{i}_k \ldots i_r)$$

$$= \sum_i \omega_i \left\{ \sum_{k=0}^{r} (-1)^{k+1} f(i i_0 \ldots \hat{i}_k \ldots i_r) + f(i_0 \ldots i_r) \right\}$$

$$+ \sum_{k=0}^{r} (-1)^k \sum_i \omega_i \; f(i i_0 \ldots \hat{i}_k \ldots i_r)$$

$$= f(i_0 \ldots i_r) \sum_i \omega_i - \sum_i \sum_{k=0}^{r} (-1)^k \omega_i f(i_0 \ldots \hat{i}_k \ldots i_r)$$

$$+ \sum_i \sum_{k=0}^{r} (-1)^k \omega_i f(i i_0 \ldots \hat{i}_k \ldots i_r)$$

$$= f(i_0 \ldots i_r) \; .$$

Note that this proof hinges upon the fact that $C^r(X,U,C^\infty)$ is a module over globally defined C^∞ functions.

Chapter 7. Differential Forms

§1. Ring of differential forms in a domain

A. Definition 39. Let $D \subset \mathbb{R}^n$, D open. A differential form in D is a formula sum $\sum_{r=0}^{n} \sum_{i_1<i_2<\dots<i_r} a_{i_1\dots i_r} dx_{i_1} \wedge \dots \wedge dx_{i_r}$ (the symbol "$\wedge$" is read as "wedge"), where the $a_{i_1\dots i_r}$ are functions defined in D, possibly zero (in which case the term $dx_{i_1} \wedge \dots \wedge dx_{i_r}$ may be omitted from the sum). Identify $1 \cdot dx_{i_1} \wedge \dots \wedge dx_{i_r}$ and $dx_{i_1} \wedge \dots \wedge dx_{i_r}$.

Note that the collection of differential forms in D constitute a module (Abelian group written additively), closed under multiplication by functions in D, of dimension 2^n. The addition is the natural one. We denote this module by R_D.

We shall say that a monomial $a_{i_1\dots i_r} dx_{i_1} \wedge \dots \wedge dx_{i_r}$ is of degree r of r-dimensional if it is the sum of monomials of degree at most r, and pure r-dimensional if it is a sum of monomials of degree precisely r. Observe that any form in D may be written uniquely as a sum of pure r-dimensional forms, $0 \leq r \leq n$. We now introduce a multiplication "*" in R_D, giving R_D a ring structure, as follows:

Define $(a_{i_1\dots i_r} dx_{i_1} \wedge \dots \wedge dx_{i_r}) * (b_{j_1\dots j_k} dx_{j_1} \wedge \dots \wedge dx_{j_k})$

$$= \begin{cases} 0 & \text{if } i_K = j_L \text{ for some } K,L \\ \varepsilon a_{i_1\dots i_r} b_{j_1\dots j_k} dx_{q_1} \wedge \dots \wedge dx_{q_{k+r}} & \text{otherwise} \end{cases}$$

where each q_k is an i_K or a j_L; $q_k < q_{k+1}$ and ε is the sign of the permutation $\pi : (i_1 \dots i_r\ j_1 \dots j_k) \to (q_1 \dots q_{k+r})$. Define "*" on all R_D by postulating that the distributive law holds. The multiplication "*" is clearly associative; hence R_D is a ring. We now denote this multiplication by "$\wedge$", for obvious reasons. (R_D is also called the ring of exterior differential forms).

We note that R_D may also be defined as follows: R_D is a ring with operators $\mathbb{C}^D = \{\text{functions defined in } D\}$, generated by the symbols $dx_1,\ldots,dx_n$, where

$$1\cdot dx_i = dx_i$$
$$dx_i \wedge dx_j = -dx_j \wedge dx_i \quad .$$

We remark that, if α,β are pure dimensional

$$\alpha\wedge\beta = (-1)^{\deg\alpha\cdot\deg\beta}\ \beta\wedge\alpha \quad .$$

Remark. It is to be understood in the sequel that all functions are C^∞.

B. We now introduce a cohomology structure to R_D, called the d-cohomology, or de Rham cohomology.

Define the ring homomorphism $d: R_D \to R_D$ as follows:

On zero forms, i.e., on functions defined in D, set

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j \quad ;$$

and on monomials:

$$d(a\,dx_{j_1}\wedge\ldots\wedge dx_{j_r}) = (da)\wedge(dx_{j_1}\wedge\ldots\wedge dx_{j_r}) \quad .$$

Extend d to R_D linearly. Note that d is not an operator-homomorphism, i.e. $d(f\alpha) \neq f d\alpha$, where f is a function on D and $\alpha \in R_D$.

Example (1). Let $D \subset \mathbb{R}^3$, and observe that $d\alpha$ on a zero form acts like a "gradient," on a pure 1-form like "curl," and on a pure 2-form like "divergence."

Lemma 1. i) $d^2\alpha = 0$ for every $\alpha \in R_D$

ii) $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^r\alpha\wedge d\beta$, for every pure r-dimensional form α.

Proof of this lemma is left as an exercise. Hint: it suffices to consider only monomials α,β.

Definition 40. A form $\alpha \in R_D$ is said to be closed if $d\alpha = 0$.

A form $\alpha \in R_D$ is said to be <u>exact</u> if $\alpha = d\beta$ for some $\beta \in R_D$.

It is clear that the closed forms form a subgroup R_D^C of R_D; and that the exact forms are a subgroup R_D^E of R_D. Hence, we form the d-cohomology group R_D^C / R_D^E = closed forms / exact forms.

<u>Lemma 2</u>. The closed forms form a subring of R_D in which the exact forms are a two-sided ideal.

(i.e. i) closed $\wedge$ closed = closed

ii) closed $\wedge$ exact = exact

iii) exact $\wedge$ closed = exact)

<u>Proof</u>. We may assume that α, β are monomials.

i) If $d\alpha = d\beta = 0$, then
$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge d\beta = 0.$$

ii) If $\beta = d\gamma$, $d\alpha = 0$,
$$d(\alpha\wedge\gamma) = d\alpha\wedge\gamma + (-1)^{\deg\alpha}\alpha\wedge d\gamma = [(-1)^{\deg\alpha}\alpha]\wedge\beta,$$
$$\text{i.e.}\quad \alpha\wedge\beta = d((-1)^{\deg\alpha}\alpha\wedge\gamma),$$

and similarly for iii), $(\beta\wedge\alpha) = d(\gamma\wedge\alpha)$.

Hence, the de Rham group is a <u>ring</u>, (the de Rham cohomology ring).

C. Now assume $D \subset \mathbb{C}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the usual way, and observe that

$$\left.\begin{aligned} z_j &= x_j + iy_j \\ \bar{z}_j &= x_j - iy_j \end{aligned}\right\} \quad j = 1,\ldots,n$$

are functions of x_j and y_j; hence we may apply "d", obtaining

$$\left.\begin{aligned} dz_j &= dx_j + idy_j \\ d\bar{z}_j &= dx_j - idy_j \end{aligned}\right\} \quad j = 1,\ldots,n\ .$$

But the dx_j, dy_j generate the ring of differential forms on D; and the above equations are solvable for dx_j, dy_j in terms of dz_j and $d\bar{z}_j$. Hence the dz_j and $d\bar{z}_j$ also generate the ring of forms, so that for any form $\alpha \in R_D$:

$$\alpha = \sum_{r=0}^{2n} \sum_{\substack{p+q=r \\ i_1<\dots<i_p \\ \bar{j}_1<\dots<\bar{j}_q}} \alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

Now any element may be written uniquely as a sum of pure dimensional forms; and any pure dimensional form of degree r may be written uniquely as a sum of monomials of "bi-degree" $(p,r-p)$, $p = 0,\dots,r$ where

Definition 41. We say the monomial $\alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ is of type or bidegree (p,q); we denote $dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ by α_{pq}. Hence any form in R_D may be written uniquely as a sum of monomials of distinct bidegrees (p,q):

$$\alpha = \sum_{r=0}^{2n} \sum_{p+q=r} \alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} \alpha_{pq} \quad .$$

Hence, the α_{pq} form a basis for the module R_D. Let A^{pq} denote the subset of forms of bidegree (p,q) and note that $A^{pq} \wedge A^{p'q'} = A^{p+p',q+q'}$. We now introduce homomorphisms $\partial : A^{pq} \to A^{p+1,q}$ and $\bar{\partial} : A^{pq} \to A^{p,q+1}$ as follows:

$$\partial a = \sum_{j=1}^{n} \frac{\partial a}{\partial z_j} dz_j \quad ,$$

$$\bar{\partial} a = \sum_{j=1}^{n} \frac{\partial a}{\partial \bar{z}_j} d\bar{z}_j \quad ;$$

where a is a zero form, and extend ∂, $\bar{\partial}$ to R_D as before.

Now $d = \partial + \bar{\partial}$, as can easily be verified. Hence $d^2 = 0 = (\partial+\bar{\partial})^2 = \partial^2 + \bar{\partial}^2 + (\partial\bar{\partial}+\bar{\partial}\partial)$. Note that

$$(\partial+\bar{\partial})^2 A^{pq} = \partial^2 A^{pq} + \bar{\partial}^2 A^{pq} + (\partial\bar{\partial}+\bar{\partial}\partial) A^{pq}$$

$$= A^{p+2,q} + A^{p,q+2} + 2A^{p+1,q+1} \quad ;$$

so that $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

We now define ∂-closed, $\bar{\partial}$-closed, ∂-exact, and $\bar{\partial}$-exact forms, and form the associated cohomology groups:

$$\frac{\partial\text{-closed forms}}{\partial\text{-exact forms}}$$

and

$$\frac{\bar{\partial}\text{-closed forms}}{\bar{\partial}\text{-exact forms}} .$$

We may again verify that these groups are in fact cohomology <u>rings</u>.

Observe that, if we restrict the coefficients to be holomorphic functions in D, $\bar{\partial}$ becomes trivial and $d = \partial$. Hence, we can also form the cohomology ring:

$$\frac{\text{closed holomorphic forms}}{\text{exact holomorphic forms}}$$

where we define a holomorphic form as follows:

<u>Definition 42</u>. The form $\sum \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$ is said to be <u>holomorphic</u> if $q = 0$ and the $\alpha_{i_1 \ldots i_p}$ are holomorphic functions.

§2. <u>Differential forms on manifolds</u>

Let D and Δ be domains in R^n and R^m respectively, and let f be a diffeomorphism from D onto Δ. Denote a point of D by $x = (x_1, \ldots, x_n)$ and a point of Δ by $\xi = (\xi_1, \ldots, \xi_m)$. Then $f(x) = \xi$, or $\xi_j = f_j(x_1, \ldots, x_n)$, $j = 1, \ldots, m$. There is an induced mapping f^* associated with f which maps differential forms in Δ into differential forms in D; if $= \sum a_{j_1 \ldots j_r}(\xi)\, d\xi_{j_1} \wedge \ldots \wedge d\xi_{j_r}$, then $f^*\alpha = \sum a_{j_1 \ldots j_r}(f(x))\, df_{j_1} \wedge \ldots \wedge df_{j_r}$ and

1. f^* preserves degrees
2. $f^*(\alpha+\beta) = f^*\alpha + f^*\beta$
3. $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$
4. $df^*\alpha = f^* d\alpha$.

Let D, Δ be domains in $\mathbb{C}^n$, and let f be a holomorphic mapping from D to Δ. Let $(z_1, \ldots, z_n)$ denote a point of D

and $(\xi_1,\ldots,\xi_n)$ denote a point of Δ. The induced mapping f^* (as above) has the properties

1. f^* preserves bidegrees
2. $f^*(\alpha+\beta) = f^*\alpha + f^*\beta$
3. $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$
4. $\bar{\partial} f^*\alpha = f^*\bar{\partial}\alpha$ and $\partial f^*\alpha = f^*\partial\alpha$.

For the case of holomorphic forms, since f is holomorphic and a holomorphic function of a holomorphic function is holomorphic, f^* takes holomorphic forms into holomorphic forms.

Let X be a differentiable manifold of dimension n. We define a differential form on X as follows:

Definition 43. A differential form on X is a rule which defines a differential form in every coordinate patch. Each coordinate patch P_α is diffeomorphic to a domain D in $\mathbb{R}^n$ (by definition). A differential form on X associates with every coordinate patch P_α a differential form α in D_α such that if $P_{\alpha_1} \cap P_{\alpha_2} \neq \phi$ then the images of this intersection in D_{α_1} and D_{α_2} are diffeomorphic and the induced map on forms takes α_1^2 into α_2.

§3. Poincaré Lemmas

The Poincaré Lemmas state that in sufficiently "nice" domains any closed form not containing a 0-form is exact. More precisely,

Theorem 23.

(Ta) Let $D = \{(x_1,\ldots,x_n) \mid |x_i| < R_i \leq \infty\} \subset \mathbb{R}^n$, and let α be a pure r-dimensional form, $r > 0$, in D. If $d\alpha = 0$ then $\alpha = d\beta$ for some β .

(Tb) Let $D = \{(z_1,\ldots,z_n) \mid |z_j| < R_j \leq \infty\} \subset \mathbb{C}^n$, and let α be a pure r-dimensional holomorphic differential form, $r > 0$, in D. If $d\alpha = 0$ then $\alpha = d\beta$ for some holomorphic form β .

(Tc) Let D be the domain in (Tb), and let α be a (p,q)-form in D, $q \geq 1$. If $\bar{\partial}\alpha = 0$ then $\alpha = \bar{\partial}\beta$ for some $(p,q-1)$-form β.

Proof.

Lemma. Let i,j be fixed numbers, $i \neq j$, $i,j = 1,2,\ldots,n$.

(La) Let D be the domain in (Ta). If ϕ is a C^{∞} function in D, then there is a C^{∞} function ψ in D such that $\phi = \partial\psi/\partial x_i$ and if $\partial\phi/\partial x_j = 0$ then $\partial\psi/\partial x_j = 0$.

Proof. Define $\psi(x_1,\ldots,x_n) = \int_0^{x_i} \phi(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)\, dt.$

(Lb) Let D be the domain in (Tb). If ϕ is a holomorphic function in D, then there is a holomorphic function ψ in D such that $\phi = \partial\psi/\partial z_i$ and if $\partial\phi/\partial z_j = 0$ then $\partial\psi/\partial z_j = 0$.

Proof. Since ϕ is holomorphic in D,
$\phi = \sum_{k=0}^{\infty} a_k(z_1,\ldots,\hat{z}_i,\ldots,z_n)z_i^k$. Define

$$\psi(z_1,\ldots,z_n) = \sum_{k=0}^{\infty} \frac{a_k(z_1,\ldots,\hat{z}_i,\ldots,z_n)z_i^{k+1}}{k+1}.$$

(Lc) Let $D_1 = \{(z_1,\ldots,z_n) \mid |z_j| < r_j < \infty\} \subset \mathbb{C}^n$. If ϕ is a C^{∞} function in a neighborhood of $\bar{D}_1$, then there is a C^{∞} function ψ in perhaps a smaller neighborhood of $\bar{D}_1$ such that $\phi = \partial\psi/\partial\bar{z}_i$ and if ϕ is holomorphic in z_j then so is ψ.

Proof. Define

$$\psi(z_1,\ldots,z_n) = -\frac{1}{\pi} \iint_{|\zeta|<r_i+\varepsilon} \frac{\phi(z_1,\ldots,\hat{z}_i,\ldots,z_n)}{\zeta - z_i}\, d\xi\, d\eta .$$

(a) (Proof of Ta)). Use induction on k where $\alpha = \sum a_{j_1\ldots j_r}\, dx_{j_1}\ldots dx_{j_r}$ contains no terms with $dx_{k+1}, dx_{k+2},\ldots,dx_n$. For $k = 1$, $\alpha = a_1 dx_1$ and $d\alpha = 0$ means that $\partial a_1/\partial x_i = 0$, $i \geq 2$. By (La) there is a b_1 with $\partial b_1/\partial x_i = 0$, $i \geq 2$ and $\alpha = \partial b_1/\partial x_1\, dx_1$. Take $= b_1$. Assume the theorem to be true for some $k-1$, and let α be a pure r-form with no terms involving $dx_{k+1},\dots,dx_n$. Then $\alpha = (-1)^{r-1}(dx_k \wedge \rho) + \sigma$ where ρ and σ are pure differential forms of dimension $(r-1)$ and r respectively and ρ and σ do not involve dx_j with $j \geq k$. Then $0 = d\alpha = (-1)^r(dx_k \wedge d\rho) + d\sigma$. Since σ contains no terms with $dx_k, dx_{k+1},\dots,dx_n$; $d\sigma$ contains only terms with either dx_j, $j \leq k$ or involving $dx_1,\dots,dx_{k-1},dx_{k+1},\dots,dx_n$. Hence $d\sigma$ has no terms with both dx_k and dx_j, $j > k$. Thus $d\rho$ contains no terms with dx_j, $j > k$, which means that the coefficients of ρ do not depend on $x_{k+1},\dots,x_n$. Now, define $\hat{\rho}$ as follows: Replace each coefficient of ρ by a function whose derivative with respect to x_k is this coefficient; call this new form $\hat{\rho}$. The coefficients of $\hat{\rho}$ do not depend on $x_{k+1},\dots,x_n$. (The existence of $\hat{\rho}$ and its independence of $x_{k+1},\dots,x_n$, is given by (La).) Hence $d\hat{\rho} = (dx_k \wedge \rho)$ + terms involving dx_j, $j < k$. Therefore $\alpha - (-1)^{r-1} d\hat{\rho}$ does not contain $dx_k, dx_{k+1},\dots,dx_n$. But $d(\alpha - (-1)^{r-1} d\hat{\rho}) = d\alpha = 0$ (and $(\alpha - (-1)^{r-1}d\hat{\rho})$ is a pure r-form. By the induction hypothesis, then $\alpha - (-1)^{r-1} d\hat{\rho} = d\gamma$ which means that $\alpha = d((-1)^{r-1}\hat{\rho} + \gamma)$.

(b) (Proof of (Tb)). The proof is identical with that of (Ta) with the x's replaced by z's and α a holomorphic form. (Lb) gives the existence of a holomorphic form $\hat{\rho}$ and its independence of $z_{k+1},\dots,z_n$.

(c) (Proof of (Tc)). 1. Let D_0 be an open polydisc, $D_0 \subset\subset D$. Then α is defined in a neighborhood N of $\bar{D}_0$ and satisfies $\bar{\partial}\alpha = 0$ in N. We claim that there is a form β, in perhaps a smaller neighborhood of $\bar{D}_0$, such that $\alpha = \bar{\partial}\beta$. For the proof, use induction on $d\bar{z}_k$ and an argument analogous to that in (a).

2. Construct open polydiscs $D_j \subset\subset D_{j+1}$ whose union is D. By 1., for each D_j there exists a form β_j such that $\alpha = \bar{\partial}\beta_j$ in a neighborhood of $\bar{D}_j$.

(i) Assume that α has bidegree (p,1), then β_j has bidegree (p,0); i.e. $\beta_j = \sum b^j_{i_1\dots i_p} dz_{i_1}\dots dz_{i_p}$, and

$\bar\partial\beta_j = \alpha$ in $\bar{D}_j$. For fixed k and $j > k$, $(\beta_j-\beta_k)$ is a holomorphic form on D_k since $\bar\partial(\beta_j-\beta_k) = 0$ on D_k. Therefore the coefficients of $(\beta_j-\beta_k)$ can be approximated as closely as desired by polynomials, and hence the form $(\beta_j-\beta_k)$ can be approximated by a form P_{jk} whose coefficients are these polynomials. Choose $\varepsilon_j > 0$ such that $\sum \varepsilon_j < \infty$, and define $|\beta_j-\beta_k| = \sum |b^j_{\tau_1\ldots\tau_p} - b^k_{\tau_1\ldots\tau_p}|$. Construct $\hat\beta_j$ as follows: $\hat\beta_1 = \beta_1$, $\hat\beta_2 = \beta_2 - P_{21}$ where $|\hat\beta_2-\beta_1| = |(\beta_2-\hat\beta_1)-P_{21}| < \varepsilon_1$ on D_1, $\hat\beta_3 = \beta_3 - P_{32}$ where $|\hat\beta_3-\hat\beta_2| = |(\beta_3-\hat\beta_2)-Q_{32}| < \varepsilon_2$ on D_2 and $Q_{32} = Q_{21} + P_{32}$, etc. $\bar\partial\hat\beta_j = \bar\partial\beta_j = \alpha$ on $\bar{D}_j$. Since $|\hat\beta_j-\hat\beta_{j+1}| < \varepsilon_j$ on D_j, $\lim_{j\to\infty} \hat\beta_j = \beta$ exists uniformly on compact subsets of D, the coefficients of β are C^∞ functions and $\bar\partial\beta = \alpha$ in D. (Cf. Part 2 of the proof of Theorem 19.)

(ii) If α has bidegree (p,q), $q \geq 2$ then each β_j has bidegree $(p,q-1)$. $\bar\partial(\beta_{j+1}-\beta_j) = 0$ in a neighborhood of $\bar{D}_j$. Therefore $\beta_{j+1}-\beta_j = \bar\partial\gamma_j$ for some form γ_j, in a neighborhood N_j of $\bar{D}_j$, by 1. of (c). Let ω_j be a real valued C^∞ function $\equiv 1$ on $\bar{D}_j$ and $\equiv 0$ outside a neighborhood M_j of $\bar{D}_j$, $M_j \subset\subset N_j$. Set $\hat\gamma_j = \omega_j\gamma_j$. Then $\hat\gamma_j$ is defined in D_{j+1} and $\beta_{j+1} - \beta_j = \bar\partial\hat\gamma_j$ on $\bar{D}_j$. Let $\hat\beta_1 = \beta_1$, $\hat\beta_2 = \beta_2 - \bar\partial\hat\gamma_1, \ldots,$ and in general $\hat\beta_j = \beta_j - \bar\partial(\hat\gamma_1+\hat\gamma_2+\ldots+\hat\gamma_{j-1})$. Then $\hat\beta_{j+1}-\hat\beta_j = (\beta_{j+1}-\beta_j) - \bar\partial(\hat\gamma_1+\ldots+\hat\gamma_j) + \bar\partial(\hat\gamma_1+\ldots+\hat\gamma_{j-1}) = (\beta_{j+1}-\beta_j) - \bar\partial\hat\gamma_j = 0$, and $\bar\partial\hat\beta_j = \bar\partial\beta_j = \alpha$ on $\bar{D}_j$. Letting $j \to \infty$ we obtain the desired β.

Chapter 8. Canonical Isomorphisms

§1. De Rham's Theorem

A. Definition 44a. Let X be a differentiable manifold with covering U. We say U is simple with respect to differential forms, or d-simple, if it is open, locally finite, and, for the intersection of any finite number of sets of the cover $u_0 \cap \ldots \cap u_r$, the Poincaré lemma for "d" holds.

Theorem 24a. Let X be a differentiable manifold. Then there exist arbitrarily fine d-simple coverings (i.e. every covering of X has a d-simple refinement).

Proof. We shall first prove this theorem assuming $X \subset \mathbb{R}^n$, X open. The case of an arbitrary manifold X is treated at the end of this section.

Assume $X \subset \mathbb{R}^n$. Note that in any box $\{|x_i - a_i| < r_i,$ $i = 1,\ldots,n\}$, Poincaré's lemma holds by Theorem 23. Furthermore, the intersection of any finite number of boxes is again a box; so it suffices to refine any covering U to a locally finite covering by boxes and this is easily done.

Lemma 1a. $H^r(X,U,\Omega^p) = 0$; $r > 0$, $p \geq 0$, where Ω^p denotes the sheaf of germs of p-forms, and U is locally finite.

Proof. The sections of Ω^0 over U are C^∞ functions, so $H^r(X,U,\Omega^0) = H^r(X,U,C^\infty) = 0$, $r > 0$, by Theorem 22. Since any element of Ω^p, when multiplied by a C^∞ function, remains in Ω^p, we may establish this lemma by constructing a homomorphism $\Theta : C^r(X,U,\Omega^p) \to C^{r-1}(X,U,\Omega^p)$, $r > 0$, so that if $f \in C^r(X,U,\Omega^p)$, then $f = \Theta\delta f + \delta\Theta f$ precisely as before. Hence every cocycle is a coboundary.

Corollary. $H^r(X,\Omega^p) = 0$; $r > 0$, $p \geq 0$, Ω^p as above.

Theorem 25a. Let X be a differentiable manifold, U a d-simple covering. Then the following groups are canonically isomorphic, where $\bar{\Omega}^p$ denotes the sheaf of germs of closed p-forms in X:

(i) $H^1(X,U,\bar{\Omega}^p \simeq \dfrac{\text{closed p+1-forms}}{\text{exact p+1-forms}}$, $p \geq 0$.

(ii) $H^{r+1}(X,U,\bar{\Omega}^p) \simeq H^r(X,U,\bar{\Omega}^{p+1})$, $r > 0$, $p \geq 0$.

(iii) $H^r(X,U,\mathbb{C}) \simeq \dfrac{\text{closed r-forms}}{\text{exact r-forms}}$, $r > 0$.

Before proving this theorem (following A. Weil), we introduce the notion of coelements.

B. Definition 45. A coelement f^{rp} of bidegree (r,p) is an r-cochain on a (fixed) covering with coefficients which are pure dimensional differential forms of degree p, i.e. if $u_{j_0},\dots,u_{j_r}$ are distinct sets of the covering with nonempty intersection, then $f^{rp}(j_0\dots j_r)$ assigns to this intersection a pure differential form of degree p defined there.

The coelements form a vector space over $\mathbb{C}$.

Define $df^{rp} = g^{r,p+1}$, where g assigns to each intersection "d" of the form which f assigns; $d^2f = 0$.

Define $\delta f^{rp} = h^{r+1,p}$ in the usual way; $\delta^2 = 0$. Clearly $d\delta = \delta d$ (for δ "adjusts" the domain, and d the range, of the coelements).

Coelements f for which $df = 0$ are cochains with closed forms as coefficients, and if $df = \delta f = 0$, the coelements are cocycles with closed forms as coefficients.

If U is a d-simple covering, f^{rp} a coelement, $p > 0$, then $df^{rp} = 0$ implies $f = dg$. If $r > 0$ and $\delta f^{rp} = 0$, then $f = \delta g$ by Lemma 1a.

We say that coelements $f^{r+1,p}$ and $\tilde{f}^{r,p+1}$ are associated if there is a coelement g^{rp} such that $f = \delta g$ and $\tilde{f} = dg$.

C. Proof of Theorem 25a.

Note first that (i) and (ii) imply (iii), for

$$\frac{\text{closed (p+1)-forms}}{\text{exact (p+1)-forms}} \simeq H^1(X,U,\bar{\Omega}^p) \simeq \dots \simeq H^{1+s}(X,U,\bar{\Omega}^{p-s}) \simeq \dots$$
$$\simeq H^{1+p}(X,U,\bar{\Omega}^o) , \qquad p \geq 0 ,$$

and $\bar{\Omega}^o = \mathbb{C}$.

Note also that

$$H^r(X,U,\bar{\Omega}^p) = \frac{\{f^{rp} \mid df^{rp} = \delta f^{rp} = 0\}}{\{\delta h^{r-1,p} \mid dh^{r-1,p} = 0\}} \quad ; \quad r > 0 \; .$$

(i) We associate a closed (p+1)-form on X (or, equivalently, a coelement $\tilde{f}^{o,p+1}$ of bidegree $(0,p+1)$) to each cocycle class in $H^1(X,U,\bar{\Omega}^p)$ as follows: Let f^{1p} be any cocycle; $df = \delta f = 0$. Using Lemma 1a, there exists a g^{op} such that $\delta g = f$. Set $\tilde{f}^{o,p+1} = dg$, and note that $d\tilde{f} = 0$; hence, we have assigned a closed (p+1)-form. (Also, $\delta\tilde{f} = \delta dg = d\delta g = df = 0$.) Denote by $\{\tilde{f}\}$ the class in $\frac{\text{closed (p+1)-forms}}{\text{exact (p+1)-forms}}$ containing $\tilde{f}$. We make the following assertions:

α) $\{\tilde{f}\}$ does not depend on the choice of g.

β) $\{\tilde{f}_1\} = \{\tilde{f}_2\}$ if $\{f_1\} = \{f_2\}$, i.e. if $f_1 - f_2 = \delta h$, $dh = 0$.

γ) The class mapping $\{f\} \to \{\tilde{f}\}$ is an isomorphism.

Proof of α) Assume $f = \delta g_1 = \delta g_2$, and set $\tilde{f}_1 = dg_1$; $\tilde{f}_2 = dg_2$. α) asserts that $\tilde{f}_1 - \tilde{f}_2 = dh^{0p}$ where h^{0p} is globally defined on X; i.e. $\delta h = 0$. But $\tilde{f}_1 - \tilde{f}_2 = d(g_1 - g_2)$ and $\delta(g_1 - g_2) = 0$.

Proof of β) Suppose $f_1^{1p} - f_2^{1p} = \delta h^{0p}$, where $\delta f_1 = \delta f_2 = df_1 = df_2 = 0$ and $dh = 0$. Now $f_2 = \delta g_2$, $f_1 = \delta g_1 = \delta g_2 + \delta h = \delta(g_2 + h)$. Hence $\tilde{f}_1 = dg_1 = dg_2 + dh$, $\tilde{f}_2 = dg_2$, and $\tilde{f}_1 - \tilde{f}_2 = dh = 0$.

Proof of γ) It is clear that the association map $\{f\} \to \{\tilde{f}\}$ is a homomorphism. It is one-to-one for, assume $\{f\} \to 0$; i.e. $f = \delta g$, and $dp^{0p} = 0$. Now, $dg^{0p} = 0$ means $g^{0p} = dh^{0,p-1}$ so that $f = \delta g = \delta dh$: hence $\{f\} = 0$. Furthermore, it is onto, for, assume $\tilde{f}$ is any closed (p+1)-form. Then $d\tilde{f} = 0$, so $\tilde{f} = dg$ by d-simplicity of $\tilde{U}$. Define $f = \delta g$. Then $df = d\delta g = \delta dg = \delta\tilde{f} = 0$, as $\tilde{f}$ is globally defined, and $\delta f = \delta^2 g = 0$.

(ii) The proof in this case is essentially the same; let $f^{r+1,p}$ satisfy $df^{r+1,p} = \delta f^{r+1,p} = 0$. Then there exists a g^{rp} such that $\delta g = f$. Set $\tilde{f}^{r,p+1} = dg$, and observe $d\tilde{f} = \delta\tilde{f} = 0$. With a similar notation, we prove α), β), and γ).

α) Assume $f = 0 = \delta g$; then $\{dg\} = 0$, for $\delta g^{rp} = 0$ implies $g^{rp} = \delta h^{r-1,p}$, and $dg = d\delta h^{r-1,p} = \delta dh$. Now dh is closed, so $\{dg\} = \{\delta(dh)\} = 0$ in $H^r(X,U,\bar{\Omega}^{p+1})$.

β) $f^{r+1,p} = \delta h^{rp}$ implies $\{dh^{rp}\} = 0$, as $dh^{rp} = 0$.

γ) We again have a homomorphism $\{f^{r+1,p}\} \to \{\tilde{f}^{r,p+1}\}$. It is one-to-one for, if $f = \delta g^{rp}$ and $dg^{rp} = 0$, then $g = dh^{r,p-1}$, and $f = \delta g = \delta dh$, so $\{f\} = 0$. It is onto, for assume $\tilde{f}^{r,p+1}$ satisfies $d\tilde{f} = \delta\tilde{f} = 0$. Then $\tilde{f} = dg^{rp}$ by simplicity of U. Set $f = \delta g^{rp}$. Then $df = d\delta g = \delta dg = \delta\tilde{f} = 0$ and $\delta f = 0$. Clearly $\{f\} \to \{\tilde{f}\}$.

D. <u>Lemma 2a</u>. Let X be a differentiable manifold; U,V d-simple coverings and V a refinement of U. Then the following diagrams commute, where $\bar{\Omega}^p$ denotes the sheaf of germs of d-closed p-forms

i) $$\begin{array}{ccc} H^{r+1}(X,U,\bar{\Omega}^p) & \longrightarrow & H^r(X,U,\bar{\Omega}^{p+1}) \\ \downarrow & & \downarrow \\ H^{r+1}(X,V,\bar{\Omega}^p) & \longrightarrow & H^r(X,V,\bar{\Omega}^{p+1}) \end{array}$$

ii) $$\begin{array}{ccc} H^1(X,U,\bar{\Omega}^p) & \searrow & \\ & & \dfrac{\text{d-closed } (p+1)\text{-forms}}{\text{d-exact } (p+1)\text{-forms}} \\ H^1(X,V,\bar{\Omega}^p) & \nearrow & \end{array}$$

iii) $$\begin{array}{ccc} H^r(X,U,\mathbb{C}) & \searrow & \\ & & \dfrac{\text{d-closed } r\text{-forms}}{\text{d-exact } r\text{-forms}} \\ H^r(X,V,\mathbb{C}) & \nearrow & \end{array}$$

where $r > 0$ and $p \geq 0$.

<u>Proof</u>. For i), this lemma states that one obtains the same result by first mapping a coelement to any associate and then restricting the domain of definition; or by first restricting the domain and then associating it. The other commutativity claims are disposed of as easily.

Theorem 26a. Let X be a differentiable manifold for which there exist arbitrarily fine d-simple coverings. Then the following groups are canonically isomorphic:

(I) (de Rham): $H^r(X,\mathbb{C}) \simeq \dfrac{\text{d-closed } r\text{-forms}}{\text{d-exact } r\text{-forms}}$, $r > 0$

(II) (Leray): $H^r(X,U,\mathbb{C}) \simeq H^r(X,\mathbb{C})$, $r > 0$,

for any d-simple covering U.

Proof. $H^r(X,\mathbb{C})$ is a direct limit of groups $H^r(X,U,\mathbb{C})$; U any covering of X, where the class of all coverings is directed by "is a refinement of." By Theorem 24a, the d-simple coverings are cofinal, hence it suffices to consider only d-simple coverings in the direct limit. By (iii) of Theorem 25a, $H^r(X,U,\mathbb{C}) \simeq \dfrac{\text{d-closed } r\text{-forms}}{\text{d-exact } r\text{-forms}}$ for any d-simple U; hence I and II.

We now complete Theorem 24a. Note that Theorem 25a does not require the existence of arbitrarily fine d-simple coverings. Now let U be any covering of X. Let V be a locally finite refinement of U such that each $v \in V$ lies entirely in a coordinate patch and the intersection of any finite number of v's is contractible to a point. (Such a covering can be constructed using Whitney's imbedding theorem.) Since every finite intersection of sets of V is diffeomorphic to an open set in $\mathbb{R}^n$, the de Rham theorem applies. If the r th cohomology group of such an open set in $\mathbb{R}^n$ with complex coefficients is trivial, then every closed r-form is exact: that this cohomology is trivial is a known result.

§2. Dolbeault's Theorem

This section is Section 1 applied to complex manifolds and the operator $\bar{\partial}$.

A. Definition 44b. Given a complex manifold X and covering U, we say that U is simple with respect to (p,q) differential forms, $q \geq 1$, or $\bar{\partial}$-simple if it is open, locally finite, and, for the intersection of any finite number of sets of the covering, the Poincaré lemma for $\bar{\partial}$ holds.

Theorem 24b. Let X be a complex manifold. Then there exist arbitrarily fine $\bar{\partial}$-simple coverings.

Proof. As before, assume $X \subset \mathbb{C}^n$, open. Once again use Theorem 23, which establishes the Poincaré lemma for $\bar{\partial}$, for coverings by polydiscs $\{|z_j - a_j| < R_j\}$.

The completion of this theorem for a manifold is remarked on at the end of this section.

Lemma 1b. $H^r(X,U,\Omega^p) = 0$, $r > 0$; where Ω^p now denotes the sheaf of germs of forms of type $(0,p)$, and U is locally finite.

Proof. We may again use a partition of unity argument as in Lemma 1a.

Theorem 25b. Let X be a complex manifold, U a $\bar{\partial}$-simple covering. Then, if $\bar{\Omega}^p$ denotes the sheaf of germs of $\bar{\partial}$-closed forms of type $(0,p)$, there exist canonical isomorphisms between the following groups:

i) $H^1(X,U,\bar{\Omega}^p) \cong \dfrac{\bar{\partial}\text{-closed forms of type } (0,p+1)}{\bar{\partial}\text{-exact forms of type } (0,p+1)}$,

ii) $H^{r+1}(X,U,\bar{\Omega}^p) \cong H^r(X,U,\bar{\Omega}^{p+1})$

iii) $H^r(X,U,\mathcal{O}) \cong \dfrac{\bar{\partial}\text{-closed forms of type } (0,r)}{\bar{\partial}\text{-exact forms of type } (0,r)}$

where $r > 0$, $p \geq 0$.

Proof. i) and ii) proceed precisely as in Theorem 25a. iii) is implied by i) and ii), also as before, when one notes that a $\bar{\partial}$-closed form of type $(0,0)$ is a holomorphic function, and conversely; i.e. $\bar{\Omega}^0 = \mathcal{O}$.

Lemma 2b. Let X be a complex manifold; U,V $\bar{\partial}$-simple coverings where V is a refinement of U. Then the following diagrams commute, where $\bar{\Omega}^p$ denotes the sheaf of germs of $\bar{\partial}$-closed forms of type $(0,p)$, and $r > 0$, $p \geq 0$:

i) $$\begin{array}{ccc} H^{r+1}(X,U,\bar{\Omega}^p) & \longrightarrow & H^r(X,U,\bar{\Omega}^{p+1}) \\ \downarrow & & \downarrow \\ H^{r+1}(X,V,\bar{\Omega}^p) & \longrightarrow & H^r(X,V,\bar{\Omega}^{p+1}) \end{array}$$

ii) $$\begin{array}{ccc} H^1(X,U,\bar{\Omega}^p) & \searrow & \\ \downarrow & & \dfrac{\bar{\partial}\text{-closed forms of type }(0,p+1)}{\bar{\partial}\text{-exact forms of type }(0,p+1)} \\ H^1(X,V,\bar{\Omega}^p) & \nearrow & \end{array}$$

iii) $$\begin{array}{ccc} H^r(X,U,\mathcal{O}) & \searrow & \\ \downarrow & & \dfrac{\bar{\partial}\text{-closed forms of type }(0,r)}{\bar{\partial}\text{-exact forms of type }(0,r)} \\ H^r(X,V,\mathcal{O}) & \nearrow & \end{array}$$

Theorem 26b. Let X be a complex manifold for which there exist arbitrarily fine $\bar{\partial}$-simple coverings. Then there exist canonical isomorphisms between

I. (Dolbeault) $H^r(X,\mathcal{O}) \simeq \dfrac{\bar{\partial}\text{-closed forms of type }(0,r)}{\bar{\partial}\text{-exact forms of type }(0,r)}$, $r > 0$.

II. (Leray) $H^r(X,U,\mathcal{O}) \simeq H^r(X,\mathcal{O})$, $r > 0$.

Corollary. If $D \subset \mathbb{C}^n$, D a polydisc, then $H^r(D,\mathcal{O}) = 0$ for all $r > 0$.

We shall eventually use this corollary to establish the result for any region of holomorphy.

Remark. The general Dolbeault theorem reads as follows: $H^r(X$, sheaf of germs of holomorphic forms of degree $s) \simeq \dfrac{\bar{\partial}\text{-closed forms of type }(s,r)}{\bar{\partial}\text{-exact forms of type }(s,r)}$.

However, we shall require only the restricted result of 26b, I.

B. In order to complete Theorem 24b, we require the result that in domains of holomorphy all the above cohomology groups are trivial (proof in Chap.11). Assuming this result, we have the Poincaré lemma with respect to $\bar{\partial}$ for holomorphy domains, so that any locally finite covering by domains of holomorphy is $\bar{\partial}$-simple. Hence it suffices to establish that arbitrarily fine coverings by domains of holomorphy exist.

§3. Complex de Rham theorem

Once again, we attempt to repeat 1 for complex manifolds X, holomorphic forms, and the operator d.

Definition 44c. Given a complex manifold X and covering U, we say that U is ∂-simple or simple with respect to holomorphic forms, if it is open, locally finite, and, for the intersection of any finite number of sets of the covering, the Poincaré lemma for ∂ holds. (Recall that $\partial = d$ on holomorphic forms.)

Theorem 24c. Let X be a complex manifold. Then there exist arbitrarily fine ∂-simple coverings.

Proof. For $X \subset \mathbb{C}^n$, open, the proof is again immediate and proceeds as before.

At this point, however, we find that no Lemma 1c exists. Hence, we must modify Theorem 25c as follows:

Theorem 25c. Let X be a complex manifold, U a ∂-simple covering, for which $H^r(X,U,\Omega^p) = 0$, $r > 0$, where Ω^p denotes the sheaf of germs of holomorphic p-forms. Let $\bar{\Omega}^p$ denote the sheaf of germs of closed holomorphic p-forms. Then the following groups are canonically isomorphic:

i) $H^1(X,U,\bar{\Omega}^p) \simeq \dfrac{\text{closed holomorphic }(p+1)\text{-forms}}{\text{exact holomorphic }(p+1)\text{-forms}}$

ii) $H^{r+1}(X,U,\bar{\Omega}^p) \simeq H^r(X,U,\bar{\Omega}^{p+1})$

iii) $H^r(X,U,\mathbb{C}) \simeq \dfrac{\text{closed holomorphic }r\text{-forms}}{\text{exact holomorphic }r\text{-forms}}$

where $p \geq 0$, $r > 0$.

Proof. As before, under the remark that $\bar{\Omega}^0 = \mathbb{C}$.

Lemma 2c. We state here merely that the analogous commutativity lemma is valid, assuming the missing Lemma 1c for all manifolds and coverings used.

Theorem 26c. (Complex de Rham). Let X be a complex manifold for which there exist arbitrarily fine ∂-simple coverings and such that $H^r(X,U,\Omega^p) = 0$, $r > 0$, for all ∂-simple coverings U. Then, there exists a canonical isomorphism between: $H^r(X,\mathbb{C}) \simeq \frac{\text{closed holomorphic } r\text{-forms}}{\text{exact holomorphic } r\text{-forms}}, r>0.$

(We shall not need the complex Leray theorem.)

Let us assume that we have already proven that in a domain of holomorphy the cohomology with holomorphic coefficients is trivial (not closed forms). Then the hypothesized Lemma 1c holds, so that the theorems of this section hold for domains of holomorphy.

We remark that the group $\frac{\text{closed holomorphic } r\text{-forms}}{\text{exact holomorphic } r\text{-forms}}$ is clearly trivial for $r > n$, where n is the dimension of the manifold. Hence

Theorem 27. Let X be a complex manifold of dimension n, such that $H^r(\underline{X},U,\Omega^p) = 0$, $r > 0$ for all ∂-simple coverings U. Then $H^r(X,\mathbb{C}) = 0$, $r > n$.

This theorem gives a topologically necessary condition for a differentiable manifold X of real dimension $2n$ to be a complex manifold.

Chapter 9. The Multiplicative Cousin Problem

§1. The Multiplicative Problem, formulated

A. This second Cousin problem is a generalization of the Weierstrass problem in one complex variable:

Given a domain $D \subset \mathbb{C}$, a discrete set of points, a_ν, b_ν and positive integers n_ν, m_ν, find a function f, meromorphic in D, with zeroes at a_ν of order n_ν, and poles at b_ν of order m_ν.

We now formulate the multiplicative problem, referred to as C.II in the sequel:

Multiplicative Cousin Problem. Let X be a complex manifold $U = \{u_i\}$, $i \in I$, an open covering, and let functions F_i be defined and meromorphic in u_i, such that F_i/F_j is holomorphic in $u_i \cap u_j$. Does there exist a function $F \not\equiv 0$, defined and meromorphic in X, such that F/F_i is holomorphic in u_i?

Note. C.II is precisely C.I, written multiplicatively. As in the Weierstrass problem, where it is sufficient to find a function with given zeroes of given order, we shall find we need only consider holomorphic functions F_i. We shall also show that C.II is not always solvable. As before, we shall formulate C.II using sheaves and cohomology groups.

B. Definition 46. Let X be a complex manifold, and let $\mathcal{M}$ denote the sheaf, over X, of germs of meromorphic functions under multiplication, where $\mathcal{M} = \bigcup_{x \in X} \mathcal{M}_x$, and $\mathcal{M}_x$ consists of the germs of meromorphic functions at x, and is a multiplicative group.

We topologize $\mathcal{M}$ as we did $\mathcal{O}$, by defining a subbasis for the topology utilizing the topology of X, as follows: Let $m \in \mathcal{M}$; then $m \in \mathcal{M}_x$ and $g \in m$ is defined in a neighborhood N of x. For each $y \in N$, let $\{g\} \in \mathcal{M}_y$ be the equivalent class of meromorphic functions in $\mathcal{M}_y$ containing g. Then the sets $N_g = \{\{g\} \in M_y \mid y \in N\}$ for each choice of $g \varepsilon m$, and for each $m \varepsilon \mathcal{M}$, form the subbasis of $\mathcal{M}$.

Let $\mathcal{F}$ denote the subsheaf of invertible holomorphic elements of $\mathcal{M}$; such that $\mathcal{F} \subset \mathcal{M}$ and $\mathcal{F}_x \subset \mathcal{M}_x$ for each $x \varepsilon X$. Form the quotient groups $\mathcal{M}_x / \mathcal{F}_x$, and set $\mathcal{M}/\mathcal{F} = \bigcup_{x \varepsilon X} \mathcal{M}_x/\mathcal{F}_x$ with the quotient topology.

The sheaf of germs of divisors of X is the quotient sheaf $\mathcal{M}/\mathcal{F}$.

Note that elements of $\mathcal{M}/\mathcal{F}$ are equivalence classes of germs of meromorphic functions, where germs represented by two functions F_1, F_2 are equivalent (at x) if F_1/F_2 is a local unit, i.e. if F_1/F_2 is holomorphic and non-vanishing in a neighborhood of x.

We note also that a set of Cousin data associated with the covering U may be regarded as a section of $\mathcal{M}/\mathcal{F}$ over X.

Definition 47. A divisor on X is a section over X of $\mathcal{M}/\mathcal{F}$, i.e. is an equivalence class of sets of Cousin data, in the sense that two sets of Cousin data are equivalent if their "mesh" is a set of Cousin data. A divisor on X is integral (positive) if all germs are germs of holomorphic functions.

Definition 48. A divisor α on X is principal if there exists a meromorphic function F defined in X such thatthe divisor it defines $(F) = \alpha$.

Hence, we may state C.II in the following equivalent way: given a divisor on X, is it principal?

Lemma 1. If every integral divisor of the complex manifold X is principal, then every C.II is solvable.

Proof. It is enough to show that every divisor is a quotient of integral divisors. Let $p \varepsilon X$, and let F_p be a meromorphic function defined in a neighborhood N_p of p such that (F_p) is the restriction of α to N_p. $F_p = \phi_p/\psi_p$, where ϕ_p and ψ_p are holomorphic functions defined in N_p

and coprime at p, and hence in a neighborhood of p, say N_p. Let $q \in N_p$, and $F_q = \phi_q/\psi_q$ the corresponding function at q, defined in N_q. We shall be done if $\phi_p \sim \phi_q$ and $\psi_p \sim \psi_q$, where defined, where "$\sim$" means equivalence modulo $\mathcal{F}$. Now $F_p \sim F_q$, hence $\phi_p/\psi_p \sim \phi_q/\psi_q$, so $\phi_p\psi_q \sim \psi_p\phi_q$, at q. But $\phi_p\psi_q$ and $\psi_p\phi_q$ are holomorphic, and are equivalent in a neighborhood of q. We may choose this neighborhood so that ϕ_q, ψ_q are coprime. But ϕ_p divides $\psi_p\phi_q$; hence ϕ_p divides ϕ_q. Similarly, ϕ_q divides ϕ_p, as ϕ_p, ψ_p were coprime in N_p. Therefore ϕ_p/ϕ_q is a unit, i.e. $\phi_p \sim \phi_q$, and similarly $\psi_p \sim \psi_q$. Thus $(\phi_p \mid p \in X)$ and $(\psi_p \mid p \in X)$ are integral divisors.

C. Theorem 28. Let X be a complex manifold such that C.II is always solvable. Then so is the strong Poincaré problem, i.e. every globally defined meromorphic function is the ratio of two holomorphic functions, coprime at every point of X.

Proof. Let F be the global meromorphic function. Then $(F) = \alpha/\beta$, where α and β are principal integral divisors; for as was shown in the proof of Lemma 1, on any complex manifold X every divisor is a quotient of integral divisors, and since C.II is solvable every integral divisor is principal; hence $F/f/g = \omega_1/\omega_2$ is a local unit. Therefore $F/f/g$ is holomorphic and equivalent to 1 at each point of X; so $F/f/g$ is a global unit, say G. Thus $F = fG/g$, where fG and g are coprime at each point of X.

Theorem 29. Let X be a complex manifold such that C.II is always solvable, and Y a regularly imbedded analytic hypersurface. Then Y may be globally presented.

Proof. Exercise for the reader.

§2. The Multiplicative Cousin Problem is not always solvable

The following example is due to Oka.

Let $X \subset \mathbb{C}^2$ be defined as follows:

$X = \{(z_1,z_2) \mid 3/4 < |z_j| < 5/4, \ j = 1,2\}$. Note that X, as a product domain, is a domain of holomorphy. This shows, incidentally, that "C.I implies C.II" is false.

Set $Y = \{z_1 - z_2 - 1 = 0\} \cap X$, and note that Y is a closed subset of X consisting of two distinct components, for $(z_1,z_2) = (x_1+iy_1, x_2+iy_2) \in Y$ implies $0 < x_1 < 1$, where $x_1 - 1 = x_2$. Hence $y_1 = y_2 \neq 0$; but Y is nonempty and $(z_1,z_2) \in Y$ implies $(\bar{z}_1,\bar{z}_2) \in Y$. Hence, we see also that the components of Y lie in $(\operatorname{Im} z_1 > 0, \ \operatorname{Im} z_2 > 0)$ and $(\operatorname{Im} z_1 < 0, \ \operatorname{Im} z_2 < 0)$.

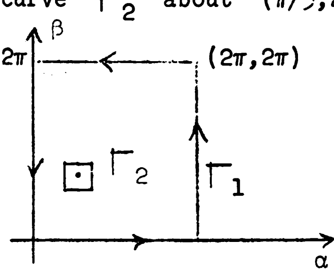
Now define the divisor γ of X as follows:

$$\gamma = \begin{cases} (z_1 - z_2 - 1), & \text{for } (\operatorname{Im} z_1 > 0, \ \operatorname{Im} z_2 > 0) \\ 1 & \text{outside the upper component of } Y \end{cases}$$

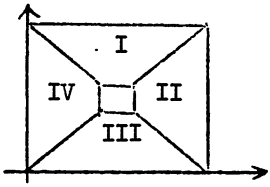
This clearly defines a set of Cousin data, for which, we claim, the Cousin problem has no solution. For, assume there exists a solution $F(z_1,z_2)$, and consider its restriction to $\{|z_1| = 1, \ |z_2| = 1\}$:

$$g(\alpha,\beta) = F(e^{i\alpha}, e^{i\beta}) .$$

Now $g(\alpha,\beta)$ is a continuous, periodic function of both α and β. Furthermore, g has precisely one zero, for the upper component of $Y \cap \{|z_1|=1, |z_2|=1\} = \{(e^{i\pi/3}, e^{i2\pi/3})\}$, and $g \sim 1$ elsewhere. Now consider the edge curve Γ_1 in the α,β plane as indicated in the figure, and the edge curve Γ_2 about $(\pi/3, 2\pi/3)$ within Γ_1 and oriented similarly.



Since g is periodic in α and β it is clear that arg $g(\alpha,\beta)$ can be defined as a single-valued function along Γ_1. Furthermore, by connecting the two curves as in the second figure, we obtain the following:



$$\int_{\Gamma_1} d \log g = \int_{\Gamma_2} d \log g \;;$$

i.e. $\arg g(\alpha,\beta)$ is also single-valued along Γ_2.

Now $F(z_1,z_2) = h(z_1,z_2)\cdot(z_1-z_2-1)$, where $h(z_1,z_2)$ is a unit. In the region enclosed by Γ_2, we may define a single-valued branch of the log; so

$$\int_{\Gamma_2} d \log g = \int_{\Gamma_2} d \log [-e^{i\beta} + e^{i\alpha} - 1] \,.$$

We may calculate this latter integral explicitly. Set

$$\begin{cases} \beta = \beta' + \frac{2\pi}{3} \\ \alpha = \alpha' + \frac{\pi}{3} \;; \end{cases}$$

obtain

$$\int_{\Gamma_2} d \log (-e^{i\beta} + e^{i\alpha} - 1) = \int_{\Gamma_2'} d \log (-e^{i\beta' + \frac{i2\pi}{3}} + e^{i\alpha' + \frac{i\pi}{3}} - 1),$$

where Γ_2' now encircles the origin in the (α',β')plane, and $-e^{i\beta'+2\pi i/3} + e^{i\alpha'+\pi i/3} - 1$ has a zero at $(0,0)$. Set $-e^{i\beta'+2\pi i/3} + e^{i\alpha'+\pi i/3} - 1 = u + iv$. Now

$$\left|\frac{\partial(u,v)}{\partial(\alpha',\beta')}\right| = -\sin(\beta'-\alpha'+\pi/3) \;.$$

Hence, if Γ_2 was chosen small enough, $\left|\frac{\partial(u,v)}{\partial(\alpha',\beta')}\right| < 0$ in the region enclosed by Γ_2'. So

$$\int_{\Gamma_2'} d \log (u+iv) = \int_{\Gamma_2'} d \log (\alpha' + i\beta') \neq 0 \,,$$

and this contradition establishes our claim.

§3. The solution of the Multiplicative Cousin Problem for polydiscs

Theorem 30 (Cousin). Let $D \subset \mathbb{C}^n$ be a polydisc. Then C.II is always solvable.

We prove this theorem twice; the first proof, given in this section, is due to Cousin. The second is given in §4, and gives more.

Lemma 2. Let Δ_1, Δ_2 be box domains in $\mathbb{C}^n$ defined as follows:

$$\Delta_1 = \{(z_1, \dots, z_n) \,|\, a_1 \le x_1 \le a_2, b_1 \le y_1 \le b_2; \alpha_j \le x_j \le \alpha_j', \beta_j \le y_j \le \beta_j', \ j=2,\dots,n\}$$

$$\Delta_2 = \{(z_1, \dots, z_n) \,|\, a_2 \le x_1 \le a_3, b_1 \le y_1 \le b_2; \alpha_j \le x_j \le \alpha_j', \beta_j \le y_j \le \beta_j', \ j=2,\dots,n\}$$

Let G_1, G_2 be open sets (in $\mathbb{C}^n$) containing Δ_1, Δ_2 respectively, and F_1, F_2 meromorphic (holomorphic) functions in G_1, G_2, respectively, such that $F_1/F_2 \sim 1$ in $G_1 \cap G_2$. Then there exists a domain G such that $\Delta_1 \cup \Delta_2 \subset G \subset (G_1 \cup G_2)$, and a function F, meromorphic (holomorphic) and defined in G such that $F_i/F \sim 1$ in $G_i \cap G$, $i = 1,2$.

In other words, if C.II is solvable for neighborhoods of Δ_1 and Δ_2, it is solvable for a neighborhood of $\Delta_1 \cup \Delta_2$.

Proof. It is easily seen that $G_1 \cap G_2$ contains a neighborhood G_0 of $\Delta_1 \cap \Delta_2$, and that we may choose this neighborhood to be simply connected, for $\Delta_1 \cap \Delta_2$ is.

Now, in a simply connected domain every nonvanishing holomorphic function F may be written as e^f, where f is holomorphic, for f is the solution of the following set of equations:

$$\begin{cases} \dfrac{\partial f}{\partial z_j} = \dfrac{1}{F}\dfrac{\partial F}{\partial z_j} \\[2ex] \dfrac{\partial f}{\partial \bar{z}_j} = 0 \ . \end{cases}$$

Hence, $F_1/F_2 = e^f$ in G_0. Define

$$f_1(z, z_2, \dots, z_n) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta, z_2, \dots, z_n)}{\zeta - z}\, d\zeta$$

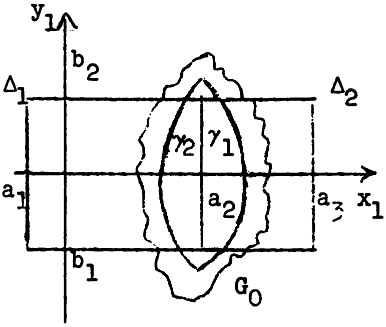
$$f_2(z,z_2,\ldots,z_n) = -\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta,z_2,\ldots,z_n)}{\zeta-z}\, d\zeta ,$$

where γ_1 and γ_2 are curves in the projection of G_0 on the z_1-plane defined as follows: Let C be a simple closed curve in the z_1-plane lying in the projection of G_0 on that plane, positively oriented, and containing the line $\{a_2 = x_1,\ b_1 \leq y_1 \leq b_2\}$ in its interior. Set

$$\gamma_1 = \{z \mid z \in C,\ \mathrm{Re}\ z_1 \geq a_2\}$$

$$\gamma_2 = \{z \mid z \in C,\ \mathrm{Re}\ z_1 \leq a_2\} .$$

The situation is indicated in the figure.



Now f_1 is holomorphic in a neighborhood G^1 of Δ_1, and f_2 is holomorphic in a neighborhood G^2 of Δ_2. Take G to be a domain satisfying $(\Delta_1 \cup \Delta_2) \subset G \subset \{(G^1 \cap G_1) \cup (G^2 \cap G_2)\}$. For $z_1 \in$ interior $(\gamma_1+\gamma_2)$, $f = f_1-f_2$; and hence $f = f_1-f_2$ wherever everything is defined so that $F_1/F_2 = e^f = e^{f_1-f_2}$ there.

Now define

$$F = \begin{cases} F_1\, e^{-f_1} & \text{in } G^1 \cap G_1 \\ F_2\, e^{-f_2} & \text{in } G^2 \cap G_2 . \end{cases}$$

F is meromorphic (holomorphic) in each neighborhood and has the same values in $(G^1 \cap G_1) \cap (G^2 \cap G_2)$ by construction.

<u>Lemma 3</u>. Let $\Delta = \{(z_1,\ldots,z_n) \mid \alpha_j \leq x_j \leq \alpha_j',\ \beta_j \leq y_j \leq \beta_j' ;\ j = 1,\ldots,n\}$ be a bounded box, $\Delta \subset G$, a domain in $\mathbb{C}^n$. Let α be a divisor in G. Then there exists a neighborhood G_1 of Δ in G, and a function F defined and meromorphic in G_1 such that $(F) = \alpha$ on G_1; i.e. given C.II for $G \supset \Delta$, C.II is solvable in a neighborhood of Δ.

<u>Proof</u>. We cover G by boxes whose edges are parallel to

the axes as follows: For each point $z \varepsilon G$, there exists a closed box Δ_z^1, containing z and contained in some set associated with the divisor α (i.e. contained in some set u in which the restriction of α to u is principal). Let Δ_z^2 be an open box containing z and such that $\Delta_z^2 \subset \Delta_z^1$. Now Δ is compact; hence Δ is contained in the union of a finite number of open boxes $\Delta_1^2, \ldots, \Delta_N^2$. Hence $\Delta \subset \bigcup_{i=1}^N \Delta_i^1 \subset G$, and $G_1 = \bigcup_{i=1}^N \Delta_i^1$ is a closed neighborhood of Δ as it contains $\bigcup_{i=1}^N \Delta_i^2$, which is open. We claim that the restriction of α to G_1 is principal; we prove this indirectly. Let $\Delta^1 = \{(z_1, \ldots, z_n) \mid x_1 \leq (\alpha_1' - \alpha_1)/2\}$, $\Delta^2 = \{(z_1, \ldots, z_n) \mid (\alpha_1' - \alpha_1)/2 \leq x_1\}$. Then $\Delta^i \cap \Delta_j^1$ is a box, for $i = 1,2$; $j = 1, \ldots, N$. Set $G_{11} = \{\Delta^1 \cap G_1\}$, $G_{12} = \{\Delta^2 \cap G_1\}$. Then G_{1i} is covered by $\{\Delta^i \cap \Delta_j^1 \mid j=1,\ldots,N\}$, $i = 1,2$ and we have "halved" the problem with respect to x_1. Note that $G_{11} \cup G_{12} = G_1$. Now, if α is principal in G_{11} and in G_{12}, it is principal in G_1; since

$$(F) = \begin{cases} (F_1) \text{ on } G_{11} \\ (F_2) \text{ on } G_{12} \end{cases}$$

is a divisor in G_1 by lemma 2. Hence, assume C.II is not solvable in G_1. Then it is not solvable in either G_{11} or G_{12}; choose one such, and call it G_2. Proceed by halving G_2 with respect to y_1, obtaining $G_3 \subset G_2$, and so on, each time halving G_n with respect to the "next" coordinate, where "next" is with respect to the following sequence: $x_1, y_1, x_2, \ldots, x_n, y_n, x_1, y_1, \ldots$. Now the diameters of the closed nested sets G_m form a null sequence, and so contain a point which is contained in some set Δ_i^1. Hence, for some integer M, $G_M \subset \Delta_i^1$. But C.II is therefore solvable in G_M as Δ_i^1 is contained in a set associated with α; and this contradiction establishes the lemma.

Remark. We could also prove this lemma directly by forming a neighborhood of Δ with a mesh fine enough so that α restricted to each box of the mesh is principal. Construct F by "pasting" the boxes of the mesh following Lemma 2.

B. Proof of Theorem 30. It is clear that it suffices to prove this theorem for box domains Δ. Let Δ_i be a sequence of closed boxes such that $\bigcup_i \Delta_i = \Delta$ and $\Delta_i \subset\subset \Delta_{i+1} \subset\subset \Delta$. By the above, there exists an F_i meromorphic in a neighborhood of Δ_i such that $(F_i) = \alpha$ there. Hence F_{i+1}/F_i is a unit in Δ_i for each i. Clearly, we would be done if the infinite product $F_1(F_2/F_1)(F_3/F_2)\ldots$ converged normally in Δ to a function F; for at each point $z \in \Delta$, $z \in \Delta_n$ for some n and there exists an N such that $(F_{N+1}/F_N)(F_{N+2}/F_{N+1})\ldots$ converges uniformly to a unit in Δ_n. Hence $(F_1/F)\,(F_2/F_1)\ldots(F_N/F_{N-1}) = F_N/F \sim 1$ in Δ_n. Generally, the product will not converge. However, our functions are only defined up to units, and so we replace F_i by $\hat{F}_i = F_i e^{u_i}$, u_i a polynomial, so that the product formed from these $\hat{F}_i$ does converge normally, precisely as in the proof of the Weierstrass theorem in one complex variable.

Corollary. C.II is solvable in any domain which is the product of simply connected domains in $\mathbb{C}$.

Exercise. Use the method of the above theorem to solve C.I for a polydisc.

§4. Characteristic classes (From C.II to C.I)

A. In the following, we consider only coverings U such that u_i and $u_i \cap u_j$ are simply connected; call such coverings distinguished.

We now associate a cohomology group to C.II, as we did for C.I. However, our coefficients are now $\mathcal{J}$, so that our cocycles will be multiplicative.

C.II'. Let X be a complex manifold, and U any covering of X with meromorphic functions F_i defined in u_i such that F_i/F_j is a unit in $u_i \cap u_j$. Set

$$F_{ij} = F_i/F_j .$$

Now F_{ij} is a 1-cochain, for $F_{ij}F_{ji} = 1$ in $u_i \cap u_j$. It is also a cocycle, for $F_{ij}F_{jk}F_{ki} = 1$ in $u_i \cap u_j \cap u_k$.

The Cousin problem is now: Given a cocycle F_{ij}, is it a coboundary; i.e. is the class of F_{ij} the unit in the (multiplicative) cohomology group $H^1(X,U,\mathcal{J})$? It is clear that C.II' implies C.II.

Now assume that the covering is distinguished. We may thus define

$$F_{ij} = e^{2\pi i f_{ij}}, \quad i < j$$

and

$$F_{ji} = e^{2\pi i(-f_{ij})}, \text{ with } f_{ii} = 0 .$$

Now observe that $f_{ij} = - f_{ji}$; i.e. $\{f_{ij}\}$ is an additive 1-cochain, with holomorphic coefficients. However, this is not necessarily a cocycle, for

$$f_{ij} + f_{jk} + f_{ki} = \frac{1}{2\pi i} \log 1 = m_{ijk} ,$$

an integer. But $\{m_{ijk}\}$ is a 2-cocycle (with integral coefficients); the antisymmetry condition is easily verified, and $\delta m_{ijk} = 0$ for m_{ijk} is a coboundary of a 1-cochain. Note that m_{ijk} is not uniquely determined by the Cousin problem, for the f_{ij} are only determined up to an additive integer. However, let f_{ij} be replaced by $f_{ij} + n_{ij}$. Then m_{ijk} is replaced by $m_{ijk} + (n_{ij} + n_{jk} + n_{ki})$, and so the class of m_{ijk} in $H^2(X,U,\mathbb{Z})$ is unchanged. We are thus led to:

Definition 49. The cohomology class of m_{ijk} in $H^2(X,U,\mathbb{Z})$, for distinguished covers U, is called the characteristic class of the Cousin problem.

B. The above procedure gives us a mapping of the group of divisors into the group of characteristic classes, for which a commutative lemma holds.

This map is clearly homomorphic. We may regard it as a map of $H^1(X,U,\mathcal{J}) \to H^2(X,U,\mathbb{Z})$, for distinguished covers U, for, suppose the cocycle F_{ij} of C.II' is a coboundary; i.e. $F_{ij} = F_i/F_j$. Then $f_{ij} = f_i - f_j$ and $m_{ijk} = 0$, i.e. the map $F_{ij} \to \{m_{ijk}\}$ is a map of classes $\{F_{ij}\} \to \{m_{ijk}\}$.

There is a commutativity lemma in this case which we also shall not state explicitly. Furthermore, distinguished coverings are cofinal in the set of all coverings (easily seen for domains in space) so that we summarize in the following theorem, which makes no reference to particular coverings:

Theorem 31. (Oka-Serre). There exists a canonical homomorphism of the group of divisors of X into $H^2(X,\mathbb{Z})$, with the following properties, where α denotes a divisor and $c(\alpha)$ its class:

i) If α is principal then $c(\alpha) = 0$.

ii) If $H^1(X,\mathcal{O}) = 0$ and $c(\alpha) = 0$, then α is principal.

Proof. Note that i) states that C.II solvable implies $H^2(X,U,\mathbb{Z}) = 0$, and ii) states that if C.I is solvable, then this condition is sufficient. We remark also that i) implies that if two divisors differ by a principal divisor, they have the same class. It is clear that the map is homomorphic, for a multiplication of divisors (on a fixed U) induces an addition of the associated 1-cochains and hence also of the m_{ijk}.

i) Now let $\alpha = (F)$. Then $F_i \sim F$ on u_i for each i, so that $F_{ij} = F_i/F_j \sim 1$. Hence we may choose $f_{ij} = 0$ for every i,j.

ii) $c(\alpha) = 0$ implies $m_{ijk} = n_{ij} + n_{jk} + n_{ki}$, where the n_{ij} are integral. Now redefine f_{ij} by setting $\hat{f}_{ij} = f_{ij} - n_{ij}$. Note that $F_{ij} = e^{f_{ij}2\pi i} = e^{\hat{f}_{ij}2\pi i}$. But now the $\hat{f}_{ij}$ is a cocycle, for

$$\hat{f}_{ij} + \hat{f}_{jk} + \hat{f}_{ki} = m_{ijk} - (n_{ij} + n_{jk} + n_{ki}) = 0.$$

Hence the $\hat{f}_{ij}$ are also coboundaries as $H^1(X,\mathcal{O}) = 0$, so $\hat{f}_{ij} = f_i - f_j$, and now we are done, for $F_{ij} = e^{2\pi i f_i}\, e^{-2\pi i f_j} = F_i/F_j$, so $F_j e^{-2\pi i f_j} = F_i e^{-2\pi i f_i}$. Hence the function $F = F_i e^{-2\pi i f_i}$ in u_i is globally defined, and clearly solves C.II.

<u>Corollary 1</u>. Let X be a complex manifold. If $H^2(X,\mathbb{Z}) = H^1(X,\mathcal{O}) = 0$, then C.II is solvable.

<u>Corollary 2</u>. Cousin II is solvable in any polydisc X.

<u>Proof</u>. We have shown C.I is solvable in any polydisc, i.e. that $H^1(X,\mathcal{O}) = 0$; $H^2(X,\mathbb{Z}) = 0$ is a well-known topological result.

Chapter 10. Runge Regions

Runge regions are regions in which Runge's theorem can be generalized: the theorem states that in a simply connected domain in the finite plane a holomorphic function can be expressed as a normally convergent series of polynomials. In other words, given an $\varepsilon > 0$, a compact subset K of a simply connected domain D, and a holomorphic function f in D, there is a polynomial p such that $|f-p| < \varepsilon$ on K. Note that in a multiply connected domain it may be impossible to represent a holomorphic function by an infinite sum of polynomials.

§1. Preliminaries

Let X and Y be complex manifolds of the same dimension with $Y \overset{\text{open}}{\subset} X$.

Definition 50. Y has the Runge property relative to X if every holomorphic function in Y can be represented as a series of functions holomorphic in X converging normally in Y, i.e. given $K \overset{\text{compact}}{\subset} Y$, and $\varepsilon > 0$, for every holomorphic function f in Y there is a holomorphic function g in X such that $|f-g| < \varepsilon$ on K.

Lemma α. Let X be a complex manifold of dimension n and $H^q(X, \mathcal{O}) = 0$ for all $q > 0$. Let $f_1, \ldots, f_r$ be holomorphic functions in X such that if, at $p \in X$, $f_1 = f_2 = \ldots = f_k = 0$ for k fixed, $k = 1, \ldots, r$, then the rank of the matrix $(\partial f_i / \partial z_\ell)$ is $\min(k,n)$, where $i = 1, \ldots, k$; $\ell = 1, \ldots, n$; and z_ℓ are suitable local parameters. Let $Y_j = \{p \in X \mid f_j(p) = 0\}$. Then
(1) $Z = Y_1 \cap Y_2 \cap \ldots \cap Y_r$ is a regularly imbedded manifold of dimension $n-r$ and $H^q(Z, \mathcal{O}) = 0$ for all $q > 0$.
(2) Every holomorphic function on Z is the restriction of a function holomorphic in X.

Proof. (1) Define $Z_0 = X$; $Z_1 = Y_1$; $Z_2 = Y_1 \cap Y_2$; $\ldots$; $Z_r = Z$. Z_{j+1} is a globally presented, regularly imbedded, analytic hypersurface in Z_j. By Theorem 20 (p. 73), and since Z_0 has trivial cohomology groups of all positive dimensions, so has Z_1. But then so has Z_2, etc. Hence $H^q(Z,\mathcal{O}) = 0$ for $q > 0$.

(2) Since $H^1(Z_{r-1},\mathcal{O}) = 0$, the first Cousin problem is solvable in Z_{r-1}. Since $Z = Z_r$ is a globally presented regularly imbedded analytic hypersurface in Z_{r-1}, by the extension theorem, (Theorem 17, p. 59), every holomorphic function in Z_r can be continued holomorphically to Z_{r-1}. Similarly, every holomorphic function in Z_{r-1} can be continued holomorphically to Z_{r-2}, and so on, down to $Z_0 = X$.

Lemma β. Let $X^{\text{open}} \subset \mathbb{C}^n$ and let $X_j^{\text{open}} \subset X$, $X_j \subset\subset X_{j+1}$, $X = \bigcup X_j$, $H^q(X_j,\mathcal{O}) = 0$ for all $q > 0$, and let X_j have the Runge property with respect to X_{j+1}. Then

(1) Every X_j has the Runge property with respect to X.

(2) $H^q(X,\mathcal{O}) = 0$ for $q > 0$.

Proof. (1) Fix j. Let $K_j^{\text{compact}} \subset X_j$, $\varepsilon > 0$ be given, and let f be any function holomorphic in X_j. Choose $\{\varepsilon_i\}$ with $\sum \varepsilon_i = \varepsilon$. Let $\{K_i\}$ $i \geq j$ be a sequence of compact sets $K_i \subset\subset K_{i+1}$, $K_i \subset X_{i+1}$, $\bigcup K_i = X$, and K_i is the given set K_j for $i = j$. By hypothesis, there is a function g_1 holomorphic in X_{j+1} with $|f-g_1| < \varepsilon_1$ on K_j. Similarly, for $K_{j+1}^{\text{compact}} \subset X_{j+1}$, since X_{j+1} has the Runge property with respect to X_{j+2}, there is a g_2 holomorphic in X_{j+2} with $|g_2-g_1| < \varepsilon_2$ on K_{j+1}, etc. Since $\sum \varepsilon_i < \infty$, $\lim\limits_{k\to\infty} g_k = g$ exists uniformly on compact subsets K of X, because every $K \subset K_\ell$ for some ℓ. g is therefore holomorphic in X and satisfies $|f-g| \leq |f-g_\ell| + |g_\ell-g_{\ell+1}| + \ldots < \varepsilon_\ell + \varepsilon_{\ell+1} + \ldots \leq \varepsilon$ on K_j.

(2) By Dolbeault's theorem, $H^q(X,\mathcal{O}) = 0$ for $q > 0$ if and only if $\dfrac{\bar\partial \text{ closed } (0,q) \text{ forms}}{\bar\partial \text{ exact } (0,q) \text{ forms}} = 0$ for $q > 0$, i.e., if and only if for a differential form α in X of type $(0,q)$ with $\bar\partial\alpha = 0$ there is a β of type $(0,q-1)$ such that $\alpha = \bar\partial\beta$.

(a) Let $q = 1$, and let α, defined in X, be a $(0,1)$ form with $\bar{\partial}\alpha = 0$. By Dolbeault's theorem for X_j, there are differentiable functions $\beta_1, \beta_2, \ldots$ defined in $X_1, X_2, \ldots$ respectively, such that $\bar{\partial}\beta_j = \alpha$ in X_j. Consider the sum: $\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_2) + \ldots$. In X_1 and X_2, $\bar{\partial}(\beta_2 - \beta_3) = 0$ and hence $\beta_2 - \beta_3$ being holomorphic can be approximated on any compact subset of X_2 by a holomorphic function P_{23} on X. Set $\hat{\beta}_2 = \beta_2$, $\hat{\beta}_3 = \beta_3 - P_{23}$, etc., using the Runge trick as before (cf. proof of Theorem 23, p. 88).

(b) Let $q > 1$, and let α, defined in X, be of type $(0,q)$ with $\bar{\partial}\alpha = 0$. Again, there are β_j in X_j such that $\bar{\partial}\beta_j = \alpha$ in X_j, where β_j has type $(0,q-1)$. Since $\bar{\partial}(\beta_3 - \beta_2) = 0$ in X_2, $(\beta_3 - \beta_2) = \bar{\partial}\hat{\gamma}_2$ in X_2. Let $\hat{\beta}_3 = \beta_3 - \bar{\partial}\hat{\gamma}_2$, etc. (cf. p. 88).

§2. Polynomial Polyhedra

Definition 51. Let $p_1, \ldots, p_r$ be polynomials in $z_1, \ldots, z_n$. Let $A = \{z \mid |p_j(z)| < 1$ for $j = 1, \ldots, r\}$. A is an open set. If $A \subset\subset \mathbb{C}^n$, then A is called a polynomial polyhedron (of dimension n).

Note that a polynomial polyhedron is a region of holomorphy.

Theorem 32. (Oka-Weil). Let X be a polynomial polyhedron of dimension n, then

(1) $H^q(X, \mathcal{O}) = 0$ for all $q > 0$

(2) If $f(z)$ is holomorphic in X then $f = \sum q_j$ where the q_j are polynomials in $z_1, \ldots, z_n$ and the p_k used to define X, and the sum converges normally in X.

Proof. (1) X is a bounded set and therefore lies in a polydisc; assume X lies in the unit polydisc, i.e. $X \subset (|z_j| < 1,\ j = 1, \ldots, n)$. Consider $\mathbb{C}^{n+r}$ where r is the number of polynomials defining X. In $\mathbb{C}^{n+r}$ consider $\Sigma = (z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_r) \mid |z_j| \leq 1,\ j = 1, \ldots, n,$ $\zeta_i = p_i(z),\ i = 1, \ldots, r\}$. Σ is closed. Define $\Sigma_0 = \Sigma \cap (|z_j| < 1,\ |\zeta_i| < 1)$. Σ_0 is closed in the open polydisc. Consider the analytic hypersurfaces $0 = f_i(z_1,\dots,z_n,\zeta_1,\dots,\zeta_r) = \zeta_i - p_i(z_1,\dots,z_n)$. The f_i are defined everywhere and are clearly holomorphic functions in C^{n+r}. With $(|z_j| < 1,\ |\zeta_i| < 1)$ as the complex manifold X in lemma α and noting that the Jacobian of the f_i has maximal rank everywhere because $\partial f_i/\partial\zeta_k = \delta_{ik}$, the hypothesis of lemma α is satisfied and since $Z = \sum_o$, $H^q(\sum_o, \mathcal{O}) = 0$, for all $q > 0$ and every holomorphic function on $\sum_o$ is a restriction of a holomorphic function in the open polydisc $(|z_j| < 1,\ |\zeta_i| < 1)$. But a holomorphic function in the open polydisc can be written as a power series. Hence every holomorphic function on $\sum_o$ is the restriction of a series, $\sum a_{j_1\dots j_n i_1\dots i_r} z_1^{j_1}\dots z_n^{j_n}\zeta_1^{i_1}\dots\zeta_r^{i_r}$, converging normally in $\sum_o$.

We claim that $\sum_o$ is holomorphically equivalent to X. For, define the mapping $(z_1,\dots,z_n) \to (z_1,\dots,z_n, p_1(z),\dots,p_n(z))$. It is of rank n and one-to-one. The preimage of $\sum_o$ is X. Hence $H^q(X,\mathcal{O}) = H^q(\sum_o,\mathcal{O}) = 0$ for all $q > 0$.

(2) We have already obtained that every holomorphic function on $\sum_o$ is a restriction of a series $\sum a_{j_1\dots i_r} z_1^{j_1}\dots\zeta_r^{i_r}$ converging normally in $\sum_o$. But <u>on</u> $\sum_o$, $\zeta_k = p_k(z)$ and thus the above series is a series in <u>only</u> the z_j, converging normally in X.

§3. <u>Runge domains</u>

<u>Definition 52</u>. Let $K \subset\subset \mathbb{C}^n$. The <u>polynomial hull</u> of K, $K^* = \{z_0 \mid \text{for every polynomial } p \text{ with } |p(z)| \le 1 \text{ on } K,\ |p(z_0)| \le 1\}$.

<u>Definition 53</u>. $X^{\text{open}} \subset \mathbb{C}^n$ is <u>polynomially convex</u> if $K \subset\subset X$ implies that $K^* \subset\subset X$.

<u>Definition 54</u>. A <u>Runge region</u> is a region of holomorphy in which every holomorphic function can be expanded in a

normally convergent series of polynomials.

Theorem 33. Let $X^{\text{open}} \subset \mathbb{C}^n$. The following statements are equivalent:

(1) X is polynomially convex.

(2) $X = \bigcup X_j$, where the X_j are polynomial polyhedra, $X_j \subset\subset X_{j+1}$.

(3) X is a Runge region.

Proof. (1) implies (2). Since X is polynomially convex, X is holomorphically convex. By the Cartan-Thullen theorem, X is a region of holomorphy. Continue the proof by adapting the proof for analytic polyhedra, Theorem 7 and its corollary, (p.25).

(2) implies (3). By the Oka-Weil Theorem, Theorem 32, each polynomial polyhedron is Runge in the next one, and $H^q(X_j, \mathcal{O}) = 0$ for all $q < 0$. Apply lemma β. $H^q(X, \mathcal{O}) = 0$ for $q > 0$. Hence by Theorem 21 (p. 75), X is a region of holomorphy. Let f be holomorphic in X and $K^{\text{compact}} \subset X$. Then $K \subset X_j$, for some j, and by the Oka-Weil Theorem, f can be approximated as closely as desired by a polynomial in K. Hence X is a Runge region.

(3) implies (1). Since X is a Runge region it is a region of holomorphy and therefore is holomorphically convex, i.e. if $K \subset\subset X$, then $\hat{K} \subset\subset X$ where $\hat{K}$ is the hull of K with respect to holomorphic functions on X. We claim that $\hat{K} = K^*$ and hence $K^* \subset\subset X$. Indeed, $\hat{K} \subset K^*$ since the family of all holomorphic functions on X is larger than the family of polynomials. It remains to show that $K^* \subset \hat{K}$. Let $z_0 \in K^*$ and let f be a holomorphic function in X such that $|f(z)| \leq 1$ for $z \in K$. For every $\varepsilon > 0$, there is a polynomial $p(z)$ satisfying $|p(z)-f(z)| \leq \varepsilon$ on $K \cup \{z_0\}$, since X is a Runge region. But then $|p(z)| \leq 1 + \varepsilon$ on K, and since $z_0 \in K^*$, $|p(z_0)| \leq 1 + \varepsilon$. Hence $|f(z_0)| \leq 1 + 2\varepsilon$, and because ε is arbitrary, $|f(z_0)| \leq 1$, i.e. $z_0 \in \hat{K}$.

Chapter 11. Cohomology of Domains of Holomorphy

§1. Fundamental Lemma, stated

(1) Let $K^{\text{compact}} \subset \mathbb{C}^n$. Let K^* denote the polynomial hull of K; $K^* = \{z \mid |P(z)| \leq \max_K |P(\zeta)|$ for every polynomial $P\}$. Note that K^* is bounded and compact and that

$$K^* = \bigcap \{D \mid D \text{ is a polynomial polyhedron, } K \subset D\}$$

(Proof easy).

(2) Recall that an analytic polyhedron $D \subset \mathbb{C}^n$ was defined as follows: there exists an open set $G \subset \mathbb{C}^n$ and functions $f_1, \ldots, f_\nu$ holomorphic in G such that $D \subset\subset G$ and $D \subset \{z \mid z \in G,\ |f_j(z)| < 1,\ j = 1, \ldots, \nu\}$. Since $D \subset\subset G$, hence bounded, we shall assume that $G \subset \{\|z\| < 1\}$, the unit polydisc.

(3) Let D be an analytic polyhedron. Then Σ, the Oka image of the closure of D, is defined as follows

$$\Sigma = \{(z,\zeta) \mid |z_i|, |\zeta_j| \leq 1,\ z \in G,\ \zeta_j = f_j(z_1, \ldots, z_n)\ ;$$
$$i = 1, \ldots, n;\quad j = 1, \ldots, \nu\}.$$

$\Sigma \subset \mathbb{C}^{n+\nu}$, and is closed.

We now state the fundamental lemma.

Fundamental Lemma (Oka). Let Σ be as above. Then $\Sigma = \Sigma^*$.

We shall prove the lemma in this chapter; the proof of a more general form of the lemma will come later (p. 196).

§2. Applications of the Fundamental Lemma

A. Let $D \subset\subset G^{\text{open}} \subset \mathbb{C}^n$ be an analytic polyhedron. Observe that the Oka mapping of $\text{cl } D \to \Sigma \subset \mathbb{C}^{n+\nu}$,

$$(z_1, \ldots, z_n) \to (z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_\nu)\ ,$$

given by: $z_i = z_i$, $\zeta_j = f_j(z_1, \ldots, z_n)$; is defined on all of G: call the image of G under this mapping "Σ_1". Note that $\sum$ is closed and $\subset \sum_1$. Hence there exists an $\varepsilon > 0$ such that if $\operatorname{dist}[(z_1,\ldots,z_n,\zeta_1,\ldots,\zeta_v), \sum] < \varepsilon$, then $(z_1,\ldots,z_n) \in G$; where the distance of a point $p \in \mathbb{C}^m$ from a set $S \subset \mathbb{C}^m$ is defined as $\inf_{s\in S}\|p-s\| = \inf_{s\in S}\{\max_{i=1,\ldots,m} |p_i-s_i|\}$ Let $\sum^\varepsilon$ denote the ε-neighborhood of $\sum$, defined by $\sum^\varepsilon = \{w \mid \operatorname{dist}(w, \sum) < \varepsilon\}$.

Applying the fundamental lemma, there exists a polynomial polyhedron $\Delta_1 \subset \mathbb{C}^{n+v}$, such that $\sum \subset\subset \Delta_1 \subset \sum^\varepsilon$. Let $\Delta = \Delta_1 \cap \{|z_i| < 1, |\zeta_j| < 1\}$, and note that Δ is also a polynomial polyhedron. We now claim that $\sum_0 = \sum \cap \{|z_i| < 1, |\zeta_j| < 1\}$, the Oka image of D, is the intersection of globally presented, regularly imbedded hypersurfaces of Δ, satisfying the maximal rank condition of lemma α, (p. 110). Indeed, the hypersurfaces of Δ, $Y_j = \{\zeta_j - f_j(z_1,\ldots,z_n) = 0\}$, $j = 1,\ldots,v$ satisfy $\bigcap_{j=1}^{v} Y_j = \sum_0$ and also the maximal rank condition, for the matrix $(\delta_{jk}) = \left(\frac{\partial(\zeta_j - f_j(z_1,\ldots,z_n))}{\partial\zeta_k}\right)$ is of maximal rank. Hence lemma α applies.

Since $\sum_0$ is holomorphically equivalent to D, we have established:

Theorem 34. If D is an analytic polyhedron, then $H^q(D,\mathcal{O}) = 0$, $q > 0$.

B. Using the Oka mapping, we may consider any function f, holomorphic in D, as a holomorphic function on $\sum_0$. By the above argument and lemma α, f is the restriction of a function g holomorphic on Δ. Now by Theorem 32 (p. 112), $g = \sum \psi_j(z_1,\ldots,z_n,\zeta_1,\ldots,\zeta_v)$, normally convergent on Δ, where the ψ_j are polynomials. This series therefore converges normally on $\sum_0$, hence on D: $f(z_1,\ldots,z_n) = \sum \psi_j(z_1,\ldots,z_n,f_1,\ldots,f_v)$, and we have also established the following:

Theorem 35 (Oka-Weil Approximation Theorem). Let D be an analytic polyhedron in G defined by $f_1,\ldots,f_v$; and let f be holomorphic in D. Then

$$f = \sum \psi_j(z_1,\ldots,z_n,f_1,\ldots,f_v)$$

where the ψ_j are polynomials and the series converges normally.

Corollary. If D is an analytic polyhedron in G, then any function holomorphic in D may be represented by a sequence of holomorphic functions defined in G and normally convergent in D, i.e. D has the Runge property relative to G.

Theorem 36 (Oka). If D is a region of holomorphy, then $H^q(D,\mathcal{O}) = 0$, $q > 0$.

Corollary. C.I is solvable in every region of holomorphy.

Remark. Note that this theorem completes the proof of the de Rham theorems of Chapter 8.

Proof (of Theorem 36). Note that every analytic polyhedron is a region of holomorphy. Exhaust D by a sequence of analytic polyhedra $D_j \subset\subset D_{j+1} \overset{\text{open}}{\subset} D$, $\bigcup_{j=1}^{\infty} D_j = D$; compare corollary to Theorem 7, p. 25. Now, by Theorem 34, $H^q(D_j,\mathcal{O}) = 0$, $q > 0$, $j = 1,2,\ldots$; and by the Corollary to Theorem 35, D_j has the Runge property relative to D_{j+1}. Hence we may apply lemma β.

§3. Preparation for the proof of the fundamental lemma

A. Proposition. Let $K \overset{\text{compact}}{\subset} \mathbb{C}^n$, and let K^* denote its polynomial hull. Let $G \overset{\text{open}}{\subset} \mathbb{C}^n$, and $[a \le t \le b] \subset T \overset{\text{open}}{\subset\subset} \mathbb{C}$. Let $F(z,t)$ be holomorphic in $G \times T$, continuous in $\bar{G} \times \bar{T}$, and such that $\{z \mid z \in \text{bdry } G,\ F(z,t) = 0,\ t \in \bar{T}\} \cap K^* = \phi$, and $F(z,t) \ne 0$ on $(K \cap G) \times T$. Then either, for each $t \in [a,b]$ there exists $z \in K^* \cap G$ such that $F(z,t) = 0$; or $F(z,t) \ne 0$ on $(K^* \cap G) \times [a,b]$.

Proof. We may assume T to be a polynomial polyhedron. Let U be an open neighborhood of K^*, which we may also take to be a polynomial polyhedron, for $(K^*)^* = K^*$. Further, since $F(z,t) \ne 0$ in $(K^* \cap \text{bdry } G) \times T$, we may choose U such that $F(z,t) \ne 0$ in $(U \cap \text{bdry } G) \times T$. Therefore there is an open set $H \subset\subset G$ such that $F(z,t) \ne 0$ in $U \cap (G - \text{cl } H) \times T$.

Now $U \times T$ is a polynomial polyhedron, hence C.I is solvable (cf. Theorem 32). We are able to find a meromorphic function $G(z,t)$ in $U \times T$ with poles at the zeroes of $F(z,t)$.

To this end, we pose the following problem: In $(U \cap G) \times T$, set $f_1 = 1/F(z,t)$; in $(U \cap (C^n - \mathrm{cl}\ H)) \times T$, set $f_2 = 1$. This is a well-posed problem, for $F(z,t) \neq 0$ in $(U \cap (G - \mathrm{cl}\ H)) \times T$. Let $G(z,t)$ be the solution of this problem. Set $Y_t = \{z \mid z \in U \cap G;\ F(z,t) = 0\}$. Assume the Proposition is false; i.e. that $Y_{t_0} \cap K^* \neq \phi$ for some $t_0 \in [a,b]$; $Y_{t_1} \cap K^* = \phi$ for some $t_1 \in [a,b]$. Now $\{t \mid Y_t \cap K^* \neq \phi\}$ is closed; hence there exist numbers $b^0, b^1 \in [a,b]$ such that for $b^0 < t < b^1$, $Y_t \cap K^* = \phi$ and $Y_{b^0} \cap K^* \neq \phi$.
Choose $t \in (b^0, b^1)$. Then there exists an open neighborhood V of K^* such that $K^* \subset V \subset U$, and $Y_t \cap V = \phi$. We may assume V is a polynomial polyhedron, as any smaller neighborhood has the same property of non-intersection. Consider $G(z,t)$ for this fixed t; it is holomorphic in V, and can therefore be expanded in a normally convergent series of polynomials in V (Theorem 32). Now $G(z,t)$, for $a \leq t \leq b$, is continuous in K; so $|G|$ has a maximum here: $\underset{z \in K;\ a<t<b}{|G(z,t)|} \leq A$. But $G(z,t) = \lim P_j(z)$, normally in V. Let $\varepsilon > 0$ be given, and let N be sufficiently large so that $j > N$ implies $|G-P_j| < \varepsilon$ on K. Hence $|P_j(z)| < A + \varepsilon$ on K for every $j > N$, so $|P_j| \leq A + \varepsilon$ on K^*, implying $|G(z,t)| \leq A + 2\varepsilon$ for $z \in K^*$. But there exists a $\zeta \in Y_{b^0} \cap K^*$, and $\lim_{t \downarrow b^0} |G(\zeta,t)| \leq A + 2\varepsilon$, although G has a pole at (ζ, b^0); and this contradiction establishes the proposition.

B. We now establish some properties of subharmonic functions; their definition may be restated as follows:

Let $D^{\text{open}} \subset \mathbf{C}$. Then $\phi(z)$ is <u>subharmonic in D</u> if:

(i) $-\infty \leq \phi(z) < \infty$

(ii) ϕ is upper semicontinuous; i.e. $\overline{\lim}_{z_n \to b} \phi(z_n) \leq \phi(b)$

(iii) for every $z \in D$, and each disc $\overline{D(z,\rho)} = \{\zeta \mid |\zeta - z| \leq \rho\} \subset D$,

$$\phi(z) \leq \frac{1}{2\pi\rho} \int_{|\zeta-z|=\rho} \phi(\zeta)\, d\zeta \ .$$

Note. (i) and (ii) imply ϕ is measurable, for ϕ may be approximated from above by continuous functions.

Proposition 1. Assume ϕ to be subharmonic in D; then ϕ possesses the following properties:

(1) ϕ is bounded from above on compact sets

(2) If $\phi \equiv -\infty$ in a neighborhood of some point z of D, then $\phi \equiv -\infty$ on the component of D containing z.

(3) (Maximum principle). Let $K \overset{\text{compact}}{\subset} D$. Then $\phi(z) \leq A$ on bdry K implies $\phi(z) \leq A$ in K.

(4) Let $\overline{D(z_0,r)} \subset D$, and assume $\phi \equiv$ constant in $D(z_0,r)$. Then $\phi \equiv$ constant on $\overline{D(z_0,r)}$.

Proof. (1) follows from (ii) and (2), (3) from (iii). We prove (4): We may assume $z_0 = 0$, $r = 1$, and $\phi \equiv 0$ on $D(0,1) = \{|z| < 1\}$. Then $\phi \geq 0$ on $\{|z| = 1\}$, using (ii). Assume, e.g. $\phi(1) > 0$; and let $1 > \varepsilon > 0$, $R > 1$ such that $\overline{D(0,1-\varepsilon)} \subsetneqq \overline{D(0,1)} \subsetneqq \overline{D(0,R)} \subset D$. Then $\phi \leq A$ in $\overline{D(0,R)}$. Define

$$u(z) = A \frac{\log |z| - \log |1-\varepsilon|}{\log R - \log |1-\varepsilon|}.$$

Then u is harmonic in $\overline{D(0,R)} - \overline{D(0,1-\varepsilon)}$, $\phi \leq u$ and $u = 0$ for $|z| = 1 - \varepsilon$. Hence, $\phi(1) \leq A \log \frac{1}{1-\varepsilon} / \log \frac{R}{1-\varepsilon}$; letting $\varepsilon \downarrow 0$, we obtain $\phi(1) = 0$.

Proposition 2. If $\phi \geq 0$ and $\log \phi$ is subharmonic, then ϕ is subharmonic.

Proof. (i) and (ii) follow from the monotonicity of "log", and (iii) from the inequality:

$$\log \left[\frac{1}{2\pi r} \int_{|\zeta - z_0| = r} \phi(\zeta)\, d\zeta\right] \geq \frac{1}{2\pi r} \int_{|\zeta - z_0| = r} \log \phi(\zeta)\, d\zeta .$$

Proposition 3. Let $\phi(z) \geq 0$ be defined and upper semicontinuous in $D \subset \mathbb{C}$ such that $\log \phi$ is not subharmonic at $z_0 \in D$, and $\phi(z_0) \neq 0$. Then there exists a disc $\overline{D(z_0,r)} \subset D$ and a function ψ holomorphic in $D(z_0,r) \subset D$, and continuous in $\overline{D(z_0,r)}$, such that $\phi|\psi| < 1$ on $\{|z-z_0|=r\}$ and $\phi(z_0)\ |\psi(z_0)| = 1$.

Proof. We may assume $z_0 = 0$. Let $r > 0$ be chosen such that $\overline{D(0,r)} \subset D$ and

$$\log\phi(0) > \frac{1}{2\pi r} \int_{|z|=r} \log \phi(z)\, dz \quad .$$

Using upper semicontinuity, let ϕ_n be continuously differentiable in D, $\phi_n \downarrow \phi$. Let $h_n(z)$ be the harmonic function for which $\log \phi_n(z) = h_n(z)$ on $|z|=r$. Let $\tilde{h}_n$ be the conjugate harmonic function, and set
$\psi_n = e^{-h_n - i\tilde{h}_n}$.

Now on $\{|z|=r\}$, $\log \phi|\psi_n| = (\log \phi) - h_n = \log \phi - \log \phi_n < 0$, i.e. $\phi|\psi_n| < 1$.

If $\phi(0)\ |\psi_n(0)| = K \geq 1$, then $\phi \frac{1}{K} |\psi_n| \leq \phi|\psi_n|$, so $\frac{1}{K} |\psi_n(z)|$ is the required function.

If $\phi(0)\ |\psi_n(0)| < 1$, then $\ln \phi(0) + \ln |\psi_n(0)| < 0$, i.e. $\ln \phi(0) < h_n(0) = \frac{1}{2\pi r} \int_{|z|=r} h_n(z)\, dz = \frac{1}{2\pi r}\int_{|z|=r} \ln \phi_n(z)\, dz$, therefore $\ln \phi(0) \leq \lim_{n\to\infty} 1/2\pi r \int_{|z|=r} \ln \phi_n(z)\, dz$, so

$\ln \phi(0) \leq \frac{1}{2\pi r} \int_{|z|=r} \ln \phi(z)\, dz$, contradicting choice of disc.

§4. Proof of the fundamental lemma

Statement. Let $G^{open} \subset\subset \mathbb{C}^n$ and assume (for the sake of simplicity) that $G \subset\subset (|z_j|<1)$.
Let $f_1(z_1,\ldots,z_n),\ldots,f_v(z_1,\ldots,z_n)$ be holomorphic in G and continuous on the closure of G. Denote by
$\tilde{D} = \{(z_1,\ldots,z_n) \mid (z_1,\ldots,z_n) \in G$ and $|f_i(z_1,\ldots,z_n)| \leq 1$ for $i = 1,\ldots,v\}$: $\tilde{D}$ is compact. Let $\sum = \{(z_1,\ldots,z_n,\zeta_1,\ldots,\zeta_v) \in \mathbb{C}^{n+v} \mid |z_j| \leq 1,\ |\zeta_i| \leq 1,\ (z_1,\ldots,z_n) \in G$ and $\zeta_i = f_i(z_1,\ldots,z_n)$ for $i = 1,\ldots,v\}$, the Oka image of $\tilde{D}$. Then $\sum = \sum^*$.

Proof. Let $(z_2,\ldots,z_n) = Z$, $(\zeta_1,\ldots,\zeta_v) = \zeta$ and hence

$(z_1,z_2,\dots,z_n,\zeta_1,\dots,\zeta_\nu) = (z_1,Z,\zeta)$. Let $\sum(z)$ mean $\sum \cap [z_1=z]$ and $\sum^*(z)$ mean $\sum^* \cap [z_1=z]$. Denote by $S = \{z \mid$ there is a (Z,ζ) for which $(z,Z,\zeta) \in \sum^*\}$: S is a closed bounded set. Let $\Omega_0 = \{z \mid$ there is a (Z,ζ) for which $(z,Z,\zeta) \in \sum^*$ and $(z,Z) \notin G\}$: Ω_0 is a closed set $\subset S$. Call the complement of Ω_0, Ω : Ω is an open unbounded set.

1) For $z \in \Omega$ and $j = 1,\dots,\nu$, set

$$R_j(z) = \begin{cases} 0 & \text{if } z \notin S \\ \max |\zeta_j - f_j(z,Z)| \text{ over all} & \\ \quad (z,Z,\zeta) \in \sum^* \text{ if } z \in S . & \end{cases}$$

<u>If, for fixed $\hat{z}$ and all</u> $j = 1,\dots,\nu$, $R_j(\hat{z}) = 0$, <u>then</u> $\sum(\hat{z}) = \sum^*(\hat{z})$. Proof: If $\hat{z} \notin S$, then $\sum^*(\hat{z}) = \phi$ and, since $\sum \subset \sum^*$ and hence $\sum(\hat{z}) \subset \sum^*(\hat{z})$, $\sum(\hat{z}) = \phi$. If $\hat{z} \in S$, then for all $j = 1,\dots,\nu$, $\zeta_j = f_j(\hat{z},Z)$ for every $(\hat{z},Z,\zeta) \in \sum^*$. Since $\hat{z} \in \Omega$, $\hat{z} \notin \Omega_0$, and so $(\hat{z},Z) \in G$ for every $(\hat{z},Z,\zeta) \in \sum^*$. Hence for every $(\hat{z},Z,\zeta) \in \sum^*(z)$, $\zeta_j = f_j(\hat{z},Z)$ and $(\hat{z},Z) \in G$. Therefore $(\hat{z},Z,\zeta) \in \sum(\hat{z})$.

2) $\log R_j(z)$ <u>is subharmonic for each</u> $j = 1,\dots,\nu$. Proof: It is enough to consider R_1. $R_1 \geq 0$ and is upper semicontinuous, i.e. $\lim\limits_{z_j \to z} R_1(z_j) \leq R_1(z)$. For, if $z \notin S$ then $R_1(z) = 0$. If $z \in S$ and $z^j \to z$, then for large j, either $z^j \notin S$ and then $\lim\limits_{z^j \to z} R_1(z^j) = 0 \leq R_1(z)$, or $z^j \in S$ and then $R_1(z^j) = \max |\zeta_1 - f_1| = |\zeta_1^j - f_1(z^j,Z^j)|$ for some $(z^j,Z^j,\zeta^j) \in \sum^*$. By the compactness of $\sum^*$, there is a subsequence $\{(z^j,Z^j,\zeta^j)\}$ converging to a point $(z,\hat{Z},\hat{\zeta}) \in \sum^*$ and $\lim\limits_{z^j \to z} R_1(z^j) = |\hat{\zeta}_1 - f_1(z,\hat{Z})| \leq \max |\zeta_1 - f_1(z,Z)| = R_1(z)$. Hence R_1 is upper semicontinuous.

Now, assume that $\log R_1$ is not subharmonic at some $z_0 \in \Omega$, then there is a closed disc in Ω, $\{|z-z_0| \leq \rho\}$, and a function $\psi(z)$ holomorphic in the open disc and continuous on its closure, such that $R_1(z)\ |\psi(z)| < 1$ on $|z-z_0| = \rho$, and therefore $< 1 - 2\varepsilon$ for some $1/4 > \varepsilon > 0$, and $R_1(z_0)\ |\psi(z_0)| = 1$. Define $F(z,Z,\zeta,t) = (\zeta_1 - f_1(z,Z))\psi(z) - (1+t)e^{i\alpha}$ for all ζ, $(z,Z) \in G$, $|z-z_0| < \rho$, and $t \in T = \{t \in \mathbb{C} \mid |t-t_0| < \varepsilon$ and $t_0 \in [0,a]\}$, where α and a are fixed real numbers to be determined later. We claim that this F satisfies the hypothesis of the proposition with $K = \sum \subset \mathbb{C}^{n+\nu}$, $K^* = \sum^*$, $T = T$ and G in the proposition, call it G_p, $= G \cap (|z-z_0| < \rho)$ and all ζ. Indeed:

a) F is holomorphic in $G_p \times T$ and continuous in $\mathrm{cl}\ G_p \times \mathrm{cl}\ T$.

b) If $\Gamma = \{z \mid z \in \mathrm{bdry}\ (G \cap (|z-z_0|<\rho))$, $F(z,Z,\zeta,t)=0$ for some $t \in \mathrm{cl}\ T\}$, then $\Gamma \cap \sum^* = \phi$.

Proof: $\mathrm{Bdry}\ (G \cap (|z-z_0|<\rho)) = ((|z-z_0| \leq \rho) \cap \partial G) \cup (|z-z_0|=\rho) \cap G)$.

(i) If $(z,Z) \in \mathrm{bdry}\ G$ and $|z-z_0| \leq \rho$, then, since $(|z-z_0| \leq \rho) \subset \Omega = (\mathbb{C}-\Omega_0)$ and $(z,Z) \notin G$, there is no ζ for which $(z,Z,\zeta) \in \sum^*$. Therefore, for these boundary points, $\Gamma \cap \sum^* = \phi$.

(ii) If $(z,Z) \in ((|z-z_0|=\rho) \cap G)$ and $\Gamma \cap \sum^* \neq \phi$ then there is a $\tilde{z} \in (\Gamma \cap S)$ and hence for this $\tilde{z}$, $R_1(\tilde{z}) = \max |\zeta_1 - f_1(\tilde{z},Z)|$. Since $|\tilde{z}-z_0| = \rho$, $R_1(\tilde{z})\ |\psi(\tilde{z})| < 1-2\varepsilon$. But then $1-2\varepsilon > (\max |\zeta_1 - f_1(\tilde{z},Z)|)\ |\psi(\tilde{z})| \geq |\zeta_1 - f_1(\tilde{z},Z)|\ |\psi(\tilde{z})|$ while $|1+t|\ |e^{i\alpha}| \geq 1-\varepsilon$ and thus $F(\tilde{z},Z,\zeta,t) \neq 0$, i.e. $\tilde{z} \notin \Gamma$; a contradiction. Hence $\Gamma \cap \sum^* = \phi$.

c) $F(z,Z,\zeta,t) \neq 0$ for $(z,Z,\zeta) \in \sum$ and $t \in \mathrm{cl}\ T$ because on $\sum$, $\zeta_1 = f_1$ and therefore $|F| = (1+t) > 0$.

However, the conclusion of the proposition is violated. For, by choosing $\underline{a}$ sufficiently large, $|(1+a)|$ can be made greater than $|(\zeta_1 - f_1)\psi|$, since the latter is bounded, so that $F \neq 0$ at $t = a$. But for $t = 0$, $F = (\zeta_1 - f_1)\psi - e^{i\alpha} = 0$ at a point of $\sum^* \cap G$: for at z_0, $R_1|\psi| = 1$ implies that $R_1(z_0) \neq 0$ and hence $z_0 \in S$ and $R_1(z_0) = \max |\zeta_1 - f_1(z_0,Z)| = |\hat{\zeta}_1 - f_1(z_0,\hat{Z})|$ for some $(z_0,\hat{Z},\hat{\zeta}) \in \sum^*$.

Thus $|\hat{\zeta}_1 - f_1(z_0,\hat{Z})|\ |\psi(z_0)| = 1$ or $(\hat{\zeta}_1 - f_1(z_0,\hat{Z}))\ \psi(z_0) = e^{i\theta}$. Choose $\alpha = \theta$. Hence $\log R_1(z)$ is subharmonic, and R_1 is then subharmonic by Proposition 2.

Let Ω_1 be an unbounded component of Ω. ($\Omega_1 \neq \phi$ exists because Ω contains the complement of S, a compact set in $\mathbb{C}$.)

3) $\sum(z) = \sum^*(z)$ for $z \in \Omega_1$.

Proof: Firstly, there is a $z \in \Omega_1$, for which $R_j(z) = 0$ for all $j = 1,\ldots,\nu$. For, since Ω_1 is unbounded and S is bounded, $(\Omega_1 \cap (\mathbb{C}-S)) \neq \phi$. Therefore there is a $z \in \Omega_1$ and $z \notin S$ and for this z, $R_j(z) = 0$ for all j. Secondly, since $(\Omega_1 \cap (\mathbb{C}-S))$ is an open set, there is a neighborhood of this z contained in $(\Omega_1 \cap (\mathbb{C}-S))$ in which $R_j = 0$ for all j. By 2), since R_j vanishes in a neighborhood of z, it vanishes in the largest component containing z, i.e. in Ω_1. By 1), then, $\sum(z) = \sum^*(z)$ for $z \in \Omega_1$.

If we can prove that $\Omega_1 = \mathbb{C}$, then we are done. So, assume $\Omega_1 \neq \mathbb{C}$, then Ω_1 has a boundary point ξ. ξ is also a boundary point of Ω_0, and since Ω_0 is closed, $\xi \in \Omega_0$.

4) $\sum(\xi) \neq \phi$.

Proof: Since $\xi \in \Omega_0 \subset S$, either $\xi \in \text{int } S$ or $\xi \in \text{bdry } S$.

a) If $\xi \in \text{int } S$, then there is a sequence of points $\xi^j \to \xi$, $\xi^j \in (S \cap \Omega_1)$. Since $\xi^j \in S$, there are points $(\xi^j, Z^j, \zeta^j) \in \sum^*$, and by 3), $(\xi^j, Z^j, \zeta^j) \in \sum$. By the compactness of $\sum$, there is a convergent subsequence (ξ^j, Z^j, ζ^j), $\xi^j \to \xi$, $Z^j \to Z$, $\zeta^j \to \zeta$ and $(\xi, Z, \zeta) \in \sum$.

b) If $\xi \in \text{bdry } S$ and $\sum(\xi) = \phi$, then for every point $(\hat{\xi}, Z, \zeta)$ with first coordinate sufficiently close to ξ, $\sum(\hat{\xi}) = \phi$, by compactness, i.e. there is an $\varepsilon > 0$ such that if $|\hat{\xi} - \xi| < \varepsilon$ then $\sum(\hat{\xi}) = \phi$. Since $\xi \in \text{bdry } S$, there is a point ξ_1 with $|\xi_1 - \xi| < \varepsilon$ and $\xi_1 \notin S$. Consider the function $F(z, Z, \zeta, t) = z - ((1-t)\xi + t\xi_1)$ for

$t \in T = \{t \in \mathbf{C} \mid |t-t_0| < \varepsilon_1$, and $t_0 \in [0,1]\}$, where ε_1 is chosen such that $(1-t)\xi + t\xi_1$ for $t \in \overline{T}$ is contained in $(|\hat{\xi}-\xi|<\varepsilon)$. F is holomorphic for all (z,Z,ζ,t). $F \neq 0$ for $z \in \sum$ and $t \in \overline{T}$ because $z = (1-t)\xi + t\xi_1$ lies in the disc of radius ε about ξ and by assumption, then, $\sum(z) = \phi$. Hence, F satisfies the hypothesis of the proposition. However, for $t = 0$, $F = z - \xi$, and since $\xi \in S$, $[F=0] \cap \sum^* = [z=\xi] \cap \sum^* = \sum^*(\xi) \neq \phi$; while for $t = 1$ $F = z - \xi_1$ and since $\xi_1 \notin S$, $[F=0] \cap \sum^* = [z=\xi_1] \cap \sum^* = \sum^*(\xi_1) = \phi$, contradicting the conclusion of the proposition. Thus $\sum(\xi) \neq \phi$.

5) $\sum(z)^* = \sum(z)$.

Proof: Use induction on n. Either $n = 1$, or $n = k+1$ and the fundamental lemma holds for $n \leq k$. If $n = 1$, then $\sum(z)$ is a point, namely the point $(z,f_1(z),\ldots,f_v(z))$, and a point is its own polynomial hull. If $n = k+1$, then $\sum(z) = \{(z_1,\ldots,z_{k+1},\zeta_1,\ldots,\zeta_v) \mid |z_j| \leq 1$, $|\zeta_i| \leq 1$, $(z_1,Z) \in G \cap [z_1=z]$ and $\zeta_i = f_i(z,Z)$ for $i = 1,\ldots,v\}$ and therefore is the Oka image of an analytic polyhedron in $\mathbf{C}^k$. By the induction hypothesis, then, $\sum(z) = \sum(z)^*$.

6) There is an $\varepsilon > 0$, a number b, $0 < b < 1$, and N polynomials $P_j(Z,\zeta)$ such that

a) if $|z-\xi| < \varepsilon$ and for all j, $|P_j(Z,\zeta)| < 1$, then $(z,Z) \in G$ and

b) if $|z-\xi| < \varepsilon$ and $(z,Z,\zeta) \in \sum$ then for all j, $|P_j(Z,\zeta)| < b$.

Proof. a) If $(\xi,Z,\zeta) \in \sum$ then $(\xi,Z) \in G$, and there is a point $(\xi,Z,\zeta) \in \sum$ by 4). Since G is open, there is a neighborhood of (ξ,Z,ζ) such that every point in this neighborhood has its first n-coordinates in G; i.e. there is a $\delta > 0$ such that if the distance from (z,Z,ζ) to $\sum(\xi)$ is less than δ, then $(z,Z) \in G$. But since

$\sum(\xi) = \sum(\xi)^*$, any neighborhood of $\sum(\xi)$ contains a smaller neighborhood which is a polynomial polyhedron. Hence we can find an ε_1, and polynomials P_j with the required property a).

b) At every point $(\xi, Z, \zeta) \in \sum$, $|P_j(Z,\zeta)| < 1$, $j = 1,\ldots,N$. We claim that for z sufficiently close to ξ, and $(z,Z,\zeta) \in \sum$, $|P_j(Z,\zeta)| < 1$ for all j. Suppose not, then there is a sequence $\xi^k \to \xi$, and Z^k, ζ^k such that $(\xi^k, Z^k, \zeta^k) \in \sum$ and $\max_j |P_j(\xi^k, Z^k, \zeta^k)| \geq 1$ for each k. Since $\sum$ is compact there is a subsequence (ξ^k, Z^k, ζ^k) which converges to a point $(\xi, \hat{Z}, \hat{\zeta}) \in \sum$. Hence in the limit we obtain a point $\in \sum(\xi)$ where at least one of the polynomials P_j has absolute value ≥ 1; a contradiction. Therefore there is an ε_2 neighborhood N of ξ such that $|P_j(Z,\zeta)| < 1$ for all j and $(z,Z,\zeta) \in (\sum \cap \{z \in N\})$. Choose $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then if $|z-\xi| < \varepsilon$ and $(z,Z,\zeta) \in \sum$, $|P_j(Z,\zeta)| < 1$ for all j, and since $|P_j|$ is continuous, there is a number $0 < b < 1$ for which $|P_j(Z,\zeta)| < b$.

7) For $|z-\xi| < \varepsilon$, set

$$\phi(z) = \begin{cases} b & \text{if } z \notin S \\ \max\,(b, |P_j(Z,\zeta)|) & \text{over } (z,Z,\zeta) \in \sum^* \\ & \text{and } j = 1,\ldots,N \text{ if } z \in S. \end{cases}$$

$\log \phi(z)$ <u>is subharmonic</u>.

Proof: $\phi(z) \geq b > 0$ and is upper semicontinuous. (The proof is similar to that of part 2).) Assume that $\log \phi(z)$ is not subharmonic. Then there is a disc, $C = (|z-\hat{\xi}| \leq \rho)$, $\subset (|z-\xi| < \varepsilon)$ and $\psi(z)$ holomorphic in int C and continuous on C satisfying $\phi(z)\,|\psi(z)| < 1$ on $|z-\hat{\xi}| = \rho$, and thus is $< 1-2\varepsilon$ for some $\varepsilon > 0$, and $\phi(\hat{\xi})\,|\psi(\hat{\xi})| = 1$. By definition, $\phi(z) \geq b$. Therefore on $|z-\hat{\xi}| = \rho$, $|\psi(z)| < 1/b$. Since ψ is holomorphic in int C, $|\psi(\hat{\xi})| < 1/b$ and hence $\phi(\hat{\xi}) > b$. This means that $\hat{\xi} \in S$ and $\phi(\hat{\xi}) = \max |P_j|$. Pick out one polynomial assuming the maximum value, say P_1: $\phi(\hat{\xi}) = |P_1(\hat{Z},\hat{\zeta})|$, $(\hat{\xi},\hat{Z},\hat{\zeta}) \in \sum^*$. Let $F(z,Z,\zeta,t) = P_1(Z,\zeta)\,\psi(z) - e^{i\alpha}(1+t)$ for all z, ζ, $|z-\hat{\xi}| < \rho$, and

$t \in T = \{t \in \mathbf{C} \mid |t-t_0| < \varepsilon$ and $t_0 \in [0,a]\}$ where α and a will be determined later. F satisfies the hypothesis of the proposition.

a) If $z \in (|z-\hat{\xi}|=\rho)$ and $(z,Z,\zeta) \in \sum^*$ then $F \neq 0$ since then $z \in S$ and $\phi = \max(b,|P_j|)$ and satisfies $\phi|\psi| < 1$ or $\phi|\psi| < 1-2\varepsilon$. But all $|P_j| < \phi$ implies that $|P_1|\ |\psi| < 1 - 2\varepsilon$ on $|z-\hat{\xi}| = \rho$. But $|e^{i\alpha}|\ |1+t| \geq 1-\varepsilon$.

b) If $(z,Z,\zeta) \in (\sum \bigcap (|z-\hat{\xi}|<\rho))$, then $F \neq 0$, for $|z-\xi| < \varepsilon$ implies by 6b), that $|P_j| < b < 1$ for all j. But $|\psi(z)| < 1/b$ here too and therefore $|P_1\psi| < 1$ and thus $< 1-2\varepsilon$, while $|1+t| \geq 1-\varepsilon$. However, F contradicts the conclusion of the proposition. For $\sum^* \subset (|z_j| \leq 1$, $|\zeta_1| \leq 1)$ means that P_1 is bounded on $\sum^*$, and for $|z-\hat{\xi}| < \rho$, ψ is bounded; hence for a large enough choice of a, $|1+a|$ can be made $> |P_1|\ |\psi|$ so that $F(z,Z,\zeta,a) \neq 0$ for $(z,Z,\zeta) \in \sum^*$. For $t = 0$, $1 = \phi(\hat{\xi})\ |\psi(\hat{\xi})| = |P_1(\hat{Z},\hat{\zeta})|\ |\psi(\hat{\xi})|$. Hence at the point $(\hat{\xi},\hat{Z},\hat{\zeta})$, there is a σ such that $P_1\psi = e^{i\sigma}$. Take $\alpha = \sigma$. Then $F = 0$ at $(\hat{\xi},\hat{Z},\hat{\zeta},0)$ and $(\hat{\xi},\hat{Z},\hat{\zeta}) \in \sum^*$.

8) Consider only $z \in (|z-\xi|<\varepsilon)$. Since $\xi \in \Omega_0$, there is a point with first coordinate ξ belonging to $(\sum^* - \sum)$, since there is a point $\in \sum^*(\xi)$ with $(z,Z) \notin G$. At this point $\phi = \max(b,|P_j|)$ and at least one polynomial is ≥ 1, i.e. $\phi(\xi) \geq 1$, for if not, then $(z,Z) \in G$ by 6a). Now $\xi \in$ boundary Ω_1 means that in $(|z-\xi|<\varepsilon)$, there is an $\eta \in \Omega_1$ with $\phi(\eta) = b$. For

a) If $\eta \notin S$, $\phi(\eta) = b$.

b) If $\eta \in S$, consider any $(\eta,Z,\zeta) \in \sum^*$. Since $\eta \in \Omega_1$, $(\eta,Z,\zeta) \in \sum$ by 3). By 6b), $|P_j(Z,\zeta)| < b$ for all j at (η,Z,ζ), and hence $\phi(\eta) = b$.

Next, consider the set of points (in $(|z-\xi| < \varepsilon)$) where $\phi \geq 1$. This set is closed because ϕ is upper semicontinuous, and at η, $\phi < 1$. This means that there is a closed set with C^1 boundary $\Delta \subset (|z-\xi|<\varepsilon)$, containing η, such that $\phi < 1$ in int Δ and $\phi \geq 1$ at some boundary point of Δ. Now in int Δ, $\phi(z) < 1$. Therefore either $z \notin S$ and hence $z \notin \Omega_0$, a subset of S, or $z \in S$. If $z \in S$, then since $\phi(z) < 1$, $|P_j(Z,\zeta)| < 1$ for all j and all $(z,Z,\zeta) \in \sum^*$, and by 6a), $(z,Z) \in G$. Therefore $z \notin \Omega_0$. Hence all points in int $\Delta \in (\mathbb{C}-\Omega_0)$. But int Δ is connected and therefore lies in a component of $\mathbb{C}-\Omega_0$. But $\eta \in \Omega_1$. Thus int $\Delta \subset \Omega_1$. Then by 3), in int Δ, $\phi \equiv b$. Let S be a closed disc $\subset \Delta$ and tangent to $\partial\Delta$ at a point p where $\phi \geq 1$. Then $\phi \equiv b$ in int S and $\phi(p) \geq 1$. But a subharmonic function which is $\equiv b$ in an open disc is $\equiv b$ on the boundary of the disc (by property (4) of subharmonic functions, p. 119). But $\phi \geq 1$ at a boundary point of Δ; a contradiction. Hence $\Omega_1 = \mathbb{C}$ and by 3), then, $\sum(z) = \sum^*(z)$ for all $z \in \mathbb{C}$, i.e. $\sum = \sum^*$.

Chapter 12. Some Consequences of the Approximation Theorem

§1. Relative convexity

Definition 55. Let $D_0 \subset D$, both open in $\mathbb{C}^n$. We say that D_0 is D-convex if, for every $K \subset\subset D_0$, $K_1 \subset\subset D_0$, where K_1 denotes the hull of K with respect to D, i.e.

$$K_1 = \{z_0 \in D_0 \mid |f(z_0)| \leq \sup |f(z)|, \quad z \in K, \text{ for every } f \text{ holomorphic in } D\} .$$

Note that $K_1 \supset \hat{K}$, the hull of K with respect to functions holomorphic in D_0.

Theorem 37. Let $D_0 \subset D$, both open in $\mathbb{C}^n$. Then the following statements are equivalent:

1) D_0 is D-convex.

2) There exists a sequence $P_j \subset\subset P_{j+1}$, $\bigcup_j P_j = D_0$, where each P_j is an analytic polyhedron in D.

3) D_0 is a region of holomorphy and (D_0, D) is a Runge pair.

Note. This is in some sense an extension of Theorem 33 of Chapter 10, §3; where $D = \mathbb{C}^n$.

Proof. 1) implies 2): Let $K \subset\subset D_0$. There exists an open set $D_1 \subset\subset D_0$ such that $K_1 \subset\subset D_1$. Now, for every $\xi \in \text{bdry } D_1$ there exists a function f_ξ such that $|f_\xi| < 1$ on K_1 and $|f_\xi(\xi)| > 1$, where f_ξ is holomorphic in D. Hence, there exists a neighborhood N_ξ D of ξ, such that $|f_\xi| > 1$ on N_ξ. But $D_1 \subset\subset D_0$; hence bdry D_1 is compact and a finite number of neighborhoods $N_1, \ldots, N_r$ suffice. Let $P = \{z \mid z \in D, \ |f_j(z)| < 1, \ j = 1, \ldots, r\}$. Now P is an analytic polyhedron in D, with $P \subset\subset D_1$, and $K \subset K_1 \subset\subset P$; hence every compact subset of D_0 may be enclosed in an analytic polyhedron in D.

So, let $K_{\alpha_1} \subset K_{\alpha_2} \subset \ldots \subset D_0$ be a sequence of compact sets such that $\bigcup_i K_{\alpha_i} = D_0$. As above, there exists an analytic polyhedron in D, $P_1 \supset K_{\alpha_1}$. Now, cl $P_1 \cup K_{\alpha_2}$ is compact; so there exists an analytic polyhedron in D, $P_2 \supset \text{cl}\, P_1 \cup K_{\alpha_2}$, and so on.

2) implies 3). Each P_j is Runge in P_{j+1}. Hence lemma β (p. 111) is applicable. Therefore D_0 is a region of holomorphy (by Theorem 21, p. 75) and (D_0, D) is a Runge pair.

3) implies 1). Let $K \subset\subset D_0$, and let $\hat{K}$, K_1 be as in definition 55; $\hat{K} \subset K_1$. We claim $\hat{K} = K_1$. Assume $q \notin \hat{K}$; we must show $q \notin K_1$. Now $q \notin \hat{K}$ implies that there exists a function $f(z)$, holomorphic in D_0, such that $|f(z)| < 1$ on K and $|f(q)| > 1$; in fact, there exists an $\varepsilon > 0$ such that $|f(z)| < 1 - 2\varepsilon$ on K, and $|f(q)| > 1 + 2\varepsilon$. Since (D_0, D) is a Runge pair, there exists a function g, holomorphic in D, such that $|g-f| < \varepsilon$ on the compact set $K \cup \{q\}$. But $|g(z)| = |g(z)-f(z)+f(z)| \leq |g(z)-f(z)| + |f(z)| < 1-\varepsilon$ on K; and $|g(q)| \geq |f(q)| - |g(q)-f(q)| > 1+\varepsilon$; so $q \notin K_1$.

§2. Unbounded regions of holomorphy

Lemma γ. Let $X^{\text{open}} \subset \mathbb{C}^n$; let $X_j \subset X_{j+1}$ be such that $\bigcup_j X_j = X$, where X_j is a region of holomorphy and (X_j, X_{j+1}) is a Runge pair. Then X is a region of holomorphy.

Proof. As indicated by the proof of lemma β (Chapter 10, §1), we may assume X_j is Runge in X. Recall that, for every $K \subset\subset X \subset \mathbb{C}^n$, $\Delta(K) = \Delta(\hat{K})$ if and only if X is a region of holomorphy (Theorem 5, Chapter 2, §2). Hence, assume there exists a $K \subset\subset X$ for which $\Delta(\hat{K}) < \Delta(K)$, where Δ is taken in the maximum norm; i.e. there exists a $q \in \hat{K} - K$ such that $\Delta(q) < \Delta(K)$. For j sufficiently large, $X_j \supset K \cup \{q\}$. Now for every f holomorphic on X_j, $|f| < 1$ on K implies $|f(q)| < 1$, i.e. $q \in$ [Hull of K with respect to functions holomorphic in X_j]. But X_j is a region of holomorphy, so dist $(q, \text{bdry}\, X_j) \geq$ dist $(K, \text{bdry}\, X_j)$; i.e. letting $j \to \infty$, $\Delta(q) \geq \Delta(K)$, which is a contradiction.

Note. One obtains another proof by constructing a sequence $X_j' \subset\subset X_{j+1}'$ such that $\bigcup X_j' = X$, using analytic polyhedra as in "1 implies 2" of Theorem 37.

Remark. (Oka) Let $X \subset \mathbb{C}^n$; we assume $0 \in X$ for simplicity. Let $X_r = X \cap \{\|z\| < r\}$, $\neq \phi$ by assumption. Then X is a region of holomorphy if and only if X_r is a region of holomorphy for every $r > 0$.

Proof. Assume X_r is a region of holomorphy for every $r > 0$. If X_r is Runge in X_R for $r < R$, we are done by lemma γ. Using the preceding theorem, it suffices to show X_r is X_R-convex. Let $K \subset\subset X_r$; let K_1 be the X_R hull of K. $K_1 \subset\subset X_R$ as X_R is a domain of holomorphy. But, for every $z \in K$, $\|z\| < r - \varepsilon$. Hence, in K_1 $\|z\| \leq r - \varepsilon$; where the norm is the maximum norm. Hence, $K_1 \subset\subset X_r$.

The reverse implication is clear.

§3. The Behnke-Stein Theorem

Theorem 38. (Behnke-Stein). If $X_j \subset X_{j+1} \subset \mathbb{C}^n$, and X_j is a region of holomorphy, then $X = \bigcup_j X_j$ is a region of holomorphy.

We first require a lemma:

Lemma δ. Let $D_1 \subset\subset D_2 \subset\subset D_3 \subset\subset \mathbb{C}^n$ be regions, D_3 a region of holomorphy, such that

$$\min_{\substack{z_1 \in \text{bdry } D_1 \\ z_3 \in \text{bdry } D_3}} \|z_1 - z_3\| > \max_{z_2 \in \text{bdry } D_2} \text{dist } (z_2\text{-bdry } D_3).$$

(This means that any subset of D_3 whose distance to ∂D_3 is $\geq$ the distance of cl D_1 to ∂D_3 lies in int D_2.) Then there exists an analytic polyhedron P in D_3 such that $D_1 \subset\subset P \subset\subset D_2$.

Proof. Let $K = \text{cl } D_1$, and let K_1 denote the D_3-hull of K. Then $\Delta_{D_3}(K) = \Delta_{D_3}(K_1)$, so $K_1 \subset\subset D_2$. But now there exists an analytic polyhedron P such that $K_1 \subset P \subset\subset D_2$, as in "1 implies 2" of Theorem 37.

Proof of Theorem 38. Utilizing Oka's remark, we may assume $X \subset\subset \mathbb{C}^n$. We may also assume $X_j \subset\subset X_{j+1}$ as every region of holomorphy can be written as a strictly increasing (i.e. relatively compact) sequence of analytic polyhedra (see corollary to theorem 7, Chapter 2, § 5). Define:

$$M_j = \max_{z \,\varepsilon\, \mathrm{bdry}\, X_j} \min_{\zeta \,\varepsilon\, \mathrm{bdry}\, X} \|z-\zeta\|$$

$$M_{j_1, j_2; j_1 < j_2} = \max_{z \,\varepsilon\, \mathrm{bdry}\, X_{j_1}} \min_{\zeta \,\varepsilon\, \mathrm{bdry}\, X_{j_2}} \|z-\zeta\|$$

$$m_j = \min_{z \,\varepsilon\, \mathrm{bdry}\, X_j} \min_{\zeta \,\varepsilon\, \mathrm{bdry}\, X} \|z-\zeta\|$$

$$m_{j_1, j_2; j_1 < j_2} = \min_{z \,\varepsilon\, \mathrm{bdry}\, X_{j_1}} \min_{\zeta \,\varepsilon\, \mathrm{bdry}\, X_{j_2}} \|z-\zeta\| .$$

We select a sequence $\nu_1, \nu_2, \ldots$ of subscripts as follows:

Set $\nu_1 = 1$. Choose ν_2 such that $M_{\nu_2} < m_{\nu_1}$, possible as $M_{\nu_2} \to 0$ as $\nu_2 \to \infty$.

Select $\nu_3 > \nu_2$ such that $M_{\nu_2, \nu_3} < m_{\nu_1, \nu_3}$; $M_{\nu_3} < m_{\nu_2}$, possible as $\lim_{\nu_3 \to \infty} M_{\nu_2, \nu_3} = M_{\nu_2} < m_{\nu_1} = \lim_{\nu_3 \to \infty} m_{\nu_1, \nu_3}$ and $M_{\nu_3} \to 0$ as $\nu_3 \to \infty$. Continuing, select $\nu_j > \nu_{j-1}$ such that $M_{\nu_{j-1}, \nu_j} < m_{\nu_{j-2}, \nu_j}$ and $M_{\nu_j} < m_{\nu_{j-1}}$; possible as before.

Consider now the subsequence X_{ν_j}; any three successive terms $X_{\nu_q} \subset\subset X_{\nu_{q+1}} \subset\subset X_{\nu_{q+2}}$ satisfy the conditions of lemma 8 for

$$\min_{\substack{z \,\varepsilon\, \mathrm{bdry}\, X_{\nu_q} \\ \zeta \,\varepsilon\, \mathrm{bdry}\, X_{\nu_{q+2}}}} \|z-\zeta\| = m_{\nu_q, \nu_{q+2}} > M_{\nu_{q+1}, \nu_{q+2}}$$

$$= \max_{z \,\varepsilon\, \mathrm{bdry}\, X_{\nu_{q+1}}} \min_{\zeta \,\varepsilon\, \mathrm{bdry}\, X_{\nu_{q+2}}} \|z-\zeta\| .$$

Hence, there exists a sequence $\{P_j\}$ of analytic polyhedra such that $X_{\nu_j} \subset\subset P_j \subset\subset X_{\nu_{j+1}}$, where P_j is an analytic polyhedron in $X_{\nu_{j+2}}$ and hence in $P_{j+1} \subset X_{\nu_{j+2}}$.

Appealing to the Oka-Weil theorem (theorem 35), (P_j, P_{j+1}) is a Runge pair. Now use lemma γ (or β) to obtain the desired result.

§4. Applications to the Levi problem

We recall the

Levi problem: Is every pseudoconvex domain a domain of holomorphy?

We shall reduce this problem to the consideration of "strictly pseudoconvex" domains, defined as follows:

Definition 56. Let $G^{open} \subset \mathbb{C}^n$, and let $\phi(z)$ be a real valued, C^{∞} function on G. Let ϕ be strongly plurisubharmonic i.e., $(\partial^2\phi/\partial z_i \partial \bar{z}_j) > 0$ for every $z \in G$; assume also that $\phi > 0$ near the boundary of G and $\phi < 0$ somewhere in G; more precisely, there exists an $\varepsilon > 0$ such that for every $z \in \{z \mid \text{dist}(z, \text{bdry } G) < \varepsilon, z \in G\}$, $\phi(z) > 0$ and there exists a $z_0 \in G$ such that $\phi(z_0) < 0$.

Now consider the open (nonempty) set $G_- = \{z \mid z \in G, \phi(z) < 0\}$; it is called strictly pseudoconvex; any region D for which there exists a $G^{open} \subset \mathbb{C}^n$ and strongly plurisubharmonic (real-valued, C^{∞}) function ϕ for which $D = G_-$ is called a strictly pseudoconvex region.

We remark that a strictly pseudoconvex region is pseudoconvex; consider $\log(-1/\phi)$. The strict pseudoconvexity is essentially a "smooth boundary" condition.

Example. The unit ball is strictly pseudoconvex: set $\phi = \sum_1^n z_j \bar{z}_j - 1$. The unit polydisc is not, and analytic polyhedra are generally not, strictly pseudoconvex.

Theorem 39. Every pseudoconvex region is the limit of an increasing sequence of strictly pseudoconvex regions.

Proof. Let D be pseudoconvex; i.e. there exists a real-valued, continuous, plurisubharmonic function ψ in D such that $\psi \to +\infty$ on the boundary of D. We claim that for every compact subset K of D there is a strictly pseudoconvex region D_0 satisfying $K \subset D_0 \subset\subset D$. To prove this, notice first that since $\psi \to +\infty$ on ∂D, there is a positive integer N so large that $K \subset S \subset T \subset L^{\text{compact}} \subset D$, where $S = \{z \in D \mid \psi(z) < N\}$, $T = \{z \in D \mid \psi(z) < N+1\}$ and L is any compact subset of D containing T. Give $\varepsilon > 0$ small. By Proposition 2, p. 28, there exists a C^∞ plurisubharmonic function ψ_0 on L such that $|\psi(z)-\psi_0(z)| < \varepsilon$ on L. Define $\chi(z) = \psi_0(z) + \varepsilon/M \sum_{i=1}^{n} z_i\bar{z}_i$; $M = \max_{z\in L} \sum_{i=1}^{n} z_i\bar{z}_i$. Then $\chi(z)$ is C^∞ and strongly plurisubharmonic on L. For $z \in S$, $\chi(z) < N + 2\varepsilon$; for $z \in (L-\mathrm{cl}\ T)$, $X(z) > N+1-\varepsilon$. Therefore if we set $D_0 = \{z \in \operatorname{int} L \mid \chi(z) < N + 1/2\}$, then D_0 is strictly pseudoconvex and $K \subset D_0 \subset L$, as claimed. Let K_j be an increasing sequence of compact subsets of D such that $\bigcup_j K_j = D$. For each j, let D_j be a strictly pseudoconvex region, $K_j \subset D_j \subset\subset D$. We choose a sequence ν_j of subscripts as follows: Set $\nu_1 = 1$; $D_1 \subset K_{j_1}$ for some j_1; hence $D_1 \subset\subset K_{j_1}$. Set $\nu_2 = j_1$; and so on. Note that $K_j \subset \bar{D}_{\nu_j} \subset D_{\nu_{j+1}} \ldots$ so that $D_{\nu_j} \subset\subset D_{\nu_{j+1}}$ and $\bigcup_j D_{\nu_j} = D$.

<u>Proposition</u>. The Levi problem is reduced to the following: Is every strictly pseudoconvex region with compact closure a region of holomorphy?

<u>Proof</u>. Use the preceding theorem together with the Behnke-Stein theorem.

Chapter 13. Solution of the Levi Problem

The object of this chapter is to prove the following theorem.

Theorem 40. If $D^{open} \subset \mathbb{C}^n$ is pseudoconvex, it is a region of holomorphy.

Since every pseudoconvex region is the union of an increasing sequence of strictly pseudoconvex regions (Theorem 39), and the union of an increasing sequence of regions of holomorphy is a region of holomorphy (Theorem 38), it suffices to prove

Theorem 40'. If $D^{open} \subset\subset \mathbb{C}^n$ is strictly pseudoconvex, it is a region of holomorphy.

The Levi problem has been solved first by Oka (for n = 2), then by Bremermann, Norguet and Oka for any n. There exist today many proofs of this theorem using either Oka's original method, or functional analysis methods (Ehrenpreis, Grauert, Narasimhan, Andreotti-Grauert, etc.), or partial differential equations (J. J. Kohn). The proof given here follows mainly Grauert and Narasimhan. The idea of using an "Extension Lemma" plus a result by L. Schwartz to establish finiteness of cohomology groups is due to Cartan and Serre.

§1. Reduction to a finiteness statement

Proposition 1. If $D^{open} \subset\subset \mathbb{C}^n$ is strictly pseudoconvex, then $\dim H^1(D, \mathcal{O}) < \infty$.

Let $U = \{u_i\}$ be a fixed, locally finite covering. The $\dim H^1(D, \mathcal{O}) = m < \infty$ means that given (m+1) cocycles on U, a nontrivial linear combination of them is a coboundary on a refinement of U. But a cocycle on U is a set of Cousin I data, and a cocycle induces a coboundary when the induced Cousin I is solvable. Since we can add Cousin data and multiply by constants, $\dim H^1(D, \mathcal{O}) = m < \infty$ means that a nontrivial linear combination of (m+1) CI data is solvable.

Proposition 1 implies Theorem 40'.

Proof: Recall that $D^{open} \subset\subset \mathbb{C}^n$ is strictly pseudoconvex if there is an open set $G \subset \mathbb{C}^n$ such that $D \subset\subset G$ and in G there is a real-valued C^∞ function ϕ which is strongly plurisubharmonic and $D = [\phi<0]$.

A. Lemma 1. Let $D^{open} \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex region. There is an $\varepsilon > 0$ such that if ω is a real-valued

C^∞ function in G and if $|\omega|$, $|\partial\omega/\partial z_j|$, $|\partial^2\omega/\partial z_j\partial\bar{z}_k|$ are all $\leq \varepsilon$, $D_1 = [\phi+\omega<0]$ is strictly pseudoconvex and $D_1 \subset\subset \mathbb{C}^n$.

Proof. $D_1 \subset\subset G$ since ω is small and $\phi+\omega$ is a real-valued C^∞ function in G. $\phi+\omega$ is strongly plurisubharmonic since the sum of a small quadratic form and positive definite quadratic form is a positive definite quadratic form.

Lemma 2. Let $D^{\text{open}} \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex region. There is an $\varepsilon > 0$ such that if $q \in$ bdry D and B is the polydisc of radius ε about q, there is a quadratic polynomial Q, in $z_1,\ldots,z_n$, with $Q(q) = 0$ and $Q(z) \neq 0$ in $D \cap B$.

Proof. Since $\phi \in C^\infty$ in G and is real-valued, by Taylor's theorem,

$$\phi(z) = \sum_{j=1}^{n} [A_j(z_j-q_j) + \bar{A}_j(\bar{z}_j-\bar{q}_j)] + \sum_{i,j=1}^{n} [A_{ij}(z_i-q_i)(z_j-q_j) + \bar{A}_{ij}(\bar{z}_i-\bar{q}_i)(\bar{z}_j-\bar{q}_j)] + \sum_{i,j=1}^{n} H_{ij}(z_i-q_i)(\bar{z}_j-\bar{q}_j) + R$$

where $H = (H_{ij}) = H^*$ and R involves third order terms. So,

$$\phi(z) = 2\,\mathrm{Re}\,Q(z) + (z-q)^* H\,(z-q) + R$$

where $Q(z) = \sum_{j=1}^{n} A_j(z_j-q_j) + \sum_{i,j=1}^{n} A_{ij}(z_i-q_i)(z_j-q_j)$.

Now, since the hessian of ϕ is positive definite, H is a positive definite hermitian matrix of second order derivatives of ϕ at q. Since H varies continuously with q, and at each boundary point q of D it has a positive smallest eigenvalue, the minimum eigenvalue of H over ∂D, call it α, is positive. Hence $(z-q)^* H\,(z-q) \geq \alpha\|z-q\|^2$. Similarly by estimating third order derivatives of ϕ over ∂D, we can find an $M > 0$, independent of q, such that $-M\|z-q\|^3 \leq R \leq M\|z-q\|^3$. Hence $(z-q)^* H\,(z-q) + R \geq (\alpha - M\|z-q\|)\|z-q\|^2$. Take $\varepsilon = \alpha/M$; then for $\|z-q\| < \varepsilon$, $(z-q)^* H(z-q)+R > 0$

except at q where it vanishes. But in $D \cap B$ $\phi < 0$; therefore $Q \neq 0$.

Lemma 3. Let $D^{open} \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex region. There is an $\varepsilon > 0$ such that if $q \in \text{bdry } D$ and B is the polydisc of radius $\leq \varepsilon$ about q, then $B \cap D$ is a region of holomorphy.

Proof: We will show that every boundary point is essential. Take $\varepsilon = 1/2\, \varepsilon_2$, the ε of lemma 2.
$\text{Bdry } (B \cap D) = ((\text{bdry } B) \cap D) \cup (B \cap \text{bdry } D) \cup (\text{bdry } B \cap \text{bdry } D)$.

If $\gamma \in \text{bdry } B$, then since B is a domain of holomorphy, there is a function holomorphic in B and singular at γ.

If $\gamma \in (\text{bdry } D) \cap B$, then the polynomial Q, of lemma 2, corresponding to γ, is holomorphic, $Q(\gamma) = 0$, and $Q(z) \neq 0$ in $D \cap B$. Hence $1/Q$ is holomorphic in $D \cap B$ and singular at γ.

B. Assuming proposition 1, we will show that every boundary point of a strictly pseudoconvex D is essential. Let $q \in \text{bdry } D$. Let B be the polydisc of lemma 2 and $Q(z)$ the quadratic polynomial. Let $\omega \geq 0$ be a C^∞ function whose support lies in the interior of B and $\omega(q) > 0$. Consider $\phi - t\omega$, where $t > 0$ is small. By lemma 1, if t is small enough, $D_1 = [\phi - t\omega < 0]$ is strictly pseudoconvex. $D \subset D_1 \subset\subset \mathbb{C}^n$, $q \in D_1 - D$, and since $\text{supp } \omega \subset B$, $D_1 - D \subset B$. Since D_1 is strictly pseudoconvex, by hypothesis $\dim H^1(D_1, \mathcal{O}) = m < \infty$. Construct $(m+1)$ sets of Cousin I data as follows. Take $U_1 = B \cap D_1$, $U_2 = D$, as an open covering of D_1. For each $k = 1, 2, \ldots, m+1$, consider $F_1 = 1/Q^k$, $F_2 = 0$. F_1, F_2 are meromorphic functions and because $(U_1 \cap U_2) \subset (D \cap B)$ and $Q \neq 0$ in $D \cap B$, $F_1 - F_2 = 1/Q^k$ is holomorphic in $U_1 \cap U_2$. A linear combination of these data is solvable. Therefore there are complex constants $\alpha_1, \ldots, \alpha_{m+1}$ not all zero, and a function $F(z)$ meromorphic in D_1 such that $F(z) - \sum_{j=1}^{m+1} \alpha_j\, Q(z)^{-j}$ is holomorphic in U_1 and $F(z) - 0 = F(z)$ is holomorphic in D. This function $F(z)$ is holomorphic in D and has a pole at q.

§2. Reduction to an extension property

Proposition 2. Let $D^{\text{open}} \subset\subset \mathbb{C}^n$ be strictly pseudoconvex. There is another strictly pseudoconvex region D_0 with $D \subset\subset D_0 \subset\subset \mathbb{C}^n$ such that if $\alpha = \sum a_j \, d\bar{z}_j$ is a C^∞ differential form of type $(0,1)$ in D and $\bar{\partial}\alpha = 0$, then there exists a C^∞ form $\beta = \sum b_j \, d\bar{z}_j$ in D_0 with $\bar{\partial}\beta = 0$ in D_0 and a C^∞ function χ in D, such that $\alpha - \beta = \bar{\partial}\chi$ in D.

2 means that every closed $(0,1)$ form in D is cohomologous in D to a $(0,1)$ form defined and closed in a larger region.

2 implies 1.

Proof: By Leray's theorem, the cohomology groups of D, $H^r(D,\mathcal{O})$ are isomorphic to the cohomology groups $H^r(D,U,\mathcal{O})$ of D with respect to any simple covering of D. Hence we need only consider cohomology with respect to a simple covering in order to prove 1. Let $\varepsilon_1, \varepsilon_2$ be the ε's given by Lemma 3 for D and D_0, respectively. Take $\varepsilon = 1/2 \min(\varepsilon_1, \varepsilon_2)$. Cover ∂D_0 by finitely many open polydiscs of radius ε such that (*) the closures of the polydiscs are disjoint when their interiors are disjoint. Complete this covering to a finite covering $V' = \{v_j'\}$ of D_0 by adding open polydiscs of radius ε satisfying (*) and whose closures do not intersect ∂D_0. About each v_j' take a slightly larger open polydisc v_j'' such that $v_j' \subset\subset v_j''$, if $\operatorname{cl} v_j' \cap \partial D_0 = \phi$ then $\operatorname{cl} v_j'' \cap \partial D_0 = \phi$, if $v_j' \cap v_k' = \phi$ then $v_j'' \cap v_k'' = \phi$, and such that for each j, $v_j'' \cap D_0$ is still a region of holomorphy. Set $U' = \{u_j' = v_j' \cap D\}$ and $U'' = \{u_j'' = v_j'' \cap D_0\}$. Then U' and U'' are $\bar{\partial}$-simple coverings of D and D_0, respectively, since they are finite coverings by regions of holomorphy.

With each covering we have the groups of cochains,

$C^r(D,U',\mathcal{O})$, $C^r(D_0,U'',\mathcal{O})$, and the groups of cocycles $Z^r(D,U',\mathcal{O})$ and $Z^r(D_0,U'',\mathcal{O})$. As we have already noted, these groups are also linear vector spaces. With the following notion of convergence, it can be shown that C^r, and hence Z^r, is a topological vector space: a sequence of elements of $C^r(D,U,\mathcal{O})$ converges if in each $u_1 \cap \ldots \cap u_{r+1} \neq \phi$ the sequence of holomorphic functions assigned there converges normally. In fact, these spaces are Frechet spaces, i.e. Hausdorff, locally convex, metrizable and complete under the metric.

Define the mappings

$$\delta: C^0\,(D,U',\mathcal{O}) \to Z^1(D,U',\mathcal{O})$$

to be the coboundary operator and

$$r : Z^1(D_0,U'',\mathcal{O}) \to Z^1(D,U',\mathcal{O})$$

to be the restriction map; i.e. if $z \in Z^1(D_0,U'',\mathcal{O})$ then $r(z)$ is the restriction of z to the covering U^1. δ is a continuous linear map and r is a completely continuous (i.e. compact) linear map. To show that r is indeed compact, we must show that there is a neighborhood of the origin which is mapped into a relatively compact set. A neighborhood of the origin is the set of all 1-cocycles on U'' which assign holomorphic functions f_{ij} to $u''_i \cap u''_j \neq \phi$ with $|f_{ij}| < \varepsilon$. Consider one intersection $u''_i \cap u''_j \neq \phi$ and the holomorphic functions f_{ij} assigned there with $|f_{ij}| < \varepsilon$. The image cocycles, under r, assign to $u'_i \cap u'_j$ the holomorphic functions f_{ij}, $|f_{ij}| < \varepsilon$. Take any sequence $\{f^k_{ij}\}$ of these functions, $|f^k_{ij}| < \varepsilon$, and a compact set $K \subset D_0$, $(u''_i \cap u''_j) \supset\supset K \supset\supset (u'_i \cap u'_j)$. Since the derivatives of the f^k_{ij} are uniformly bounded on K, $\{f^k_{ij}\}$ is equicontinuous and uniformly bounded on K, and hence contains a normally convergent subsequence. This shows sequential compactness of the image, but in a metric space sequential compactness implies compactness.

Now, consider the following two maps of the direct sum of $C^0(D,U',\mathcal{O})$ and $Z^1(D_0,U'',\mathcal{O})$ into $Z^1(D,U',\mathcal{O})$,

$$C^0(D,U',\mathcal{O}) \oplus Z^1(D_0,U'',\mathcal{O}) \underset{-r=B}{\overset{\delta \oplus r=A}{\rightrightarrows}} Z^1(D,U',\mathcal{O})$$

where $\delta \oplus r$ means that δ operates on C^0 and r on Z^1 and $-r$ is just the map $-r$ operating on Z^1. Both maps are continuous and linear. B is compact, and we claim that A is onto (see below).

Theorem. (L. Schwartz). Let E, F be Frechet spaces and $A, B : E \to F$ be continuous linear maps from E into F. If A is onto and B is compact then the range of $A + B$ has finite codimension, i.e. $\dim F(A+B)E < \infty$. (The proof is given in the appendix.)

Hence if A is onto, since $A + B = \delta$, this theorem implies that the space of cocycles on U' modulo coboundaries is finite dimensional.

It remains to show that A is onto.

Proof. A onto means that a cocycle in U' is cohomologous to the restriction of a cocycle in U''. Let f_{ij} be a cocycle in U' and f_{ij} the holomorphic function assigned to $u_i' \cap u_j' \neq \phi$. $f_{ij} = g_i - g_j$ where g_i, g_j are C^∞ functions on u_i', u_j' respectively; since the intermediate Cousin I problem is always solvable. In u_i', $\bar{\partial} g_i = \alpha$, a closed form independent of i because on $u_i' \cap u_j'$, $\bar{\partial} g_i = \bar{\partial} g_j$. By proposition 2, there is a closed (0,1) form β defined over D_0 and a C^∞ function χ in D such that $\alpha = \beta + \bar{\partial}\chi$ in D. In u_i'', $\beta = \bar{\partial} h_i$, $h_i \in C^\infty$, because U'' is $\bar{\partial}$-simple. Let $\hat{f}_{ij} = h_i - h_j$. $\hat{f}_{ij}$ is defined and holomorphic in $u_i'' \cap u_j''$. Hence a cocycle $\hat{f}_{ij}$ is defined on the larger covering U''. If we restrict this cocycle to U' we get a cocycle cohomologous to f_{ij}. Indeed, in $u_i' \cap u_j'$, $f_{ij} - \hat{f}_{ij} = (g_i - g_j) - (h_i - h_j) = (g_i - h_i - \chi) - (g_j - h_j - \chi)$ and $\bar{\partial}(g_i - h_i - \chi) = \alpha - \beta - \bar{\partial}\chi = 0$ and therefore $f_{ij} - \hat{f}_{ij}$ is a coboundary on U'.

§3. Proof of Proposition 2

We will prove a proposition 2' and then show that 2' implies 2.

Proposition 2'. Let $D^{open} \subset\subset \mathbb{C}^n$ be strictly pseudoconvex. Let $q \in$ bdry D. Take ε to be the ε of lemma 3. Let B_1 be the polydisc about q of radius $\varepsilon/2$, and let $\omega \geq 0$ be a C^∞ function with support in B_1 and $\omega(q) > 0$ so that $D_1 = [\phi - t\omega < 0]$ for $t > 0$ small is strictly pseudoconvex. Then every closed $(0,1)$ form in D is cohomologous in D to a $(0,1)$ form defined and closed in D_1. (We will say that D_1 satisfies 2' with respect to D.)

Proof. Let B be the polydisc about q of radius ε. Note that $D \subset D_1 \subset\subset \mathbb{C}^n$ and $D_1 - D \subset B_1$. Let σ be a C^∞ function, $\sigma \equiv 1$ in B_1 and $\sigma \equiv 0$ outside B. Let α be a closed $(0,1)$ form in D. Since $D \cap B$ is a region of holomorphy and α is closed in $D \cap B$, there is a C^∞ function χ in $D \cap B$ such that $\alpha = \bar\partial\chi$. The function $\chi\sigma$ is C^∞ in D and $\chi\sigma = \chi$ in $B_1 \cap D$. Thus $\beta = \alpha - \bar\partial(\chi\sigma)$ is a closed form in D cohomologous to α. In $B_1 \cap D$, $\beta = \alpha - \bar\partial\chi = 0$ and therefore can be continued as 0 to all of $D_1 - D$.

2' implies 2.

Proof. Let $D^{open} \subset\subset \mathbb{C}^n$ be strictly pseudoconvex. Since $D \subset\subset G^{open}$, by definition, there are open sets E, F such that $D \subset\subset E \subset\subset F \subset \mathbb{C}^n$ and the distance from D to E is greater than ε. Cover E by a finite number of polydiscs, $B_1, \ldots, B_N$, of radius $\varepsilon/2$ such that every polydisc containing a boundary point of D is centered about a boundary point of D. Let $\omega_1, \ldots, \omega_N$ be a partition of unity subordinated to the covering: $\omega_j \geq 0$ and C^∞ in E with support in B_j and $\sum_{j=1}^{N} \omega_j = 1$ at each point of E. For $\ell = 1, \ldots, N$ consider the regions $D_\ell = [\phi - \sum_{j=1}^{\ell} t_j\,\omega_j < 0]$, where the t_j are positive numbers so small that each D_ℓ is strictly pseudoconvex and $D_\ell \subset\subset \mathbb{C}^n$. Since $\sum_{j=1}^{N} \omega_j = 1$ at every boundary point of D, $\sum_{j=1}^{N} t_j\,\omega_j > 0$ there. Hence $D \subset\subset D_N$. Also,

D_1 satisfies 2' with respect to D and D_ℓ satisfies 2' with respect to $D_{\ell-1}$, $\ell = 2,\ldots,N$. Hence every closed (0,1) form in D is cohomologous in D to a closed (0,1) form in D_1 which in turn is cohomologous in D_1 (and hence D) to a closed (0,1) form in D_2, etc. up to D_N. Take for D_0, in 2, the set D_N.

Chapter 14. Sheaves

§1. Exact sequences

In the following, all groups are Abelian and all maps are homomorphisms.

A. Definition 57. A sequence is a collection of groups A_j and maps $\phi_j : A_j \to A_{j+1}$, written $\{A_j, \phi_j\}$ or:

$$\cdots \longrightarrow A_{j-1} \xrightarrow{\phi_{j-1}} A_j \xrightarrow{\phi_j} A_{j+1} \longrightarrow \cdots .$$

The sequence is said to be exact at A_j if $\operatorname{im} \phi_{j-1} = \ker \phi_j$, where $\operatorname{im} \phi_{j-1} = \{a \mid a \in A_j$, there exists $b \in A_{j-1}$ such that $\phi_{j-1} b = a\}$

$$\ker \phi_j = \{\alpha \mid \alpha \in A_j,\ \phi_j \alpha = 0\} .$$

The sequence $\{A_j, \phi_j\}$ is called exact if it is exact at A_j, for every j.

Definition 58. A collection of maps and groups is said to form a commutative diagram if all compositions of maps leading from a group A to a group B in the collection give the same result: e.g. the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\theta"'] & B \arrow[d, "\psi"] \\ & C \end{tikzcd}$$

commutes if $\psi\phi(a) = \theta(a)$ for every $a \in A$.

Remarks. 1) Clearly, $0 \to A \to 0$ is exact if and only if $A = 0$.

2) $0 \longrightarrow A \xrightarrow{\phi} B$ is exact if and only if ϕ is one-to-one. In this case, we may regard A as a subgroup of B, for $A_1 = \phi(A) \subset B$, and $\phi : A \to A_1$ is an isomorphism.

Hence the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & B \arrow[d, "id"] \\ 0 \arrow[r] & A_1 \arrow[r, "i"'] & B \end{tikzcd}$$

is commutative (i will always denote the inclusion map and id the identity map).

3) $A \xrightarrow{\phi} B \longrightarrow 0$ is exact if and only if ϕ is onto. Here, the homomorphism theorem of group theory implies $B \simeq A/\ker \phi$ ($\simeq$ denotes "is isomorphic to"). Hence, we may "factor" ϕ as follows:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & {\scriptstyle j}\searrow \quad \swarrow{\scriptstyle \phi_1} & \\ & A/\ker \phi & \end{array}$$

This diagram commutes, where here j (as always) denotes the canonical projection and is onto; and ϕ_1, the map induced by ϕ, is an isomorphism.

4) Combining 2) and 3), $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$ is called a <u>short exact sequence</u> if and only if ϕ is 1-1, ψ is onto and $\operatorname{im} \phi = \ker \psi$.

<u>Remark</u>. Utilizing the above remarks, the following diagram commutes and both horizontal sequences are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \phi\downarrow & & \downarrow \mathrm{id} & & \downarrow \psi_1^{-1} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{i} & B & \xrightarrow{j} & B/A_1 & \longrightarrow & 0 \end{array}.$$

Note that $\phi : A \to A_1$, $\mathrm{id} : B \to B$ and $\psi_1^{-1} : C \to B/A_1$ are all isomorphisms.

Isomorphic groups may be identified; hence short exact sequences should be thought of as being in the form:

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} B/A \longrightarrow 0 .$$

B. <u>Proposition 1</u>. Let $I = \{\alpha, \beta, \gamma, \ldots\}$ be directed by "$>$"; and let there be given sequences $\{A_j^\alpha, \phi_j^\alpha\}$, exact for each α, such that, for $\alpha < \beta$ the following diagram exists and is commutative:

$$\begin{array}{ccccccc} \cdots \longrightarrow & A_j^\alpha & \xrightarrow{\phi_j^\alpha} & A_{j+1}^\alpha & \longrightarrow \cdots \\ & \Big\downarrow a_j^{\alpha\beta} & & \Big\downarrow a_{j+1}^{\alpha\beta} & \\ \cdots \longrightarrow & A_j^\beta & \xrightarrow[\phi_j^\beta]{} & A_{j+1}^\beta & \longrightarrow \cdots , \end{array}$$

where the $a_j^{\alpha\beta}$ satisfy the following compatibility condition:

$$\alpha < \beta < \gamma \text{ implies } a_j^{\alpha\gamma} = a_j^{\beta\gamma} a_j^{\alpha\beta} .$$

Then the limit sequence $\{A_j = \lim A_j^\alpha , \ \phi_j = \lim \phi_j^\alpha\}$ is exact and the following diagram commutes:

$$\begin{array}{ccccccc} \cdots \longrightarrow & A_j^\alpha & \xrightarrow{\phi_j^\alpha} & A_{j+1}^\alpha & \longrightarrow \cdots \\ & \Big\downarrow i & & \Big\downarrow i & \\ \cdots \longrightarrow & A_j & \xrightarrow[\phi_j]{} & A_{j+1} & \longrightarrow \cdots . \end{array}$$

Remark. The A_j are defined as follows:

Set $S_j = \bigcup_{\alpha \varepsilon I} A_j^\alpha$; define an equivalence relation "$\sim$" on S_j as follows: $s_1 \varepsilon A_j^{\alpha_1}$, $s_2 \varepsilon A_j^{\alpha_2}$ are equivalent, $s_1 \sim s_2$, if there exists a β such that $\alpha_1 < \beta$, $\alpha_2 < \beta$ and $a_j^{\alpha_1\beta}(s_1) = a_j^{\alpha_2\beta}(s_2)$. Then $A_j = S_j/\sim$, and the group structure is canonical.

Alternatively, define a thread to be a set of elements $\{g_\alpha\}$ such that for every $\alpha \varepsilon I$, $g_\alpha \varepsilon A_j^\alpha$; and for every pair $\alpha, \beta \varepsilon I$ such that $\alpha < \beta$, $g_\beta = a_j^{\alpha\beta}(g_\alpha)$. The group structure is again the obvious one, and the group formed by the threads is denoted A_j.

The homomorphisms ϕ_j are defined in the obvious way, utilizing the ϕ_j^α and the compatibility condition. The proof of the proposition then follows from the directedness of the set I.

§2. Differential operators

A. Definition 60. Let A be an abelian group. A homomorphism $d : A \to A$ satisfying $d^2 = 0$ is called a

<u>differential operator</u>.

The sequence $A \xrightarrow{d} A \xrightarrow{d} A$ is not necessarily exact, but $\ker d \supset \operatorname{im} d$ as $d^2 = 0$. Hence we may define:

$$H(A) \equiv \ker d/\operatorname{im} d,$$

the <u>derived group of</u> A. $H(A)$ is a measure of the deviation from exactness of the above sequence, in the sense that $H(A) = 0$ if and only if the sequence is exact.

We say $x \in A$ is <u>closed</u> if $dx = 0$;

$x \in A$ is <u>exact</u> if there exists a $y \in A$ such that $x = dy$.

Denote the homology class in $H(A)$ of an element $x \in A$ by $[x]$.

<u>Definition 61</u>. If A,B are groups with differential operators, we say that $f : A \longrightarrow B$ is an <u>allowable homomorphism</u> if $fd_1 = d_2f$; i.e. if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{d_1} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[d_2]{} & B \end{array}$$

<u>Examples</u>. The group of cochains on a space, with boundary operator; continuous maps are allowable.

The additive group of differential forms, d the differential; differential maps are allowable.

Chains on a simplex, boundary operator; simplicial maps are allowable.

<u>Proposition 2</u>. An allowable map $f : A \longrightarrow B$ induces a homomorphism

$$f^* : H(A) \longrightarrow H(B) ,$$

such that if $g : B \to C$ allowable, then:

$$(gf)^* = g^* f^*$$

and

$$(id_A)^* = id_{H(A)} .$$

Proof. Define $f^*[x] = [fx]$, $x \varepsilon A$. Then f^* is a mapping of $H(A) \to H(B)$, for :

$$dx = 0 \quad \text{implies} \quad d(fx) = f(dx) = 0$$

and

$$x = dy \quad \text{implies} \quad fx = f(dy) = d(fy) .$$

B. _Proposition 3_. Let A,B,C be groups with differential operators, and let the following be a short exact sequence of allowable maps:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 .$$

Then there exists a canonical homomorphism D such that the following diagram is exact (viewed as an infinite, repeated sequence)

$$\begin{tikzcd} & H(A) \arrow[dr, "f^*"] & \\ H(C) \arrow[ur, "D"] & & H(B) \arrow[ll, "g^*"] \end{tikzcd}$$

Proof. (Recall the Weil proof of de Rham's theorem!)

Exactness at $H(B)$: (Does not need D)

i) Let $\alpha \varepsilon H(A)$; we must show $g^*f^*(\alpha) = 0$. But, let $x \varepsilon \alpha$; $g^*f^*(\alpha) = [gf(x)] = 0$ as $gf = 0$.

ii) Let $y \varepsilon B$, $dy = 0$, such that $g^*[y] = 0$. We wish to find $x \varepsilon A$, $dx = 0$, such that $f^*[x] = [y]$. Now $g^*[y] = [gy] = 0$, implies $gy = dz$, $z \varepsilon C$. But g is onto; hence there exists a $y_1 \varepsilon B$ such that $gy_1 = z$, which implies $gy = dz = dgy_1 = g(dy_1)$. Hence $y-dy_1 \varepsilon \ker g$, so there exists an $x \varepsilon A$ such that $fx = y-dy_1$, and x is closed for $dfx = f(dx) = dy = 0$, and f is one-to-one. But $f^*[x] = [y-dy_1] = [y]$, as required.

Construction of D:

Let $z \varepsilon C$, $dz = 0$. Now g is onto, so there exists a $y \varepsilon B$ such that $z = gy$. But $0 = dz = dgy = g(dy)$, so $dy \varepsilon \ker g$ implies that there exists an $x \varepsilon A$ such that $fx = dy$, and x is closed as before. Note that x is unique once y has been chosen. Set $D[z] = [x]$; D is well defined if $[x]$ is independent of y. Hence, let $y \varepsilon B$ such that $0=gy$.

Then $y \varepsilon \ker g$, so there exists a unique $x \varepsilon A$ such that $y = fx$. Therefore $dy = f\,dx$, so $D[0] = [dx] = 0$. D is clearly homomorphic.

Exactness at $H(C)$:

i) Let $y \varepsilon B$, $dy = 0$. Then $Dg^*[y] = D[gy] = [x]$, where $fx = dy$. But f is one-to-one, so $dy = 0$ implies $x = 0$.

ii) Let $z \varepsilon C$, $dz = 0$ such that $D[z] = 0$; i.e. there exists a $y \varepsilon B$ such that $z = gy$; $dy = fx$ and $D[z] = [x] = 0$. Hence, $x = dx_1$. Set $y_1 = y - fx_1$. Then y_1 is closed, for $dy_1 = dy - dfx_1 = fx - f(dx_1) = f(x-dx_1) = 0$. Furthermore, $g^*[y_1] = [gy - gfx_1] = [gy] = [z]$, as $gf = 0$.

Exactness at $H(A)$:

i) Let $z \varepsilon C$, $dz = 0$. Then $f^*D[z] = [fx]$, where $z = gy$ and $dy = fx$. But then $f^*D[z] = [dy] = 0$.

ii) Let $x \varepsilon A$, $dx = 0$ such that $f^*[x] = 0$; i.e., $fx = dy$. Set $z = gy$. Then z is closed, for $dz = dgy = gdy = gfx = 0$; and $D[z] = [x]$.

§3. Graded groups

Definition 62. A group A is called graded if, for every integral j, there exists a subgroup A_j such that $x \varepsilon A$ implies $x = x_{j_1} + \dots + x_{j_k}$; $x_{j_i} \varepsilon A_{j_i}$, $k < \infty$ and the representation is unique.

Note that this uniqueness implies $A_j \cap A_k = 0$, $j \neq k$.

An element $x_j \varepsilon A_j$ is called pure $(j-)$dimensional.

Examples. Chains and cochains on a simplicial complex are graded by their dimension.

Differential forms are graded by their degrees.

In both these cases, $A_j = 0$ for $j < 0$.

Definition 63. A differential operator d on a graded group A is said to respect the grading if there exists an integer r, called the shift of d, such that $d\,A_j \subset A_{j+r}$ for every j. (In practice, r is almost always ± 1.)

A map $f : A \to B$ of graded groups with differential operators is called <u>allowable</u> if the differential operators have the same shift and f preserves dimension.

<u>Corollary</u>. If A is a graded group with differential operator which respects the grading, then the derived group $H(A)$ is graded; and

$$H^j(A) = \frac{\{x \varepsilon A_j,\ dx=0\}}{\{dA_{j-r}\}} .$$

<u>Corollary</u>. An allowable map $f : A \to B$ of graded groups induces homomorphisms

$$f^* : H^j(A) \to H^j(B) .$$

<u>Proposition 4</u>. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of graded groups and allowable homomorphisms. Then there exist maps d such that the following sequence is exact:

$$\ldots \to H^j(A) \xrightarrow{f^*} H^j(B) \xrightarrow{g^*} H^j(C) \xrightarrow{d} H^{j+r}(A) \to \ldots$$

where r is the shift of the differential operators.

<u>Note</u>: There are $|r|$ distinct sequences.

<u>Proof</u>. Utilizing proposition 3, there exists a $D : H(C) \to H(A)$. As a map of graded groups, $D : H^j(C) \to H^{j+r}(A)$, for: suppose $z \varepsilon C$ is pure j-dimensional, $dz = 0$. Then there exists a y such that $gy = z$, and $\dim y = j$. There exists an x such that $dy = fx$ and $\dim (dy) = j+r = \dim x$. But $D[z] = [x]$, so shift $D = r =$ shift d; rename D "d"; then the exactness result of proposition 3 and the above corollaries conclude the proof.

<u>Example</u>. Let X be a topological space, $A \subset X$ a subspace. Let $C(Z)$ denote the graded group of chains over Z, with standard boundary operator, ∂. Then

$$0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{j} C(X)/C(A) \equiv C(X,A) \to 0$$

is an allowable short exact sequence of graded groups, and proposition 4 implies the exactness of the sequence

$$\ldots \to H^j(A) \xrightarrow{i_*} H^j(X) \xrightarrow{j_*} H^j(X,A) \xrightarrow{\partial} H^{j-1}(A) \to \ldots$$

§4. Sheaves and pre-sheaves

A. Recall the definition of "sheaf" (Chapter 6, §3, Defn. 36); we rephrase it as follows:

Definition 64. A sheaf of Abelian groups, $\underline{S}$, is defined as follows: Let X, the base space, be a paracompact Hausdorff space. For every $x \in X$, let S_x be an associated Abelian group called the stalk of the sheaf over x; and set $S = \bigcup_{x \in X} S_x$, whose topology is smallest such that

i) the projection map $p : S \to X$, defined by $p(s) = x$ if $s \in S_x$, is continuous and a local homeomorphism.

ii) the group operations in the stalks are continuous; i.e. $s \to -s$ is a continuous map of S into S; and $(s_1, s_2) \to s_1 + s_2$, defined on the set R of pairs (s_1, s_2) such that s_1, s_2 belong to the same stalk, is a continuous map of the subset R of $S \times S$ into S.

We remark that the stalks are discrete.

Let $Y \subset X$; then $S(Y) = \bigcup_{x \in Y} S_x$, with the induced topology, is called the induced sheaf of S over Y.

A section over X is a map $t : X \to S$, continuous, such that $p \cdot t = id_X$. A section over $Y \subset X$ is a section of $S(Y)$ over Y.

Remarks. Every sheaf has at least one section, the zero section, given by $t : x \to 0 \in S_x$.

If two sections coincide at a point, they coincide in a neighborhood of this point.

Corollary. Let $Y \overset{\text{open}}{\subset} X$. Then $t(Y)$ is open in S.

B. Definition 65. Let X be a paracompact Hausdorff space. Let $U = \{u_i\}$ be an open covering of X such that $u_1, u_2 \in U$ implies $u_1 \cap u_2 \in U$. Let $\Gamma(u_i)$ be an Abelian group associated to each $u_i \in U$, such that, if $u_i \subset u_j$ there exists a homomorphism $\gamma_{ij} : \Gamma(u_j) \to \Gamma(u_i)$ satisfying the compatibility condition:

$$u_i \subset u_j \subset u_k \quad \text{implies} \quad \gamma_{ji}\, \gamma_{kj} = \gamma_{ki} .$$

The collection $(X, \Gamma(u_i), \gamma_{ij})$ is called a prehseaf.

Proposition 5. To each presheaf there may be associated a sheaf, called the sheaf defined by the presheaf.

Proof. Take X as the base space. For each $x \in X$ the collection $\{u_i \mid u_i \in U, x \in u_i\}$ is directed; set S_x equal to the direct limit of the groups $\Gamma(u_i)$. Set $S = \bigcup_{x \in X} S_x$, with topology defined as follows: $s \in S$ implies $s \in S_x$. Now $x \in U_x \in U$; then $\{s_y \mid s_y \in S_y;\ y \in U_x\}$ is an open set and the collection of all such sets is a basis for the topology of S.

Note that the $\Gamma(u_i)$ form sections of the sheaf S defined by the presheaf.

Proposition 6. Every sheaf is defined by some presheaf.

Proof. Take $U = \{u \mid u^{\text{open}} \text{ in } X\}$. Let $\Gamma(u)$ be the sections over u; and define the γ_{ij} by restriction.

C. A subset T of a sheaf $\underline{S}$ is itself a sheaf if and only if T is open and $T_x = T \cap S_x$ is a subgroup of S_x. Then $\underline{T}$ is called a subsheaf of $\underline{S}$.

Example. $\underline{S}$ is the sheaf of germs of continuous functions and $\underline{T}$ is that subset of $\underline{S}$ consisting of all the germs of C^∞ functions.

§5. Exact sequences of sheaves and cohomology

Unless otherwise stated, all sheaves have the same fixed base space X.

Definition 66. Let $\underline{S}_1$ and $\underline{S}_2$ be two sheaves. A continuous map of $\underline{S}_1$ into $\underline{S}_2$ such that $\phi(S_{1,x}) \subset S_{2,x}$ and $\phi \mid S_{1,x}$ is a group homomorphism, is called a homomorphism of the sheaf $\underline{S}_1$ into the sheaf $\underline{S}_2$.

The subset of $\underline{S}_1$ mapped into the neutral elements of $\underline{S}_2$, $\{0 \in S_x\}$, is called the kernel of ϕ; denoted ker ϕ. The ker ϕ is an open set, for the set of neutral elements of $\underline{S}_2$ is the image of the null section of $\underline{S}_2$ over X and this is open in $\underline{S}_2$ (cf. corollary of §4, Chap. 14), and since ϕ is continuous, the preimage of the set of neutral elements of $\underline{S}_2$ is open in $\underline{S}_1$. The ker $\phi \cap S_{1,x}$ is the kernel of the group homomorphism $\phi \mid S_{1,x}$ and thus is a subgroup of $S_{1,x}$. Therefore $\ker \phi$ is a subsheaf of $\underline{S}_1$.

The image of ϕ, $\phi(\underline{S}_1) \subset \underline{S}_2$; denoted $\operatorname{im} \phi$, is a subsheaf of $\underline{S}_2$: that $\operatorname{im} \phi$ is open follows from the continuity of ϕ, the commutivity of ϕ and the projection map, i.e. $p_2\phi = p_1$, and the fact that p_1 and p_2 are local homeomorphisms, and, as before, $\operatorname{im} \phi \cap S_{2,x}$ is a subgroup of $S_{2,x}$, so that $\operatorname{im} \phi$ is a subsheaf of $\underline{S}_2$.

Hence we can form the quotient sheaves $\underline{S}_2 | \operatorname{im} \phi$ and $\underline{S}_1 | \ker \phi$, the cokernel of ϕ and coimage of ϕ, respectively.

<u>Definition 67</u>. The sequence of sheaves and sheaf homomorphisms $\underline{S}_j \xrightarrow{\phi_j} \underline{S}_{j+1} \xrightarrow{\phi_{j+1}} \underline{S}_{j+2}$ is called <u>exact</u> when, for each $x \in X$, the sequence $S_{j,x} \xrightarrow{\phi_j} S_{j+1,x} \xrightarrow{\phi_{j+1}} S_{j+2,x}$ is exact.

<u>Example</u>. Let $\underline{T}$ be a subsheaf of $\underline{S}$, and let $\underline{0}$ denote the null sheaf, i.e. the sheaf whose stalks are the trivial groups over each point. The sequence $\underline{0} \to T \xrightarrow{i} \underline{S} \to \underline{S}/\underline{T} \to \underline{0}$ is exact by definition.

We have already defined the cohomology groups, $H^q(X,\underline{S})$, $q > 0$, of a paracompact space X with coefficients in a sheaf $\underline{S}$. For convenience, define $H^q(X,\underline{S}) = 0$ for $q < 0$. Note that $H^0(X,\underline{S})$ is the group of global sections of the sheaf.

Let $\underline{S}$ and $\underline{T}$ be two sheaves and let ϕ be a homomorphism of $\underline{S}$ into $\underline{T}$. We claim that for each q, ϕ induces a homomorphism ϕ^* of $H^q(X,\underline{S})$ into $H^q(X,\underline{T})$. Consider any open covering of X, $U = \{u_i\}$. The group of cochains $C(X,U,\underline{S})$ is a graded group with differential operator (the coboundary) which respects grading (the shift is +1). Hence the derived group $H(C(X,U,\underline{S}))$ is graded and its pure dimensional parts are the cohomology groups of the covering with coefficients in $\underline{S}$. Similarly we have $C(X,U,\underline{T})$ and $H(C(X,U,\underline{T}))$. Now, an element of $C(X,U,\underline{S})$ is an assignment of sections of $\underline{S}$, and ϕ maps $\underline{S}$ continuously into $\underline{T}$, thus ϕ maps $C(X,U,\underline{S})$ into $C(X,U,\underline{T})$. ϕ is in fact an allowable homomorphism and hence induces a homomorphism of the derived groups $H(C(X,U,\underline{S}))$ and $H(C(X,U,\underline{T}))$. Taking the direct limit, we obtain the desired homomorphism ϕ^*.

Theorem 41. (Exact cohomology sequence). Let $\underline{0} \to \underline{A} \xrightarrow{\Phi} \underline{B} \xrightarrow{\Psi} \underline{C} \to \underline{0}$ be a short exact sequence of sheaves. Then there exists, canonically, an exact sequence

$$0 \to H^0(X,\underline{A}) \xrightarrow{\phi^*} H^0(X,\underline{B}) \xrightarrow{\psi^*} H^0(X,\underline{C}) \text{---}$$
$$\xrightarrow{\delta} H^1(X,\underline{A}) \xrightarrow{\phi^*} H^1(X,\underline{B}) \xrightarrow{\psi^*} H^1(X,\underline{C}) \text{---}$$
$$\xrightarrow{\delta} H^2(X,\underline{A}) \longrightarrow \ldots$$

Assume the theorem for now. (It is proved in §7, p. 158.)

Definition 68. A sheaf $\underline{S}$ is fine if and only if, for any locally finite open covering of X, $U = \{u_i\}$, $i \in I$, there exist homomorphisms η_i of $\underline{S}$ into $\underline{S}$ such that

1. $\eta_i(S_x) = 0$ for $x \notin u_i$ and
2. $\sum_{i\in I} \eta_i =$ identity.

(The sum is finite at each point because U is locally finite and η satisfies 1.)

Example. Let X be a paracompact differentiable manifold, and let $\underline{S}$ be the sheaf of germs of differential forms of degree p. Let U be a locally finite covering of X and let $\{\omega_i\}$ be a partition of unity subordinate to U. Define η_i to be multiplication by ω_i. Then $\{\eta_i\}$ are homomorphisms of $\underline{S}$ into $\underline{S}$ satisfying 1. and 2. above, so that $\underline{S}$ is a fine sheaf.

Theorem 42. If $\underline{S}$ is a fine sheaf, then $H^q(X,\underline{S}) = 0$ for all $q > 0$.

Proof. The proof is the exact analogue of the C^{∞} case: theorem 22, p. 78. Let $q > 0$ be fixed and let $U = \{u_i\}$ be a locally finite covering of X. Define the homomorphism $\theta\colon C^q(X,U,\underline{S}) \to C^{q-1}(X,U,\underline{S})$ by $\theta f(i_0 \ldots i_{q-1}) = \sum_{i\in I} \eta_i f(ii_0 \ldots i_{q-1})$. Verify that $f = \theta\delta f + \delta\theta f$ exactly as before. Hence if $\delta f = 0$, then $f = \delta\theta f$. Thus $H^q(X,U,\underline{S}) = 0$ and then the direct limit $H^q(X,\underline{S}) = 0$.

Definition 69. A <u>resolution of a sheaf</u> $\underline{S}$ is an exact sequence of sheaves $\underline{0} \longrightarrow \underline{S} \xrightarrow{\phi_{-1}=i} \underline{A}_0 \xrightarrow{\phi_0} \underline{A}_1 \xrightarrow{\phi_1} \cdots$ such that $H^q(X,\underline{A}_j) = 0$ for all $j \geq 0$ and $q > 0$.

A resolution is called a <u>fine resolution</u> when all the $\underline{A}_j$ are fine sheaves.

<u>Examples.</u>

1. Let X be a connected differentiable manifold and let $\underline{S} = \mathbb{C}$ ($S_x = \mathbb{C}$ and the topology is the discrete one). Let A_j be the sheaf of germs of differential forms of degree j. The sequence $\underline{0} \to \mathbb{C} \xrightarrow{i} \underline{A}_0 \xrightarrow{d} \underline{A}_1 \xrightarrow{d} \cdots$ is a fine resolution of $\mathbb{C}$.

<u>Proof</u>. Note that if X were not connected we would have to take for $\underline{S}$ the sheaf of germs of functions which are constant on each component of X in order that the sequence $\underline{0} \to \underline{S} \to \underline{A}_0 \to \underline{A}_1 \to \cdots$ be exact at A_0. We have already established that $H^q(X,\underline{A}_j) = 0$ for all $j \geq 0$ and $q > 0$ (cf. corollary p. 89) and that the $\underline{A}_j$ are fine sheaves. The exactness of the sequence at $\mathbb{C}$ and at $\underline{A}_0$ is immediate, and exactness at $\underline{A}_j$, $j > 0$, follows from the Poincaré lemmas.

2. Let X be a complex manifold and let $\underline{S}$ be $\mathcal{O}$ and $\underline{A}_j$ be the sheaf of germs of differential forms of type $(0,j)$. The sequence $\underline{0} \to \mathcal{O} \xrightarrow{i} \underline{A}_0 \xrightarrow{\bar\partial} \underline{A}_1 \xrightarrow{\bar\partial} \cdots$ is a fine resolution of $\mathcal{O}$.

<u>Proof</u>. As in example 1., the A_j are fine sheaves, and the exactness of the sequence at A_j, $j > 0$, follows from the Poincaré lemmas.

<u>Theorem 43</u>. (Abstract de Rham). Let $\underline{0} \to \underline{S} \xrightarrow{\phi_{-1}=i} \underline{A}_0 \xrightarrow{\phi_0} \underline{A}_1 \xrightarrow{\phi_1} \cdots$ be a resolution of a sheaf $\underline{S}$. Consider the induced cohomology sequence
$0 \longrightarrow H^0(X,\underline{S}) \xrightarrow{\phi_{-1}^*} H^0(X,\underline{A}_0) \xrightarrow{\phi_0^*} H^0(X,\underline{A}_1) \xrightarrow{\phi_1^*} \cdots$. Then $\operatorname{im}\phi_{p-1}^* \subset \ker \phi_p^*$ and $H^p(X,\underline{S}) \underset{\substack{\text{canon.}\\ \text{isom.}}}{\sim} \ker\phi_p^*/\operatorname{im}\phi_{p-1}^*$ for all $p > 0$.

Note. Applying this theorem to the above examples of resolutions of sheaves, we obtain for 1.

$$H^p(X,\mathbb{C}) \simeq \frac{\text{closed p-forms}}{\text{exact p-forms}}, \quad p > 0,$$

i.e. the de Rham Theorem (Theorem 26a, p. 93), and for 2.

$$H^p(X,\mathcal{O}) \simeq \frac{\bar\partial\text{-closed (0,p) forms}}{\bar\partial\text{-exact (0,p) forms}}, \quad p > 0,$$

i.e. the Dolbeault Theorem (Theorem 26b, p. 95).

Proof of Theorem 43. For $j = 0,1,2,\ldots$, set $\underline{B}_j = \ker \phi_j = \text{im } \phi_{j-1}$ since the resolution sequence is exact. For each j, the sequence $\underline{0} \longrightarrow \ker \phi_j \xrightarrow{i} \underline{A}_j \xrightarrow{\phi_j} \text{im } \phi_j \longrightarrow \underline{0}$ is exact by construction; rewrite it as $\underline{0} \longrightarrow \underline{B}_j \xrightarrow{i} \underline{A}_j \xrightarrow{\phi_j} \underline{B}_{j+1} \longrightarrow \underline{0}$. By the exactness theorem (Theorem 41), the sequence

$H^q(X,\underline{A}_j) \longrightarrow H^q(X,\underline{B}_{j+1}) \longrightarrow H^{q+1}(X,\underline{B}_j) \longrightarrow H^{q+1}(X,\underline{A}_j)$ is exact for $q \geq 0$ and $j \geq 0$. By hypothesis, $H^q(X,\underline{A}_j) = 0$ for $q > 0$ and $j \geq 0$. Hence $H^q(X,\underline{B}_{j+1}) \simeq H^{q+1}(X,\underline{B}_j)$ for $q > 0$ and $j \geq 0$. Then $H^p(X,\underline{S}) = H^p(X,\underline{B}_0) \simeq H^{p-1}(X,\underline{B}_1) \simeq H^{p-2}(X,\underline{B}_2) \simeq \ldots \simeq H^1(X,\underline{B}_{p-1})$.

Now $\underline{B}_p = \ker \phi_p$ is a subsheaf of $\underline{A}_p$. We claim that $X^0(X,\underline{B}_p) \simeq \ker \phi_p^*$. Indeed, the $\ker \phi_p^*$ is the set of those global sections of $\underline{A}_p$ that ϕ_p^* maps into the null section of $\underline{B}_{p+1}$, but, by the definition of ϕ_p^*, this set is precisely the set of global sections of $\underline{B}_p$.

Consider, next, the exact sequence

$$0 \longrightarrow H^0(X,\underline{B}_j) \longrightarrow H^0(X,\underline{A}_j) \longrightarrow H^0(X,\underline{B}_{j+1}) \xrightarrow{\delta} H^1(X,\underline{B}_j) \longrightarrow H^1(X,\underline{A}_j) \longrightarrow \ldots \quad \text{for } j = p-1,\ p > 0,$$

i.e.

$$0 \longrightarrow H^0(X,\underline{B}_{p-1}) \longrightarrow H^0(X,\underline{A}_{p-1}) \longrightarrow \ker \phi_p^* \xrightarrow{\delta} H^1(X,\underline{B}_{p-1}) \longrightarrow 0 \ldots .$$

Since δ is a homomorphism from $\ker \phi_p^*$ onto $H^1(X,\underline{B}_{p-1})$, $\text{im } \phi_{p-1}^* = \ker \delta \subset \ker \phi_p^*$ and $H^1(X,\underline{B}_{p-1}) \simeq \ker \phi_p^* / \ker \delta = \ker \phi_p^* / \text{im } \phi_{p-1}^*$, by exactness.

Hence $H^p(X,\underline{S}) \simeq \ker \phi_p^* / \text{im } \phi_{p-1}^*$.

§6. Applications of the exact cohomology sequence theorem

I. Let X be a complex manifold. Let

$\underline{\mathcal{O}}$: sheaf of germs of homomorphic functions

$\underline{\mathcal{M}}$: sheaf of germs of meromorphic functions

We may view $\underline{\mathcal{O}}$ as a subsheaf of $\underline{\mathcal{M}}$; let $i : \underline{\mathcal{O}} \to \underline{\mathcal{M}}$ be the inclusion. We form the exact sequence

$$\underline{0} \longrightarrow \underline{\mathcal{O}} \xrightarrow{i} \underline{\mathcal{M}} \xrightarrow{j} \underline{\mathcal{M}} / \underline{\mathcal{O}} \longrightarrow \underline{0} .$$

Recall that a section of $\underline{\mathcal{M}} / \underline{\mathcal{O}}$ over X is an equivalence class of sets of data for a C.I problem. Using the exact cohomology sequence theorem, there exists an exact sequence

$$0 \to H^0(X, \underline{\mathcal{O}}) \xrightarrow{i_*} H^0(X, \underline{\mathcal{M}}) \xrightarrow{j_*} H^0(X, \underline{\mathcal{M}}/\underline{\mathcal{O}}) \longrightarrow H^1(X, \underline{\mathcal{O}}) \longrightarrow \ldots$$

Now j_* sends a meromorphic function (a section of $\underline{\mathcal{M}}$) into the Cousin I problem it solves, hence C.I is always solvable if j_* is "onto", i.e.

Theorem I. $H^1(X, \underline{\mathcal{O}}) = 0$ implies C.I always solvable (cf. Chapter 6, §1).

II. Let X be a complex manifold, Y a globally defined hypersurface:

$$Y = [f=0] \; ; \quad f \text{ holomorphic in } X.$$

Assume f has no critical points where it vanishes [i.e. maximal Jacobian rank on Y].

Consider $\underline{\mathcal{O}}$, the sheaf of germs of holomorphic functions on X. Let $\underline{\mathcal{O}}_Y$ denote the sheaf of germs of homomorphic functions on X vanishing on Y. $\underline{\mathcal{O}}_Y$ is clearly a subsheaf of $\underline{\mathcal{O}}$; and we form (as before) the exact sequence:

$$\underline{0} \longrightarrow \underline{\mathcal{O}}_Y \xrightarrow{i} \underline{\mathcal{O}} \xrightarrow{j} \underline{\mathcal{O}} / \underline{\mathcal{O}}_Y \longrightarrow \underline{0} .$$

We claim that $\underline{\mathcal{O}} / \underline{\mathcal{O}}_Y$ is the induced sheaf of $\underline{\mathcal{O}}$ over Y. For points off Y, the stalks are trivial, for any stalk of $\underline{\mathcal{O}}_Y$ over points not in Y is identical with the corresponding stalk of $\underline{\mathcal{O}}$. For any point $y_0 \in Y$, two functions representing elements of $(\mathcal{O}/\mathcal{O}_Y)_{y_0}$ are equivalent if and only if they coincide in a neighborhood of y_0. (If they coincide in a neighborhood in Y, they coincide in a slightly larger neighborhood in X, as Y is closed); hence they represent the same germ in the induced sheaf of $\underline{\mathcal{O}}$ over Y.

Using the exact cohomology sequence theorem, we have the following exact sequence for $q \geq 0$:

$$\cdots \to H^q(X, \underline{\mathcal{O}}) \to H^q(X, \underline{\mathcal{O}}/\underline{\mathcal{O}}_Y) \to$$

$$H^{q+1}(X, \underline{\mathcal{O}}_Y) \to H^{q+1}(X, \underline{\mathcal{O}}) \to \cdots$$

Note that $H^q(X, \underline{\mathcal{O}} / \underline{\mathcal{O}}_Y) \simeq H^q(Y, \underline{\mathcal{O}})$.

We now claim that:

Theorem II. $H^q(X, \underline{\mathcal{O}}) = 0 = H^{q+1}(X, \underline{\mathcal{O}})$ for fixed $q \geq 0$ implies $H^q(Y, \underline{\mathcal{O}}) = 0$. (Cf. Chapter 6, §2, Theorem 20.)

For, using exactness, we obtain immediately:

$$H^q(X, \underline{\mathcal{O}}/\underline{\mathcal{O}}_Y) \simeq H^{q+1}(X, \underline{\mathcal{O}}_Y) ,$$

so that it is enough to show $H^{q+1}(X, \underline{\mathcal{O}}_Y) = 0$.

Let α be a cochain in $H^{q+1}(X, \underline{\mathcal{O}})$. Then $f \cdot \alpha$ is a cochain with $\underline{\mathcal{O}}_Y$ coefficients, as f is holomorphic in X. Hence, multiplication by f induces a homomorphism f^*: $H^{q+1}(X, \underline{\mathcal{O}}) \to H^{q+1}(X, \underline{\mathcal{O}}_Y)$.

Clearly f^* is onto and one-to-one; therefore an isomorphism.

III. As a last application, we obtain another old result:

Let X be a complex manifold; and let $\underline{\mathcal{O}}^*$ denote the sheaf of germs of <u>invertible</u> holomorphic functions under <u>multiplication</u>. Note that the sections are the nowhere vanishing globally defined holomorphic functions. Let $\underline{\mathcal{M}}^*$ denote the sheaf of germs of meromorphic functions under multiplication. Then $\underline{\mathcal{O}}^* \subset \underline{\mathcal{M}}^*$, and we obtain the exact sequence:

$$\underline{0} \longrightarrow \underline{\mathcal{O}}^* \xrightarrow{i} \underline{\mathcal{M}}^* \xrightarrow{j} \underline{\mathcal{M}}^*/\underline{\mathcal{O}}^* \longrightarrow \underline{0} .$$

The sections of $\underline{\mathcal{M}}^*/\underline{\mathcal{O}}^*$ are divisors in X, i.e. equivalence classes of sets of data for the C.II problem.

Using the exact cohomology sequence theorem, we obtain the exact sequence:

$$\cdots \longrightarrow H^0(X,\underline{\mathcal{M}}^*) \xrightarrow{j^*} H^0(X,\underline{\mathcal{M}}^*/\underline{\mathcal{O}}^*) \xrightarrow{\delta} H^1(X,\underline{\mathcal{O}}^*) \longrightarrow \cdots$$

Here j^* takes a meromorphic function into the C.II problem it solves: therefore any C.II problem α can be solved if $\delta\alpha = 0$.

This, however, is not particularly illuminating for we know little about $H^1(X,\underline{\mathcal{O}}^*)$; so, we imbed this group in another exact sequence involving "simpler" coefficient groups.

We have an exact sequence:

$$\underline{0} \longrightarrow \underline{\mathbb{Z}} \xrightarrow{i} \underline{\mathcal{O}} \xrightarrow{\exp} \underline{\mathcal{O}}^* \longrightarrow \underline{0} ,$$

where here $\underline{\mathbb{Z}}$ is viewed as a subsheaf of $\underline{\mathcal{O}}$, giving exactness at $\underline{\mathbb{Z}}$; "exp" is the map: $\exp(s) = e^{2\pi is}$ and the exactness at $\underline{\mathcal{O}}$ is clear since $\ker \exp = \{ s \mid e^{2\pi is} = 1 \}$; and exactness at $\underline{\mathcal{O}}^*$ follows from the fact that every nonvanishing holomorphic function is <u>locally</u> an exponential.

Hence, we obtain the exact sequence:

$$\cdots \longrightarrow H^1(X,\underline{\mathcal{O}}) \xrightarrow{\exp_*} H^1(X,\underline{\mathcal{O}}^*) \xrightarrow{d} H^2(X,\underline{\mathbb{Z}}) \longrightarrow \cdots ;$$

but this gives rise to a (canonical) map:

$$C = d \cdot \delta : H^0(X,\underline{\mathcal{M}}^*/\underline{\mathcal{O}}^*) \longrightarrow H^2(X,\underline{\mathbb{Z}}) ;$$

assigning to each divisor $D \in H^0(X,\underline{\mathcal{M}}^*/\underline{\mathcal{O}}^*)$ its <u>Chern class</u> $C(D) \in H^2(X,\underline{\mathbb{Z}})$. Clearly, if D is principal, i.e. $D \in \operatorname{im} j^*$, $C(D) = 0$ for $\delta(D) = 0$ by exactness.

Now assume $H^1(X,\underline{\mathcal{O}}) = 0$. Then $d : H^1(X,\underline{\mathcal{O}}^*) \to H^2(X,\underline{\mathbb{Z}})$ is one-to-one. Thus $C(D) = 0$ implies D is principal.

So, we have re-established the Oka-Serre Theorem:

Theorem III. There exists a map $C : H^0(X,\underline{\mathcal{M}}^*/\underline{\mathcal{O}}^*) \to H^2(X,\mathbf{Z})$ such that

1) D principal implies $C(D) = 0$

2) $H^1(X,\underline{\mathcal{O}}) = 0$ and $C(D) = 0$ implies D is principal.

(Cf. Chapter 9, §4, Theorem 31.)

§7. Proof of the exact cohomology sequence theorem

We now restate and then prove the theorem:

Theorem 44. Let $\underline{0} \to \underline{A} \xrightarrow{f} \underline{B} \xrightarrow{g} \underline{C} \to \underline{0}$ be an exact sequence of sheaves. Then there exists an exact sequence:

$$\cdots \longrightarrow H^j(X,\underline{A}) \xrightarrow{f^*} H^j(X,\underline{B}) \xrightarrow{g^*} H^j(X,\underline{C}) \xrightarrow{\delta} H^{j+1}(X,\underline{A}) \to \cdots$$

Proof. Let $U = \{u_i\}$ be a covering of X; and consider the sequence:

$$0 \longrightarrow C^q(U,\underline{A}) \xrightarrow{f} C^q(U,\underline{B}) \xrightarrow{g} C^q(U,\underline{C})$$

where $C^q(U,\underline{A})$ denotes the group of q-cochains on the covering U, and f and g denote the induced mappings. We claim this sequence is exact.

At $C^q(U,\underline{A})$ we must show $\ker f = 0$; therefore assume $\alpha \in C^q(U,A)$; $f(\alpha) = 0$. Now $f(\alpha) = 0$ means $f \cdot \alpha_{i_0 \ldots i_q}(x) = 0 \in B_x$ for all $x \in u_{i_0} \cap \ldots \cap u_{i_q}$. By exactness, f is one-to-one so $\alpha_{i_0 \ldots i_q}(x) = 0 \in A_x$ for every $x \in u_{i_0} \cap \ldots \cap u_{i_q}$; i.e. $\alpha = 0$.

At $C^q(U,\underline{B})$, we must show $\operatorname{im} f = \ker g$. Let $\alpha \in C^q(U,\underline{A})$. Then $g \cdot f(\alpha)_{i_0 \ldots i_q}(x) = gf\alpha_{i_0 \ldots i_q}(x) = 0$, since $gf = 0$, for every $x \in u_{i_0} \cap \ldots \cap u_{i_q}$; hence $g \cdot f(\alpha) = 0$ so $\operatorname{im} f \subset \ker g$.

Now let $\beta \in \ker g$; i.e. $g \cdot \beta_{i_0 \ldots i_q}(x) = 0$ for every $x \in u_{i_0} \cap \ldots \cap u_{i_q}$. For each $x \in u_{i_0} \cap \ldots \cap u_{i_q}$ there exists an $\alpha_{i_0 \ldots i_q}(x) \in A_x$ such that $f\alpha_{i_0 \ldots i_q}(x) = \beta_{i_0 \ldots i_q}(x)$. We claim that the assignment α defined by the $\alpha_{i_0 \ldots i_q}(x)$ is a cochain; i.e. that $\alpha_{i_0 \ldots i_q}$ is a section over $u_{i_0} \cap \ldots \cap u_{i_q}$.

Now $\alpha_{i_o \dots i_q}(x) \in A_x$ so $p \cdot \alpha_{i_o \dots i_q} = id_{u_{i_o} \cap \dots \cap u_{i_q}}$.
To show continuity, let $x_0 \in S_\alpha = \alpha_{i_o \dots i_q}(u_{i_o} \cap \dots \cap u_{i_q})$
$= f^{-1} \beta_{i_o \dots i_q}(u_{i_o} \cap \dots \cap u_{i_q})$, an open set for
$u_{i_o} \cap \dots \cap u_{i_q}$ is open, $\beta_{i_o \dots i_q}$ is a section and f is
continuous. Let N_{x_o} be a neighborhood of x_0 in S_α; then
$\alpha^{-1}_{i_o \dots i_q}[N_{x_o}] = \alpha^{-1}_{i_o \dots i_q} \cdot f^{-1} \cdot f(N_{x_o}) = (f \cdot \alpha_{i_o \dots i_q})^{-1} f(N_{x_o})$
$= \beta^{-1}_{i_o \dots i_q} f(N_{x_o})$ which is open for f is an open mapping.
Hence, $\operatorname{im} f = \ker g$.

We cannot complete this sequence to a short exact sequence, for g may not be onto [e.g. take sequence $C^o(X,\underline{\mathcal{O}}) \to C^o(X,\underline{\mathcal{M}}) \to C^o(X,\underline{\mathcal{M}}/\underline{\mathcal{O}})$]. So, define $C^q_a(U,\underline{C}) = g[C^q(U,\underline{B})]$, a subgroup of $C^q(U,\underline{C})$ comprised of "liftable" cochains. Hence, we now have the following short exact sequence of groups and allowable maps:

$$0 \longrightarrow C^q(U,\underline{A}) \xrightarrow{f} C^q(U,\underline{B}) \xrightarrow{g} C^q_a(U,\underline{C}) \longrightarrow 0 .$$

Hence, we obtain the exact cohomology sequence:

$$\dots \longrightarrow H^q(U,\underline{A}) \to H^q(U,\underline{B}) \to H^q_a(U,\underline{C}) \to H^{q+1}(U,\underline{A}) \to \dots$$

For refinements of U, we have the desired commutativity, so that we may appeal to the proposition of Chapter 14, §1, to obtain the exactness of the limit sequence:

$$\dots \to H^q(X,\underline{A}) \xrightarrow{f^*} H^q(X,\underline{B}) \xrightarrow{g^*} H^q_a(X,\underline{C}) \xrightarrow{\delta} H^{q+1}(X,\underline{A}) \to \dots$$

We shall therefore be done if we can show

$$H^q_a(X,\underline{C}) \simeq H^q(X,\underline{C})$$

(canonically!), and this shall be proven by showing that for each cochain ϕ in $C^q(U,\underline{C})$ for a locally finite covering U, there is a refinement V in which ϕ is liftable; then the limit groups are isomorphic, since we have an injective (one-to-one) map

$$i : C^q_a(U,\underline{C}) \longrightarrow C^q(U,\underline{C})$$

which commutes with the boundary operator and the "refinement" maps of the direct limit procedure.

(Since X is paracompact, we may restrict ourselves to the locally finite coverings U.)

Let $U = \{u_i\}$ be a given locally finite covering. Let $W = \{w_i\}$ be a new locally finite covering, refining U and such that $w_i \subset\subset u_i$ for each i, possible since paracompactness implies normality. Let $\phi \in C^q(U,\underline{C})$. To each $x \in X$ we assign an open $v_x \subset X$ such that

i) $x \in v_x$

ii) $x \in w_i$ implies $v_x \subset w_i$

iii) $x \in u_j$ implies $v_x \subset u_j$

iv) $x \notin u_j$ implies $v_x \cap w_j = \phi$

v) $x \in u_{i_0} \cap \ldots \cap u_{i_q}$ implies $\phi_{i_0 \ldots i_q} \mid v_x \in C^q_a(V,\underline{C})$.

Observe that once we establish the existence of $V = \{v_x\}$, the theorem is proved.

Let $x \in X$; using the local finiteness of U,W there exist integers $r,s < \infty$ such that $x \in w_{j_1},\ldots,w_{j_r}$; $u_{k_1},\ldots,u_{k_s}$ and no other w_i, u_k.

Set $v_x^1 = w_{j_1} \cap \ldots \cap w_{j_r} \cap u_{k_1} \cap \ldots \cap u_{k_s}$; clearly v_x^1 is an open neighborhood of x and satisfies i), ii), and iii). Note that any smaller neighborhood of x will also satisfy these.

For each $k \neq k_1,\ldots,k_s$; $x \notin u_k$, which implies $x \notin \bar{w}_k$, so there exists an open neighborhood $v_{k,x}$ of x such that $v_{k,x} \cap \bar{w}_k = \phi$. Set $v_x^2 = v_x^1 \cap \left\{ \begin{matrix} \bigcup v_{k,x} \\ k \neq k_1,\ldots,k_s \end{matrix} \right\}$, an open neighborhood of x which now satisfies i),...,iv).

To satisfy v), observe that $x \in u_{i_0} \cap \ldots \cap u_{i_q}$ can only happen if $u_{i_j} \in \{u_{k_1},\ldots,u_{k_s}\}$, and that $v_x^2 \subset u_{i_0} \cap \ldots \cap u_{i_q}$.

Now $\phi_{i_0 \ldots i_q} \mid v_x^2$ is a section of $\underline{C}$ over v_x^2. Hence $\phi_{i_0 \ldots i_q}(v_x^2)$ is open in $\underline{C}$, so $g^{-1}\,\phi_{i_0 \ldots i_q}(v_x^2)$ is an open, non-empty subset of $\underline{B}$ (g is onto). There exists a

$b_x \varepsilon g^{-1} \phi_{i_o \ldots i_q}(v_x^2)$ such that $p \circ b_x = x$, and neighborhood N_x of b_x in $\underline{B}$ such that $p : N_x \to p(N_x)$ is a homeomorphism. Let $M_x = N_x \cap g^{-1} \phi_{i_o \ldots i_q}(v_x^2)$, and set $v_x = p \circ g(M_x) \subset v_x^2$.

Now $g \mid M_x$ is one-to-one; and $g^{-1} \phi_{i_o \ldots i_q} : v_x \to M_x$ is a section of $\underline{B}$ over v_x, mapped by g onto the section $\phi_{i_o \ldots i_q} : v_x \to g(M_x)$ of $\underline{C}$.

Chapter 15. Coherent Analytic Sheaves

§1. Definitions

Definition 71. An analytic sheaf $\mathcal{F}$ is a sheaf whose base space X is a complex manifold (or subspace of one); and such that each element of the sheaf can be multiplied by the germ of a holomorphic function; more precisely, each stalk $\mathcal{F}_x$ is an $\mathcal{O}_x$-module, and this multiplication is continuous.

We have the notions of subsheaf, sheaf homomorphism, induced sheaf, factor sheaf, etc., as before.

Examples. $\mathcal{O}^r$, the sheaf of germs of (r-dimensional) vector-valued holomorphic functions is an analytic sheaf.

The sheaf of germs of continuous functions is an analytic sheaf.

The sheaves $\mathcal{O}^*$ and $\mathcal{M}^*$ are not analytic, for there is no distributive law for multiplication by germs of holomorphic functions (recall that the operation in the stalks of these sheaves is multiplication).

Definition 72. An analytic sheaf $\mathcal{F}$ is globally finitely generated if there exist a finite number of global sections $s_1, s_2, \ldots, s_k$ such that for every $x \in X$, $t \in \mathcal{F}_x$, $t = \phi_1(s_1)_x + \ldots + \phi_k(s_k)_x$ where $\phi_j \in \mathcal{O}_x$.

Examples. The sheaf $\mathcal{O}$; section "1".

The sheaf $\mathcal{O}^r$; r sections $(0,\ldots,0,1,0,\ldots,0)$.

Definition 73. An analytic sheaf is locally finitely generated if every point $x \in X$ has a neighborhood N_x such that the induced sheaf $\mathcal{F}(N_x)$ over N_x is globally finitely generated.

Let $\mathcal{F}$ be an analytic sheaf, and $s_1, \ldots, s_k$ sections of $\mathcal{F}(U)$, $U^{\text{open}} \subset X$. Let $x \in U$. If there exists a tuple $(\phi_1, \ldots, \phi_k) \in \mathcal{O}_x^k$ such that

$$\phi_1(s_1)_x + \ldots + \phi_k(s_k)_x = 0 ,$$

the tuple $(\phi_1, \ldots, \phi_k)$ is called a relation between the sections $s_1, \ldots, s_k$ at x.

Note that the collection of all such relations forms an analytic "sheaf of relations" (between $s_1,\ldots,s_k$) contained in $\underline{\mathcal{O}}^k$ (over the space U).

Definition 74. The analytic sheaf $\underline{\mathcal{F}}$ is called a coherent analytic sheaf if:

1) It is locally finitely generated.

2) For every open $U \subset X$, the sheaf of relations of any finite number of sections over U is also locally finitely generated.

Note that the definition is local.

Remark. For convenience, we will call a coherent analytic sheaf, a coherent sheaf.

§2. Oka's coherence theorem

The aim of this section is the statement and two steps of the proof of a three-step theorem due to Oka. The last section of the proof will be postponed until two theorems are established.

Theorem 45. (Oka) The sheaf of germs of vector-valued holomorphic functions is coherent; i.e. (noting the local character of coherence) let $D^{\text{open}} \subset \mathbb{C}^n$, and let $\{a_{ij}(z)\}$, $i=1,\ldots,q$, $j=1,\ldots,p$ be holomorphic functions defined in D. Let $x \in D$; then there exists an open $D_1 \subset D$, $x \in D_1$, with the following property:

For any $\zeta \in D_1$ the holomorphic solutions $(\phi_1,\ldots,\phi_p)$ of

$$* : \qquad \sum_{j=1}^{p} a_{ij}(z)\,\phi_j(z) = 0 \quad , \qquad i = 1,\ldots,q$$

defined in some neighborhood of ζ, may be written as:

$$\phi_j(z) = \sum_{\nu=1}^{L} \psi_\nu(z)\,\phi_j^\nu(z) \; , \qquad j = 1,\ldots,p$$

where the ϕ_j^ν; $j=1,\ldots,p$, $\nu=1,\ldots,L<\infty$ are a fixed finite set of solutions of *, holomorphic in a fixed neighborhood of ζ; and the ψ_ν are defined and holomorphic in some neighborhood of ζ.

Note. The ϕ_i^ν are said to form a finite psuedo basis.

Proof. We proceed by a double induction on q = number of equations and n = dimension of space. The three steps of the proof are the following:

I. The theorem is true for $n = 0$.

II. If the theorem holds for a fixed n and $< q$ equations, then it is true for q equations, where here $q > 1$.

III. If the theorem is true for some n and all q, then it is true for $n+1$ and $q = 1$.

It is clear that these steps complete the theorem; and that I holds, since a holomorphic function of no variables is a constant. We may assume $x = 0 \in D$, with no loss of generality.

II. Let us first introduce the following abbreviations:

$$(\alpha) \equiv \left\{\sum_{j=1}^{p} a_{ij}(z)\,\phi_j(z) = 0, \quad i = 1,\dots,q\right\}$$

$$(\beta) \equiv \left\{\sum_{j=1}^{p} a_{ij}(z)\,\phi_j(z) = 0, \quad i = 1,\dots,q-1\right\}$$

$$(\gamma) \equiv \left\{\sum_{j=1}^{p} a_{qj}(z)\,\phi_j(z) = 0\right\}$$

$$(\phi_j) \equiv (\phi_1,\dots,\phi_p)\,.$$

By hypothesis, (γ) has a finite pseudobasis (ϕ_j^κ), $\kappa=1,\dots,K<\infty$. Since any solution of (α) satisfies (γ), the general solution (ϕ_j) of (α) has the form:

$$\phi_j = \sum_{\kappa=1}^{K} \psi_\kappa \phi_j^\kappa\,, \qquad j = 1,\dots,p.$$

These ϕ_j must also satisfy (β), hence:

$$\begin{aligned} 0 &= \textstyle\sum_{\kappa,j} a_{ij}\, \psi_\kappa \phi_j^\kappa \\ &= \textstyle\sum_\kappa \left(\sum_j a_{ij}\phi_j^\kappa\right)\psi_\kappa\,, \qquad i = 1,\dots,q-1. \end{aligned}$$

Set $b_{i\kappa} = \sum_j a_{ij}\,\phi_j^\kappa$; these functions are known.
But the set of equations

$$0 = \sum_{\kappa} b_{i\kappa}\,\psi_{\kappa}\ , \qquad = 1,\ldots,q-1$$

has by hypothesis a finite pseudobasis (ψ_{κ}^{ρ}), $\rho = 1,\ldots,R$.
Hence:

$$\psi_{\kappa} = \sum_{\rho=1}^{R} \omega_{\rho}\,\psi_{\kappa}^{\rho}\ ,$$

So:

$$\phi_j = \sum_{\rho=1}^{R}\sum_{\kappa=1}^{K} \omega_{\rho}\,\psi_{\kappa}^{\rho}\,\phi_j^{\kappa}\ ,$$

and therefore the functions $(\sum_{\kappa=1}^{K} \psi_{\kappa}^{\rho}\,\phi_j^{\kappa})$ form a finite pseudobasis for (α); proving step II.

We interrupt the proof to establish two needed theorems.

§3. Weierstrass preparation theorem, revisited

A. Theorem 46. (Weierstrass Preparation Theorem) Let $f(Z,z)$ be holomorphic at the origin, $f(0,z) \not\equiv 0$ [where $(Z,z) = (z_1,\ldots,z_n)$, $Z = (z_1,\ldots,z_{n-1})$ and $z = z_n$], so that $f = \sum_{j=0}^{\infty} a_j(Z)\,z^j$, with $a_0(0) = \ldots = a_{s-1}(0) = 0$, while $a_s(0) \neq 0$. Then:

$$f(Z,z) = h(Z,z)\,[z^s + b_1(Z)z^{s-1} + \ldots + b_s(Z)]\ ,$$

where h is a unit, i.e. $h(0,0) \neq 0$; and $b_1,\ldots,b_s$ are holomorphic in some neighborhood of the origin with $b_1(0) = \ldots = b_s(0) = 0$.

Remark. This representation is unique, but we shall not prove or use this. In the Weierstrass preparation theorem proved earlier (4, §1, Thm 12) we had also assumed that ord $f = s$.

Proof. For $s = 0$, this theorem is a triviality; hence, take $s \geq 1$. Note that $a_0,\ldots,a_{s-1}$ have no constant term.

Let $s < N$, an integer. Set $z_j = \zeta_j^N$, $j = 1,\ldots,n-1$; and define:

$$G(\zeta_1,\ldots,\zeta_{n-1},z) = f(\zeta_1^N,\ldots,\zeta_{n-1}^N,z)\ .$$

Then G is a holomorphic function of its variables in a neighborhood of the origin and of order s, so that Theorem 12 applies. Hence:

$$G(\zeta_1,\ldots,\zeta_{n-1},z) = H(\zeta_1,\ldots,\zeta_{n-1},z)[z^s+B_1(\zeta_1,\ldots,\zeta_{n-1})z^{s-1}+ \ldots + B_s(\zeta_1,\ldots,\zeta_{n-1})]$$

and the B_i vanish at the origin. It is now enough to show that the ζ_i occur only as ζ_i^{Nn}. Take θ to be a primitive N th root of unity; then:

$$\begin{aligned} G(\zeta_1,\ldots,\theta\zeta_i,\ldots,\zeta_{n-1},z) &= H(\zeta_1,\ldots,\theta\zeta_i\ldots,\zeta_{n-1}z[z^s + B_1(\zeta_1,\ldots,\theta\zeta_i,\ldots,\zeta_{n-1})z^{s-1}+\ldots] \\ &= G(\zeta_1,\ldots,\zeta_i,\ldots,\zeta_{n-1},z) \end{aligned}$$

But the expansion of the Weierstrass Theorem 12 is unique.

<u>Corollary</u>. Let $P(Z,z) = z^s + a_1(Z)z^{s-1} + \ldots + a_s(Z)$, where the a_j are holomorphic in a neighborhood D of the origin. Let $(C,c) \varepsilon \mathbb{C}^n$, $C \varepsilon D$. Then

$$P(Z,z) = P^I(Z,z)\, P^{II}(Z,z)$$

where
$$\begin{aligned} P^I(Z,z) &= z^r + \alpha_1(Z)z^{r-1} + \ldots + \alpha_r(Z) \\ P^{II}(Z,z) &= z^t + \beta_1(Z)z^{t-1} + \ldots + \beta_t(Z) \end{aligned}$$

with β_i, α_j holomorphic in a neighborhood of C, such that $P^I(C,z) = (z-c)^r$ and $P^{II}(C,c) \neq 0$.

<u>Proof</u>. Changing variables, set:

$$Z' = Z - C, \qquad z' = z - c$$

and define: $Q(Z',z') = P(Z'+C,z'+c)$.

Then $Q(Z',z') = Q^I(Z',z')Q^{II}(Z',z')$; where Q^{II} is a unit and $Q^I(Z',z') = z'^r + a_1(Z')z'^{r-1} + \ldots + a_r(Z')$.

Set:

$$\begin{aligned} P^I(Z,z) &= Q^I(Z-C,z-c) \\ P^{II}(Z,z) &= Q^{II}(Z-C,z-c)\ . \end{aligned}$$

Then $P^{II}(C,c) \neq 0$ and $P^I(C,z) = Q^I(0,z-c) = (z-c)^r$.

It is clear that P^I is a monic polynomial.

B. Theorem 47. (Division Theorem) Let there be given a polynomial $P = z^s + a_1(Z)z^{s-1} + \dots + a_s(Z)$ with the a_j holomorphic in a neighborhood of the origin, and $a_j(0) = 0$. Let $f(Z,z)$ be holomorphic near the origin. Then:

$$f(Z,z) = q(Z,z)\, P(Z,z) + R(Z,z)$$

where R is a polynomial in z of degree $< s$, with coefficients holomorphic in a neighborhood of the origin, and q is a holomorphic function in a neighborhood of the origin. Furthermore, this representation is unique.

Proof. We first establish uniqueness. Suppose

$$\begin{aligned} 0 &= qP + R \\ &= \Big(\sum_0^\infty q_j(Z)z^j\Big)P + R\,. \end{aligned}$$

Let $\nu \geq 1$; we equate the coefficients of $z^{s+\nu}$, obtaining

$$0 = q_\nu + q_{\nu+1}a_1 + \dots + q_{\nu+s}a_s\,.$$

But $a_i(0) = 0$, hence $q_\nu(0) = 0$. But then $\partial q_\nu/\partial z_j \equiv 0$, $j = 1,\dots,n-1$. Similarly, one finds $\partial^M q_\nu/\partial^{m_1} z_1 \dots \partial^{m_{n-1}} z_{n-1} = 0$, hence $q \equiv 0$. But then $R = 0$.

We now assume $f = \sum_{j=s}^\infty f_j(Z)z^j$. This is no loss of generality, since the terms of order $< s$ in z may be included in R.

Assume $a_j(Z) = 0(\|Z\|^j)$; this may be achieved by replacing z_i by ζ_i^N, $N > j$, $i = 1,\dots,n-1$ in f and P, since $a_j(0) = 0$. The transformation back to the z_i is achieved as in Theorem 46 since we have already established uniqueness.

Now

$$z^s = A_s P + \alpha_{s1}(Z)z^{s-1} + \dots + \alpha_{ss}(Z)$$

where $A_s = 1$ and $\alpha_{sj} = -a_j$. Hence, multiplying successively by z^m, $m = 1,2,\dots$ and substituting appropriately gives:

$$z^{s+m} = A_{s+m}(Z,z)P + \alpha_{s+m,1}(Z)z^{s-1} + \dots + \alpha_{s+m,s}(Z)$$

where in fact: $A_{s+m+1} = zA_{s+m} + A_s\alpha_{s+m,1}$

$$\alpha_{s+m+1,p} = \alpha_{s+m,1}\,\alpha_{sp} + \alpha_{s+m,p+1}; \quad 0 < p < s$$

and $\alpha_{s+m+1,s} = \alpha_{s+m,1}\,\alpha_{ss}$.

Hence $A_{s+m+1} = \sum_{j=0}^{m+1} \alpha_{s+m-j,1}\, z^j$; $\alpha_{s-1,1} = 1$ is bounded in a neighborhood of the origin since $|\alpha_{sp}| < [c\|Z\|]^p$ implies $|\alpha_{s+m,p}| < (c\|Z\|)^{m+p}$, (which is easily established by an induction over m). But this means that all series in the following expression for $f(Z,z)$, obtained by substitution for z^{s+m}, converge in a neighborhood of the origin.

$$f(Z,z) = P(Z,z)\cdot\Big(\sum_{j=s}^{\infty} f_j\,A_j\Big) + z^{s-1}\Big(\sum_{j=s}^{\infty} f_j\,\alpha_{j1}\Big) + \dots + \Big(\sum_{j=s}^{\infty} f_j\,\alpha_{js}\Big) .$$

This is the required representation of f.

§4. The third step

Recall the statement of the missing step in the proof of Theorem 45:

III. If Theorem 45 is true for some n and all q, then it is true for $n+1$ and $q = 1$.

Proof. Consider the equation:

$$\sum_{i=1}^{p} a_i(z_1,\dots,z_{n+1})\, f_i(z_1,\dots,z_{n+1}) = 0 .$$

We may assume that not all $a_i \equiv 0$. Write $a_i = h_i P_i$, valid in some neighborhood of the origin, using a linear change of variables if necessary (4, §1, Property 2), where h_i is a unit and P_i is a polynomial in z_{n+1}. It suffices to consider the equation

$$\sum_{i=1}^{p} P_i\, c_i = 0 ,$$

where, by renumbering if necessary, we may assume $\alpha = \deg P_p \geq \deg P_j$.

We first show that there exists a neighborhood of the origin in which every solution may be represented as a linear combination, with holomorphic coefficients, of polynomials in z_{n+1} of bounded degrees!

Let (C,c) be any point at which the P_i are analytic. Write:

$$P_p = P^I P^{II}$$

where $P^{II}(C,c) \neq 0$ and $P^I(C,z) = (z-c)^r$, where P^I, P^{II} are polynomials in $z = z_{n+1}$, of degree $\leq \alpha$.

Let (c_i) be a holomorphic solution of: $\sum_{i=1}^{p} P_i c_i = 0$, in a neighborhood of (C,c). Using the division theorem:

$$c_i = \mu_i P^I + \hat{c}_i, \quad i = 1,\ldots,p-1;$$

where $\hat{c}_i$ is a polynomial in z with coefficients holomorphic in a neighborhood of C and $\deg \hat{c}_i < \deg P^I \leq \alpha$.

Consider:

$$\begin{pmatrix} c_1 \\ \\ \vdots \\ \\ c_{p-1} \\ c_p \end{pmatrix} = \frac{\mu_1}{P^{II}} \begin{pmatrix} P_p \\ 0 \\ \vdots \\ \\ 0 \\ -P_1 \end{pmatrix} + \frac{\mu_2}{P^{II}} \begin{pmatrix} 0 \\ P_p \\ 0 \\ \vdots \\ 0 \\ -P_2 \end{pmatrix} + \cdots + \frac{\mu_{p-1}}{P^{II}} \begin{pmatrix} 0 \\ \\ \vdots \\ 0 \\ P_p \\ -P_{p-1} \end{pmatrix} + \frac{1}{P^{II}} \begin{pmatrix} \hat{c}_1 P^{II} \\ \hat{c}_2 P^{II} \\ \vdots \\ \\ \hat{c}_{p-1} P^{II} \\ \hat{c}_p P^{II} \end{pmatrix}$$

where $\hat{c}_p$ is chosen so that this equation holds identically. Each of the column vectors, except perhaps the last, is a solution, so the last is also. Furthermore, we will have expressed (c_i) as a sum of polynomial solutions if we can show that $\hat{c}_p P^{II}$ is a polynomial in z. Note also that all entries of these polynomial solutions (except perhaps $\hat{c}_p P^{II}$) are bounded in degree by 2α. Now $\hat{c}_p$ is defined by the equation:

$$\hat{C}_p = C_p + \frac{\mu_1 P_1}{P^{II}} + \dots + \frac{\mu_{p-1} P_{p-1}}{P^{II}}$$

so

$$P^{II} \hat{C}_p = P^{II} C_p + \sum_{i=1}^{p-1} \mu_i P_i ,$$

and is holomorphic. But

$$\begin{aligned} P^I \hat{C}_p &= P^I C_p + \frac{1}{P^{II}} \left(\sum_{i=1}^{p-1} \mu_i P^I P_i \right) \\ &= P^I C_p + \frac{1}{P^{II}} \left(\sum_{i=1}^{p-1} (C_i - \hat{C}_i) P_i \right) \\ &= -\frac{1}{P^{II}} \left(\sum_{i=1}^{p-1} \hat{C}_i P_i \right) . \end{aligned}$$

Using the division theorem,

$$-\sum_{i=1}^{p-1} (P_i \hat{C}_i) = q P^I + R ,$$

where q, R are polynomials and $\deg R < \deg P^I$. Hence

$$P^{II} \hat{C}_p = q + \frac{R}{P^I} .$$

But $P^{II} \hat{C}_p$ is holomorphic, so R vanishes of order $(\deg P^I) > \deg R$; hence $R \equiv 0$. Thus:

$$P^{II} \hat{C}_p = q ,$$

a polynomial. Furthermore, $\deg (\hat{C}_p P^{II}) = \deg \left(\dfrac{\sum_{i=1}^{p-1} P_i \hat{C}_i}{P^I} \right) =$ $\deg \left(\sum_{i=1}^{p-1} P_i \hat{C}_i \right) - \deg (P^I) \leq 2\alpha$. Hence we are only interested in solutions (C_i) of the form:

$$C_i(Z,z) = \sum_{j=0}^{2\alpha} g_{ij}(Z) z^j , \qquad i = 1,\dots,p,$$

But $$P_i(Z,z) = \sum_{\ell=0}^{\alpha} \pi_{i\ell}(Z) z^\ell , \qquad i = 1,\dots,p;$$

so $$\begin{aligned} \sum_i P_i C_i &= \sum_{i=1}^{p} \sum_{\ell=0}^{\alpha} \sum_{j=0}^{2\alpha} z^{\ell+j} g_{ij} \pi_{i\ell} \\ &= \sum_{\ell=0}^{\alpha} \sum_{j=0}^{2\alpha} z^{\ell+j} \left(\sum_{i=1}^{p} g_{ij} \pi_{i\ell} \right) . \end{aligned}$$

Hence $\sum_i P_i C_i = 0$ if and only if

$$\sum_{j+\ell=s} \sum_{i=1}^{p} g_{ij}(Z)\, \pi_{i\ell}(Z) = 0 \; ; \qquad s = 0,\ldots,3\alpha \; ,$$

where the $\pi_{i\ell}$ are known functions; and this system of equations involves n variables. Hence the solutions have a finite pseudobasis by the induction hypothesis, thus completing the proof.

§5. Consequences of Oka's theorem

A. Remarks on coherent sheaves.

1. Coherence is a local property. A sheaf is coherent if and only if every $x \in X$ has a neighborhood in which the induced sheaf is coherent.

2. A subsheaf $\underline{G}$ of a coherent sheaf $\underline{\mathcal{F}}$, is coherent if and only if it is locally finitely generated. For, any section of $\underline{G}$ is a section of $\underline{\mathcal{F}}$, since G_x is a subgroup of $\mathcal{F}_x$. Hence, the sheaf of relations $\underline{R}$ of any finite number of sections of $\underline{G}$ is the sheaf of relations between these sections, considered as sections of $\underline{\mathcal{F}}$. Since $\underline{\mathcal{F}}$ is coherent, $\underline{R}$ is locally finitely generated.

B. Corollary. If $\underline{\mathcal{F}}$ is coherent and $s_1,\ldots,s_k$ are sections of $\underline{\mathcal{F}}$, then the sheaf of relations $\underline{R}(s_1,\ldots,s_k)$ is coherent.

Proof. $\underline{R}$ is a subsheaf of $\underline{\mathcal{O}}^k$, a coherent sheaf, and is locally finitely generated, by the definition of coherence of $\underline{\mathcal{F}}$.

Theorem 48. Let $\underline{\mathcal{F}}$ and $\underline{G}$ be coherent sheaves, $\underline{G}$ a subsheaf of $\underline{\mathcal{F}}$. Then the quotient sheaf $\underline{\mathcal{F}}/\underline{G}$ is coherent.

Proof. For every $x \in X$, there is a neighborhood N_x in which a finite number of sections $s_1,\ldots,s_k$ of $\underline{\mathcal{F}}(N_x)$ generate the stalk at every $y \in N_x$. The images of $s_1,\ldots,s_k$ under the natural homomorphism of $\underline{\mathcal{F}}$ into $\underline{\mathcal{F}}/\underline{G}$ are sections of $\underline{\mathcal{F}}(N_x)/\underline{G}(N_x)$ generating the factor stalk at every $y \in N_x$. Hence $\underline{\mathcal{F}}/\underline{G}$ is globally finitely generated.

To show that the sheaf of relations of any finite number of sections of $\underline{\mathcal{F}}/\underline{G}$ is locally finitely generated, consider an open set $V \subset X$, sections $t_1, \dots, t_k$ of $\underline{\mathcal{F}}(V)/\underline{G}(V)$, a point $x \in V$, and a neighborhood N_x of x in V. Construct sections $s_1, \dots, s_k$ of $\underline{\mathcal{F}}$ as follows: For each fixed j, $(t_j)_x \in \mathcal{F}_x/G_x$ is the image of an $f_j \in \mathcal{F}_x$ under the natural homomorphism h of $\underline{\mathcal{F}}$ into $\underline{\mathcal{F}}/\underline{G}$. $\underline{\mathcal{F}}$ is coherent; thus in perhaps a smaller neighborhood of x, there exists a section s_j of $\underline{\mathcal{F}}$ with $(s_j)_x = f_j$. Let t_j' be the image of s_j under h. Then $(t_j')_x = (t_j)_x$, implying that $t_j' = t_j$ near x. Therefore, there is a neighborhood U of x in which we may assume that t_j is the image of s_j, $j = 1, \dots, k$, and since G is coherent, that $r_1, \dots, r_p$ are sections of $\underline{G}(U)$ generating the stalk at every $y \in U$. Now, consider any element of $(\underline{R}(t_1, \dots, t_k))_x$: it is a relation $(\phi_1, \dots, \phi_k) \in \mathcal{O}_x^k$ with $\sum_{j=1}^{k} \phi_j (t_j)_x = 0$. But $\sum_k \phi_j (t_j)_x = 0$ means that $\sum_p \phi_j (s_j)_x \in G_x$ which means $\sum_{j=1}^{k} \phi_j (s_j)_x = -\sum_{i=1}^{p} \psi_i (r_i)_x$, so that $(\phi_1, \dots, \phi_k, \psi_1, \dots, \psi_p) \in \mathcal{O}_x^{k+p}$ is a relation between the sections $s_1, \dots, s_k, r_1 \dots, r_p$ of $\underline{\mathcal{F}}$, a coherent sheaf. Thus in some neighborhood $N_x \subset U$ of x there are sections $\Phi_1^\nu, \dots, \Phi_k^\nu, \Psi_1^\nu, \dots, \Psi_p^\nu$, $\nu = 1, \dots, N$ of $\underline{R}(s_1, \dots, r_p)(N_x)$ over N_x such that

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_k \\ \psi_1 \\ \vdots \\ \psi_p \end{pmatrix} = \sum_{\nu=1}^{N} \omega_\nu \begin{pmatrix} \Phi_1^\nu \\ \vdots \\ \Phi_k^\nu \\ \Psi_1^\nu \\ \vdots \\ \Psi_p^\nu \end{pmatrix} \quad , \quad \omega_\nu \in \mathcal{O}_x .$$

Hence $\phi_j = \sum_{\nu=1}^{N} \omega_\nu \, (\Phi_j^\nu)_x$, $j = 1, \dots, k$. Note that the vectors $(\Phi_1^\nu, \dots, \Phi_k^\nu)$, $\nu = 1, \dots, N$ <u>are</u> sections of $\underline{R}(t_1, \dots, t_k)(N_x)$ over N_x since they are continuous maps of N_x into $(\phi_1, \dots, \phi_k) \in \mathcal{O}_x^k$ such that $\sum \phi_j s_j + \sum \psi_i r_i = 0$, i.e. $\sum \phi_j s_j \in G_x$, i.e. $\sum \phi_j t_j = 0$.

Theorem 49. Suppose $\underline{A}$ and $\underline{B}$ are coherent sheaves and $f : \underline{A} \to \underline{B}$ is a homomorphism of $\underline{A}$ into $\underline{B}$. Then $\operatorname{im} f$, $\ker f$, $\operatorname{coker} f = \underline{B}/\operatorname{im} f$, and $\operatorname{coim} f = \underline{A}/\ker f$ are coherent sheaves.

Proof. It is sufficient to prove that $\operatorname{im} f$ and $\ker f$ are coherent, as the coherence of $\operatorname{coker} f$ and $\operatorname{coim} f$ is then given by Theorem 48. To establish the coherence of $\operatorname{im} f = f(\underline{A})$ and $\ker f$, we need only show that they are locally finitely generated since we already know that $\operatorname{im} f$ is a subsheaf of $\underline{B}$ and $\ker f$ is a subsheaf of $\underline{A}$ (cf. p. 151).

1. Since $\underline{A}$ is locally finitely generated, every $x \in X$ has a neighborhood N_x in which a finite number of sections $s_1,\dots,s_k$ of $\underline{A}(N_x)$ over N_x generate $(A(N_x))_y$, $y \in N_x$. Their images under f generate $((\operatorname{im} f)(N_x))_y$, $y \in N_x$. Thus $\operatorname{im} f$ is locally finitely generated.

2. For $\ker f$, take x, N_x and $s_1,\dots,s_k$ as above. Since $f(s_1),\dots,f(s_k)$ are sections of $\underline{B}(N_x)$ over N_x, $\underline{R}(f(s_1),\dots,f(s_k))$ is a coherent sheaf. Consider the following mapping $g : R_y \to A_y$ of R_y into A_y, $y \in N_x$:

$$\begin{pmatrix}\phi_1\\ \vdots\\ \phi_k\end{pmatrix} \to \sum_{i=1}^{k} \phi_i(s_i)_y .$$

Since $f(\sum \phi_i(s_i)_y) = \sum \phi_i\, f(s_i)_y = 0$, g is a homomorphism of $\underline{R}$ into $(\ker f) \mid N_x$. But every element of $((\ker f) \mid N_x)_y$ is of the form $\sum \phi_i(s_i)_y$ with $\phi_i \in \mathcal{O}_x$ and $\sum \phi_i\, f(s_i)_y = 0$. Hence g is onto. Therefore $\ker f \mid N_x$ is the image under a homomorphism of a coherent sheaf and thus is coherent, by part 1.

§6. The sheaf of ideals of a variety

Definition 75. Let X be a complex manifold. An *analytic set* V in X is a *closed* subset of X such that for every $v \in V$ there is a neighborhood N_v of v and functions $\phi_1,\ldots,\phi_k$ holomorphic in N_v such that $V \cap N_v = \{x \in X \mid \phi_1(x) = \phi_2(x) = \ldots = \phi_k(x) = 0\}$.

Definition 76. For $x \in X$ and V an analytic set in X, let $\mathcal{O}_x^V$ be the subset of $\mathcal{O}_x$ of all the germs of holomorphic functions vanishing on V. (If $x \notin V$ then $\mathcal{O}_x = \mathcal{O}_x^V$; if $V \equiv X$ then $\mathcal{O}_x^V$ is trivial.) $\mathcal{O}_x^V$ is a subgroup of $\mathcal{O}_x$ and an ideal. $\bigcup_{x\in X} \mathcal{O}_x^V$ with induced topology is a subsheaf of $\underline{\mathcal{O}}$ called the sheaf of germs of the ideals of the analytic set V, denoted $\mathcal{J}_V(X)$.

Theorem 50. (Cartan) If V is an analytic set in a complex manifold X then $\mathcal{J}_V(X)$ is coherent.

We will only prove a weaker form of this theorem.

Theorem 51. If V is a regularly imbedded, analytic subvariety of codimension k in an n-dimensional complex manifold X, then $\mathcal{J}_V(X)$ is coherent.

Proof. $\mathcal{J}_V(X)$ is a subsheaf of $\underline{\mathcal{O}}$, a coherent sheaf. Hence it suffices to show that $\mathcal{J}_V(X)$ is locally finitely generated. Let $x \in X$. If $x \notin V$ then there is a neighborhood N_x of x with $N_x \cap V = \phi$. In N_x, $\mathcal{J}_V(X) = \underline{\mathcal{O}}$, and we are done. If $x \in V$, then by the definition of V we can introduce local coordinates $z_1,\ldots,z_n$ such that $V = \{(z_1,\ldots,z_n) \in X \mid z_1 = z_2 = \ldots = z_k = 0\}$, in some neighborhood N_x of x in X. Now, if $f \in (\mathcal{J}_V(N_x))_x$, then f is the germ of a function holomorphic in N_x and vanishing on V, so that $f = \sum_{i=1}^{k} \omega_i z_i$, where the ω_i are holomorphic in N_x. Since $z_1,\ldots,z_k$ are sections of $\mathcal{J}_V(N_x)$ over N_x and the $\omega_i \in \mathcal{O}_x$, the proof is complete.

Remark. $\underline{\mathcal{O}}(X)/\mathcal{J}_V(X)$ is a coherent sheaf, by Theorem 48. Its stalk over every point off of V is trivial. On V, since $(\mathcal{J}_V(V))_x$ is trivial, its stalk is the stalk of $\underline{\mathcal{O}}(V)$. Hence, on V this sheaf can be identified with the sheaf $\mathcal{O}(V)$ of germs of holomorphic functions on V. $\underline{\mathcal{O}}(X)/\mathcal{J}_V(X)$ is the trivial extension of $\underline{\mathcal{O}}(V)$ to X.

Chapter 16. Fundamental Theorems (semi-local form)

§1. Statement of the fundamental theorems for a box (semi-local form)

Notation. By an open box in $\mathbb{C}^n$, we mean a $\{(z_1,\dots,z_n) \in \mathbb{C}^n \mid a_i < x_i < a_i'$ and $b_i < y_i < b_i'$ for all $i = 1,\dots,n$; where $z_i = x_i + iy_i$ and a_i, a_i', b_i, b_i' are real numbers or $\pm\infty\}$. By a closed box we mean the closure of a finite open box.

Theorem 52A. Let X be an open box in $\mathbb{C}^n$, $\underline{\mathcal{F}}$ a coherent sheaf over X, and $K \subset\subset X$. Then there exists an open box X_0 such that $K \subset\subset X_0 \subset\subset X$ and $\underline{\mathcal{F}}(X_0)$ is globally finitely generated.

Theorem 52B. Under the same hypothesis as in Theorem 52A, there exists an open box X_0 such that $K \subset\subset X_0 \subset\subset X$ and $H^q(X_0, \underline{\mathcal{F}}) = 0$ for all $q > 0$.

Note that these theorems hold for polydiscs as well as for boxes.

Remark. Theorem 52A implies the Fundamental Lemma (proof later, p. 196).

§2. First step of the proof

By a degenerate closed box we mean a closed box given by, say $a_i \le x_i \le a_i'$ and $b_i \le y_i \le b_i'$, $i = 1,\dots,n$ where some of the $a_i = a_i'$ and/or $b_i = b_i'$, i.e. some of the intervals degenerate into points. The number of non-degenerate intervals is the (real) dimension of the box.

For $r = 0,1,\dots,2n$ we formulate

Theorem 52A$_r$. Let X be an open box in $\mathbb{C}^n$, $\underline{\mathcal{F}}$ a coherent sheaf ofer X, and $K \subset X$ a degenerate closed box of dimension r. Then there exists an open box X_0 such that $K \subset\subset X_0 \subset\subset X$ and $\underline{\mathcal{F}}(X_0)$ is globally finitely generated.

and Theorem 52B$_r$. Under the same hypothesis as in Theorem 52A$_r$, there exists an open box X_0 such that $K \subset\subset X_0 \subset\subset X$ and

$H^q(X_0,\underline{\mathcal{F}}) = 0$ for all $q > 0$.

Theorems $52A_{2n}$ and B_{2n} are Theorems 52A and B.

Proof of Theorems $52A_r$ and B_r for $r = 0,1,\ldots,2n$. The proof consists of verifying the following three statements.

(1) Theorem $52A_0$ is true.

(2) If Theorem $52A_j$ is true for all $j \leq r$, then Theorem $52B_r$ is true.

(3) If Theorems $52A_r$ and B_r are true, then Theorem $52A_{r+1}$ is true.

First we prove that if X is an open box in $\mathbb{C}^n$ and $\underline{\mathcal{F}}$ is any sheaf, then $H^q(X,\underline{\mathcal{F}}) = 0$ for $q > 2n$. It is enough to consider $H^q(X,U,\underline{\mathcal{F}})$ for U a locally finite covering of X. Since if $X \subset E^{2n}$ we can refine U to a covering in which more than 2n+1 sets always have an empty intersection, we are done.

That (1) is true follows from the definition of a coherent sheaf as being locally finitely generated.

To prove (2), assume Theorem $52A_j$ for $j \leq r$. Let K be the given degenerate box of dimension r and $\underline{\mathcal{F}}$ the coherent sheaf over X. Then there is a box X_0 with $K \subset\subset X_0 \subset\subset X$ and $\underline{\mathcal{F}}(X_0)$ globally finitely generated; call the generating sections $s_1,\ldots,s_{m_1}$. This means that there exists a homomorphism f_1 of $\underline{\mathcal{O}}^{m_1}(X_0)$ onto $\underline{\mathcal{F}}(X_0)$. For, define $f_1 : \mathcal{O}_x^{m_1} \to \underline{\mathcal{F}}_x$, $x \in X_0$ by $\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{m_1} \end{pmatrix} \to \sum \phi_i(s_i)_x$. It is onto because the s_i generate $(\mathcal{F}(X_0))_x$ at every $x \in X_0$. Therefore, denoting $\ker f_1$ by G_1, we have the following exact sequence $\underline{0} \to \underline{G}_1 \xrightarrow{i} \underline{\mathcal{O}}^{m_1} \xrightarrow{f_1} \underline{\mathcal{F}} \to \underline{0}$. By Theorem 49, $\underline{G}_1$ is coherent. Apply Theorem $52A_r$ to $\underline{G}_1$ over X_0. We get a new X_0, call it X_0^1 and $\underline{G}_1(X_0^1)$ globally finitely generated. Then there exist m_2 generating sections ... . Denoting $\ker f_2$ by G_2, we get the exact sequence $\underline{0} \to \underline{G}_2 \to \underline{\mathcal{O}}^{m_2} \to \underline{G}_1 \to \underline{0}$. Again, $\underline{G}_2$ is coherent and we can continue this process, as far as we want, up to 4n, obtaining $\underline{G}_3, \underline{G}_4, \ldots, \underline{G}_{4n}$ and $X_0^2 \supset\supset X_0^3 \supset\supset \ldots \supset\supset X_0^{4n-1}$. Call X_0^{4n-1}, X_0. Then for X_0 and every $\ell = 0,1,\ldots,4n-1$ the sequence $\underline{0} \to \underline{G}_{\ell+1} \to \underline{\mathcal{O}}^{m_{\ell+1}} \to \underline{G}_\ell \to \underline{0}$ is exact; $\underline{G}_0 = \underline{\mathcal{J}}$. Hence the cohomology sequence, $\ldots H^k(X_0,\underline{\mathcal{O}}^{m_{\ell+1}}) \to H^k(X_0,\underline{G}_\ell) \to H^{k+1}(X_0,\underline{G}_{\ell+1}) \to H^{k+1}(X_0,\underline{\mathcal{O}}^{m_{\ell+1}}) \to \ldots$ is exact for $k \geq 1$. But $H^{k+1}(X_0,\underline{\mathcal{O}}^{m_{\ell+1}}) = H^k(X_0,\underline{\mathcal{O}}^{m_{\ell+1}}) = 0$ because $H^q(X_0,\underline{\mathcal{O}}) = 0$ for $q > 0$. Thus $H^k(X_0,\underline{G}_\ell) \underset{\text{isom.}}{\simeq} H^{k+1}(X_0,\underline{G}_{\ell+1})$ for $k \geq 1$ and $\ell = 0,1,\ldots,4n-1$. Iterating we get $H^k(X_0,\underline{G}_0) \simeq H^{2k+\ell}(X_0,\underline{G}_{k+\ell})$, $k = 1,\ldots,2n$; $\ell = 0,1,\ldots,2n$. Let $\ell = 2n$, then $H^k(X_0,\underline{G}_0) \simeq H^{2k+2n}(X_0,\underline{G}_{k+2n})$, $k = 1,\ldots,2n$. Since $2k+2n > 2n$, $H^k(X_0,\underline{G}_0) \equiv H^k(X_0,\underline{\mathcal{J}}) = 0$ for all $k > 0$.

It remains to prove (3).

§3. Reduction of (3) to Cartan's theorem on holomorphic matrices

Lemma 1. Let K be an $r+1$-dimensional degenerate closed box as in Theorem 52 A_{r+1}, given by: $K = \{\alpha_i \leq x_i \leq \hat{\alpha}_i, \beta_i \leq y_i \leq \hat{\beta}_i\}$. Assume, e.g. that $\alpha_{i_0} < \hat{\alpha}_{i_0}$ (or similarly, that $\beta_{i_0} < \hat{\beta}_{i_0}$), for some i_0. $H = \{x_{i_0} = (\alpha_{i_0} + \hat{\alpha}_{i_0})/2\}$ is a $2n-1$-dimensional hyperplane. Set $K_1 = K \cap \{(\alpha_{i_0} + \hat{\alpha}_{i_0})/2 \leq x_{i_0}\}$, $K_2 = K \cap \{(\alpha_{i_0} + \hat{\alpha}_{i_0})/2 \geq x_{i_0}\}$. Then, if Theorem A_{r+1} holds for both K_1 and K_2, it holds for K.

Note. In view of the following claim, it is enough to prove Lemma 1 using only Theorems 52A_r and B_r.

Claim. Lemma 1 implies (3).

Proof. Assume (3) is false. Order all the nondegenerate dimensions cyclically. Cut K along a first nondegenerate dimension as in Lemma 1. Then (3) is false for at least one of the two resulting boxes; choose one and call it K_1. Now cut K_1 in the second nondegenerate dimension; (3) is then false for a still smaller box. Call it K_2.

Proceed, halving each nondegenerate dimension successively, obtaining a sequence $K_j \supset K_{j+1}$ of closed, nested boxes such that in no neighborhood of any box is the sheaf induced by $\mathcal{J}$ globally finitely generated. But the K_j intersect in a point, and a point surely has such a neighborhood; and this contradiction establishes the claim.

Lemma 2. Let $\hat{K}_1$, $\hat{K}_2$ be two closed boxes, given as follows:

$$\hat{K}_1 = \left\{ \begin{matrix} \alpha_1 \le x_1 \le \hat{\alpha}_1, \ \beta_1 \le y_1 \le \hat{\beta}_1 \\ \alpha_j \le x_j \le \hat{\alpha}_j, \ \beta_j \le y_j \le \hat{\beta}_j, \quad j = 2,\dots,n \end{matrix} \right\}$$

$$\hat{K}_2 = \left\{ \begin{matrix} \alpha_j \le x_j \le \hat{\alpha}_j, \ \beta_j \le y_j \le \hat{\beta}_j, \quad j = 2,\dots,n \\ \hat{\alpha}_1 - \varepsilon \le x_1 \le \hat{\hat{\alpha}}_1, \beta_1 \le y_1 \le \hat{\beta}_1 \end{matrix} \right\}$$

where $\alpha_1 \le \hat{\alpha}_1 - \varepsilon$. Let $\mathcal{J}$ be an analytic sheaf over a neighborhood of $\hat{K}_1 \cup \hat{K}_2$. Let there be given sections $a_1,\dots,a_r$ of $\mathcal{J}$ over a neighborhood of $\hat{K}_1$ generating the stalks of $\mathcal{J}(\hat{K}_1)$ at every point; and, similarly, sections $b_1,\dots,b_s$ of $\mathcal{J}$ over a neighborhood of $\hat{K}_2$. Furthermore, assume that in some neighborhood of $\hat{K}_1 \cap \hat{K}_2$, $a_i = \sum \phi_{ij} b_j$ and $b_j = \sum \psi_{jk} a_k$; ϕ_{ij}, ψ_{jk} functions holomorphic in this neighborhood.

Then there exist sections $c_1,\dots,c_N$ over a neighborhood of $\hat{K}_1 \cup \hat{K}_2$ such that, in a neighborhood of $\hat{K}_1$, $a_i = \sum \phi_{ij} c_j$; and, in a neighborhood of $\hat{K}_2$, $b_i = \sum \psi_{ij} c_j$.

Claim. Lemma 2 implies Lemma 1.

Let K_1, K_2 be as in Lemma 1, where we take $i_0 = 1$. We first show that, in some neighborhood X_0 of $K_1 \cap K_2$, $a_i = \sum \phi_{ij} b_j$ and $b_j = \sum \psi_{ji} a_i$; where ϕ_{ij}, ψ_{ji} are holomorphic in X_0; and a_i, b_j are the generating sections over K_1 and K_2, respectively, given by the hypothesized Theorem 52 A_{r+1}. But then the sheaf over $\hat{K}_2 = \{[(\alpha_1+\hat{\alpha}_1)/2 - \varepsilon] \le x_1 \le \hat{\alpha}_1, \dots\}$ will also be generated by $b_1,\dots,b_s$ for $\varepsilon > 0$ so small that $K_1 \cap \hat{K}_2 \cap X_0$.

By the induction hypothesis, there exists a neighborhood X_0 of $K_1 \cap K_2$ such that Theorems 52A_r and B_r apply; now consider the map $\underline{\mathcal{O}}^s \to \underline{\mathcal{F}}(X_0)$ given by $(\phi_1,\ldots,\phi_s) \to \sum_{i=1}^{s} \phi_i b_i$. This is a sheaf homomorphism, and the b_i generate the stalks of $\mathcal{F}(X_0)$ at every point, so that the sequence $\underline{\mathcal{O}}^s \to \underline{\mathcal{F}}(X_0) \to \underline{0}$ is exact. We complete to a short exact sequence:

$$\underline{0} \longrightarrow \underline{G} \longrightarrow \underline{\mathcal{O}}^s \longrightarrow \underline{\mathcal{F}}(X_0) \longrightarrow \underline{0} \quad ;$$

and $\underline{G}$ is coherent. Hence, we have an exact cohomology sequence:

$$H^0(X_0,\underline{\mathcal{O}}^s) \longrightarrow H^0(X_0,\underline{\mathcal{F}}(X_0)) \longrightarrow H^1(X_0,\underline{G}) \ .$$

If X_0 is small enough, Theorem $52B_r$ applies, so $H^1(X_0,\underline{G}) = 0$. Hence $H^0(X_0,\underline{\mathcal{O}}^s) \to H^0(X_0,\underline{\mathcal{F}})$ is onto; but this means that every section of $\underline{\mathcal{F}}$ over X_0 is a linear combination of the b_i ; in particular, the a_j are. In a similar manner, the b_i are a linear combination of the a_j.

Proof of Lemma 2. Under the hypothesis of Lemma 2, $a_i = \sum \phi_{ij} b_j$ and $b_j = \sum \psi_{jk} a_k$ where ϕ_{ij}, ψ_{jk} are holomorphic in a neighborhood of $\hat{K}_1 \cap \hat{K}_2$.

We adopt the following notation:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}; \quad \text{column vectors}$$

$\phi = (\phi_{ij})$ and $\psi = (\psi_{ij})$; matrices holomorphic in a neighborhood of $\hat{K}_1 \cap \hat{K}_2$. Then the hypothesis takes the form:

$$\phi b = a \ , \quad \psi a = b.$$

Now consider the $(r+s) \times (r+s)$ matrices defined below, which satisfy the following relations:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline \psi & I_s \end{array}\right)\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ b_1 \\ \vdots \\ b_s \end{pmatrix} ; \quad \left(\begin{array}{c|c} -I_r & \phi \\ \hline 0 & I_s \end{array}\right)\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ b_1 \\ \vdots \\ b_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ b_s \end{pmatrix} ;$$

where I_j denotes the $j \times j$ unit matrix. Define

$$M = \left(\begin{array}{c|c} -I_r & \phi \\ \hline 0 & I_s \end{array}\right)\left(\begin{array}{c|c} I_r & 0 \\ \hline \psi & I_s \end{array}\right)$$

Then $M\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$, and M is a nonsingular matrix holomorphic in a neighborhood of $\hat{K}_1 \cap \hat{K}_2$.

If we can write $M = M_1^{-1}M_2$, where M_1 is a holomorphic nonsingular matrix defined in a neighborhood of $\hat{K}_2$, and M_2 a holomorphic nonsingular matrix defined in a neighborhood of $\hat{K}_1$. Then $M_2\begin{pmatrix} a \\ 0 \end{pmatrix} = M_1\begin{pmatrix} 0 \\ b \end{pmatrix}$. Set

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_{r+s} \end{pmatrix} = M_2\begin{pmatrix} a \\ 0 \end{pmatrix} = M_1\begin{pmatrix} 0 \\ b \end{pmatrix}.$$

Then the c_j are global sections generating $\mathcal{F}$ in a neighborhood of $\hat{K}_1 \cup \hat{K}_2$, since the rows of $M_2\begin{pmatrix} a \\ 0 \end{pmatrix}$ are sections generating $\mathcal{F}$ in a neighborhood of $\hat{K}_1$, $M_1\begin{pmatrix} 0 \\ b \end{pmatrix}$ sections in a neighborhood of $\hat{K}_2$; and $M_2\begin{pmatrix} a \\ 0 \end{pmatrix} = M_1\begin{pmatrix} 0 \\ b \end{pmatrix}$ in a neighborhood of $\hat{K}_1 \cap \hat{K}_2$; and both M_1, M_2 are invertible.

Hence, the proof of Theorems A and B is reduced to:

Lemma 3. (Cartan's Theorem on Holomorphic Matrices) Let M be a holomorphic nonsingular matrix defined in a neighborhood of $\hat{K}_1 \cap \hat{K}_2$. Then there exist holomorphic nonsingular matrices A, defined in a neighborhood of $\hat{K}_1$, and B, defined in a neighborhood of $\hat{K}_2$, such that $M = BA$.

§4. Proof of Cartan's Theorem on Holomorphic Matrices

A. Recall that $\|z\| \equiv \max\limits_{j=1,..,n} |z_j|$ where $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$. Let A be an $N \times N$ matrix; set

$$\|A\| \equiv \sup \frac{\|Ax\|}{\|x\|}.$$

Note that $|a_{ij}| \leq \|A\| \leq N \max |a_{ij}|$.

Let $A(z)$ denote an $N \times N$ matrix whose entries are functions holomorphic in a domain $D \subset \mathbb{C}^n$. Set:

$$|||A||| = \max_{z \in D} \|A(z)\|$$

and

$$|||A|||_\alpha = |||A||| + H ,$$

where $0 < \alpha < 1$ and H is the smallest constant such that $|a_{ij}(z)-a_{ij}(\hat{z})| \leq H|z-\hat{z}|^\alpha$.

Note that if $D_1 \subset\subset D$, domains, then $|||A|||_{\alpha,D_1} \leq k|||A|||_D$.

It is known that $||| \; |||_\alpha$ is a norm; and if $D \subset \mathbb{C}^n$, the $N \times N$ matrices whose entries are functions holomorphic in D form a Banach space under this norm.

Furthermore:

$$|||AB||| \leq |||A||| \cdot |||B||| ,$$

$$|||AB|||_\alpha \leq C \, |||A|||_\alpha \, |||B|||_\alpha .$$

(The proof of the above statements is left as an exercise.)

Note that, for any matrix A, $e^A \equiv \sum \frac{A^n}{n!}$ is dominated by $\sum \frac{|||A|||^n}{n!}$. If $|||A||| < 1$, then $\log (I+A) = A - A^2/2 + A^3/3 - A^4/4 + \ldots$ is dominated by $|||A||| + |||A|||^2/2 + \ldots$. Furthermore, e^A is always nonsingular, and $e^{\log(I+A)} = I+A$.

B. The following propositions establish Lemma 3:

<u>Proposition 1</u>. Let D be a polydisc, $D_1 \subset\subset D$ and M a holomorphic nonsingular matrix defined in D. Given $\varepsilon > 0$, there exists a nonsingular entire matrix P such that $M = PM_1$ in D_1 and $|||I-M_1|||_{D_1} < \varepsilon$.

<u>Proposition 2</u>. There exists an $\varepsilon > 0$ such that, if $|||I-M|||_\alpha < \varepsilon$ then Lemma 3 holds for M.

<u>Claim</u>. Propositions 1 and 2 imply Lemma 3.

<u>Proof</u>. By Proposition 1, $M = PM_1$, $|||I-M_1|||_{D_1} < \varepsilon$. Then in a smaller domain, $|||I-M_1|||_\alpha < \varepsilon$. Hence, $M_1 = BA$ by Proposition 2. But P is holomorphic everywhere, so $M = (PB)A$.

Proof of Proposition 1. Assume that D is a polydisc about 0. Write:

$$M(z) = M(0)\{M(0)^{-1}M(\tfrac{1}{L}z)\}\{M^{-1}(\tfrac{1}{L}z)\ M(\tfrac{2}{L}z)\} \cdots \{M^{-1}(\tfrac{L-1}{L}z)\ M(z)\} ,$$

L an integer. Now

$$|||I-M^{-1}(\tfrac{K}{L}z)M(\tfrac{K+1}{L}z)||| \leq |||M^{-1}(\tfrac{K}{L}z)|||\ |||M(\tfrac{K}{L}z)-M(\tfrac{K+1}{L}z)||| < \varepsilon$$

for L sufficiently large, $0 \leq K < L$. Hence:

$$M^{-1}\ (\tfrac{K}{L}\ z)\ M(\tfrac{K+1}{L}\ z) = e^{N_K(z)} ,$$

as the log series converges. So:

$$M(z) = M(0)\ e^{N_0(z)} \cdots e^{N_{L-1}(z)} .$$

By going to a smaller polydisc, $\hat{D}$, if necessary, for each $N_K(z)$ there exists a polynomial sequence $P_{Kj}(z)$ such that

$||P_{Kj}(z)-N_K(z)||_{\hat{D}} \to 0$. Set $M_j(z) = M(0)e^{P_{0j}(z)}\, e^{P_{1j}(z)} \cdots e^{P_{L-1,j}(z)}$.
Then $|||M_j(z)-M(z)|||_{\hat{D}} \to 0$, and $\det M_j \neq 0$ for j sufficiently large. Hence $M(z) = M_j(z)\ \{M_j^{-1}(z)\ M(z)\}$, and $M_j^{-1}(z)\ M(z)$ converges uniformly to I with j; and M_j is a matrix of entire functions, for all j.

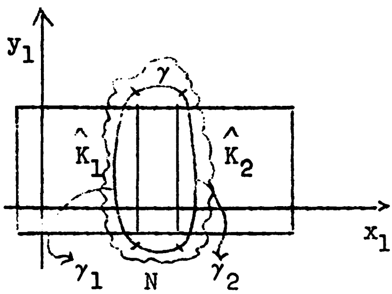
Proof of Proposition 2. Let $\hat{K}_1$, $\hat{K}_2$ be given as follows:

$$\hat{K}_1 = \begin{cases} \alpha_1 \leq x_1 \leq \hat{\alpha}_1 , & \beta_1 \leq y_1 \leq \hat{\beta}_1 \\ \alpha_j \leq x_j \leq \hat{\alpha}_j , & \beta_j \leq y_j \leq \hat{\beta}_j ; \quad j = 2,\ldots,n \end{cases}$$

$$\hat{K}_2 = \begin{cases} \alpha_j \leq x_j \leq \hat{\alpha}_j , & \beta_j \leq y_j \leq \hat{\beta}_j ; \quad j = 2,\ldots,n \\ \hat{\alpha}_1-\varepsilon \leq x_1 \leq \hat{\hat{\alpha}}_1 , & \beta_1 \leq y_1 \leq \hat{\beta}_1 \end{cases}$$

where $\alpha_1 \leq \hat{\alpha}_1-\varepsilon$, as in lemmas 2 and 3.

M is defined in a neighborhood N of $\hat{K}_1 \cap \hat{K}_2$; we write $M = M(z)$, where $z \equiv z_1$ and the dependence on $z_2,\ldots,z_n$ is suppressed. Now $M = I + X$, and $|||X|||_\alpha < \varepsilon$ where ε is to be chosen later. In the z_1-plane, let γ be a smooth analytic Jordan curve containing $\hat{K}_1 \cap \hat{K}_2$ and lying in N (more precisely, their projections on the z_1-plane). Let γ_1 and γ_2 denote disjoint closed arcs, segments of γ, such that $\gamma \cap \hat{K}_1 \subset \text{int}\,\gamma_1$ and $\gamma \cap \hat{K}_2 \subset \text{int}\,\gamma_2$, where "int γ_i" denotes the segment γ_i without its endpoints, as indicated in the diagram.



Let G denote the bounded component of the complement of γ in the z_1-plane, and $\bar{G}$ its closure. Take σ to be a real-valued C^∞ function defined on γ, $0 \le \sigma \le 1$, such that $\sigma \equiv 0$ on γ_1 and $\sigma \equiv 1$ on γ_2. For any matrix Y holomorphic in G and continuous in $\bar{G}$, define:

$$T_1(Y) = \frac{1}{2\pi i} \oint_\gamma \frac{Y(\zeta)\,\sigma(\zeta)}{\zeta - z}\, d\zeta$$

$$T_2(Y) = Y - T_1(Y) = \frac{1}{2\pi i} \oint_\gamma \frac{Y(\zeta)\{1-\sigma(\zeta)\}}{\zeta - z}\, d\zeta$$

Then $T_i(Y)$ is holomorphic in a neighborhood of $\hat{K}_i$; and the linear operator $Y \to T_i(Y)$ is bounded in $\bar{G}$; $|||T_i(Y)|||_\alpha \le c|||Y|||_\alpha$.

We wish to solve the equation:

$$\begin{aligned} I + X &= (I + T_1(Y))\,(I + T_2(Y)) \\ &= (I + T_1(Y))\,(I + Y - T_1(Y))\,, \end{aligned}$$

for some matrix Y holomorphic in G and continuous in $\bar{G}$; for $|||X|||_\alpha$ sufficiently small; i.e. we wish to solve:

$$\begin{aligned} X &= Y - T_1(Y)\,T_1(Y) + Y\,T_1(Y) \\ &= Y + F(Y)\,. \end{aligned}$$

Therefore, define $T(Y) = X - F(Y)$, for $Y \in S = \{Y \mid Y$ holomorphic in G and continuous in $\bar{G}$, $|||Y-X|||_\alpha < \varepsilon$, $|||X|||_\alpha < \varepsilon\}$. Note that $T(S) \subset S$. We claim that, if ε

is small enough, T is contracting; so that the contracting mapping principle applies and there exists a unique $Y_0 \in S$ such that $Y_0 = X - F(Y_0)$, as desired.

Now $|||T_1(Y)|||_\alpha \leq c|||Y|||_\alpha$ implies that for some constant $k(c,C)$ depending on c and C (of p. 181), $|||T(Y)-X|||_\alpha = |||F(Y)|||_\alpha \leq k|||Y|||_\alpha^2 \leq k[|||Y-X|||_\alpha + |||X|||_\alpha]^2 < 4k\varepsilon^2$; so we require $\varepsilon < 1/4k$. Furthermore:

$$\begin{aligned} |||T(Y)-T(Z)|||_\alpha &= |||F(Y)-F(Z)|||_\alpha \\ &= |||(Y-Z)T_1(Y)+ZT_1(Y-Z)+T_1(Z)T_1(Z-Y) \\ &\quad + T_1(Y)\,T_1(Z-Y)|||_\alpha \\ &\leq |||Y-Z|||_\alpha\ K(K+1)\ (|||Y|||_\alpha + |||Z|||_\alpha) \end{aligned}$$

for some constant $K(c,C)$. Now $|||Y-X|||_\alpha < \varepsilon$, hence $0 \leq |||Y|||_\alpha \leq 2\varepsilon$, so $|||T(Y)-T(Z)|||_\alpha \leq 4\varepsilon\, K(K+1)\ |||Y-Z|||_\alpha$, and we require also $\varepsilon < 1/[4K(K+1)]$.

§5. New proof of the Oka-Weil Approximation Theorem

Theorem 53. Let X be an analytic polyhedron, $X \subset\subset G^{\text{open}} \subset \mathbb{C}^n$, $X = \{z \in G \mid |f_j(z)|<1,\ j=1,\ldots,r;\ f_j$ holomorphic in $G\}$. Then, given $\phi(z)$ holomorphic in X, ϕ can be approximated on any $K \subset\subset X$ by functions holomorphic in G, in fact by polynomials in $z_1,\ldots,z_n,f_1,\ldots,f_r$.

Proof. Assume that $G \subset (|z_j|<k<1)$, and that $K \subset\subset X$ is given. Let D denote the Oka image of X; $D = \{(z,\zeta) \mid |z_i|, |\zeta_i|<1,\ z \in G,\ \zeta_i = f_i(z)\}$. By Theorem 51, $\mathcal{J}_X(D)$, the sheaf of ideals of X, is coherent. Hence by Theorem 52B, there exists an Oka image D_ε of an X_ε, $X_\varepsilon = \{z \in G \mid |f_j(z)| < 1-\varepsilon,\ j=1,\ldots,r\}$ such that $K \subset\subset X_\varepsilon \subset\subset X$ and since $\mathcal{J}_X(D_\varepsilon) = \mathcal{J}_{X_\varepsilon}(D_\varepsilon)$, $H^q(D_\varepsilon,\mathcal{J}_{X_\varepsilon}) = 0$ for all $q > 0$.

Now X_ε and X are regularly imbedded analytic subvarieties of codimension r in D_ε and D, respectively. We claim that the function $\phi(z)$ can be extended holomorphically into D_ε. Consider the sheaf $\mathcal{J}_{X_\varepsilon}(D_\varepsilon)$. It is a subsheaf of

$\underline{\mathcal{O}}(D_\varepsilon)$. Since $\underline{\mathcal{O}}(D_\varepsilon)/\mathcal{J}_{X_\varepsilon}(D_\varepsilon) = \underline{\mathcal{O}}(X_\varepsilon)$, (cf. remark at end of Chapter 15), $\underline{0} \to \mathcal{J}_{X_\varepsilon}(D_\varepsilon) \xrightarrow{i} \underline{\mathcal{O}}(D_\varepsilon) \xrightarrow{r} \underline{\mathcal{O}}(X_\varepsilon) \to \underline{0}$ is exact: i = inclusion, r = restriction. Therefore, the cohomology sequence $H^0(D_\varepsilon, \underline{\mathcal{O}}) \xrightarrow{r} H^0(X_\varepsilon, \underline{\mathcal{O}}) \to H^1(D_\varepsilon, \mathcal{J}_{X_\varepsilon})$ is exact. But $H^1(D_\varepsilon, \mathcal{J}_{X_\varepsilon}) = 0$, implying that r is onto.

Hence ϕ is the restriction of a function ψ holomorphic in D_ε. Write ψ as a power series $\sum a_{j_1 \ldots i_k} z_1^{j_1} \ldots \zeta_k^{i_k}$ converging uniformly in some D_{ε_1}, $K \subset\subset X_{\varepsilon_1} \subset\subset X_\varepsilon$. Then $\psi = \sum p_j(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_r)$ on D_{ε_1}; the p_j are polynomials. Therefore on K, $\phi = \sum p_j(z_1, \ldots, z_n, f_1, \ldots, f_r)$. Since $z_1, \ldots, z_n, f_1(z), \ldots, f_r(z)$ are holomorphic on G, so are the p_j.

§6. Fundamental Theorems for regions of holomorphy (semi-local form)

Theorem 54A. Let X be a region of holomorphy, $\underline{\mathcal{F}}$ a coherent sheaf over X and $K \subset\subset X$. Then there exists an analytic polyhedron X_1 such that $K \subset\subset X_1 \subset\subset X$ and a finite number of sections of $\underline{\mathcal{F}}(X_1)$ generate $\mathcal{F}_x$ at every $x \in X_1$.

Theorem 54B. Under the same hypothesis as in Theorem 54A, there exists an analytic polyhedron X_1 such that $K \subset\subset X_1 \subset\subset X$ and $H^q(X_1, \underline{\mathcal{F}}) = 0$ for all $q > 0$.

Proof of A. Exhause X by analytic polyhedra, $X_j \subset\subset X_{j+1} \subset\subset X$, $\bigcup X_j = X$. Pick one of the X_j, call it X_0, satisfying $K \subset\subset X_0 \subset\subset X$. Let $f_1, \ldots, f_r$ be the functions defining X_0, and let D be its Oka image. X_0 is a regularly imbedded analytic subvariety of codimension r in D.

Define the sheaf $\check{\underline{\mathcal{F}}}$ over D as follows: let

$$\check{\mathcal{F}}_x = \begin{Bmatrix} \mathcal{F}_x \text{ if } x \in X_0 \\ 0 \text{ otherwise} \end{Bmatrix},$$ and let $A \subset \check{\underline{\mathcal{F}}}$ be open if

and only if pA is open in D and $A \cap \check{\mathcal{F}}$ is open in $\check{\mathcal{F}}$ (p is the projection map in the sheaf $\check{\mathcal{F}}$). Since $\mathcal{F}(X_0)$ is coherent, in particular analytic, $\check{\mathcal{F}}$ is an analytic sheaf. We claim that $\check{\mathcal{F}}$ is coherent. Introduce local coordinates $\eta_1,\ldots,\eta_{n+r}$ in a neighborhood N in D of a point of X_0 such that $X_0 \cap N$ is given by $\eta_1 = \ldots = \eta_r = 0$. $\mathcal{F}$ is locally finitely generated. Consider its sheaf of relations $\underline{R}(s_1,\ldots,s_\ell)$, i.e. let $s_1,\ldots,s_\ell$ be sections of $\check{\mathcal{F}}$ over some open set $U \subset (D \cap N)$. A relation is a set of functions $\phi_j(\eta_1,\ldots,\eta_{n+r}) \in \mathcal{O}_x$, $x \in U$, $j=1,\ldots,\ell$, satisfying $\sum_1^\ell \phi_j(s_j)_x = 0$. But $(s_j)_x = 0$ for $x \notin X_0$, so that the ϕ_j satisfy $\sum_1^\ell \phi_j(0,\ldots,0,\eta_{r+1},\ldots,\eta_{n+r})(s_j)_x = 0$, $x \in U \cap X_0$. Thus the $\{\phi_j(0,\ldots,0,\eta_{r+1},\ldots,\eta_{n+r})\}$ are relations of $\mathcal{F}$ over $U \cap X_0$ and hence are locally finitely generated, say by $(\psi_1^\nu,\ldots,\psi_\ell^\nu)$, $\nu = 1,\ldots,N$ in a neighborhood N_x of x. Therefore as generators of the sheaf $\underline{R}$ in N_x take the $(\psi_1^\nu,\ldots,\psi_\ell^\nu)$ and add the ℓ-tuples $(\eta_i,0,\ldots,0)$, $(0,\eta_i,0,\ldots,0),\ldots,(0,\ldots,0,\eta_i)$ for $i = 1,\ldots,r$. Hence $\check{\mathcal{F}}$ is a coherent sheaf over D.

Apply Theorem 52A to $\check{\mathcal{F}}(D)$. Then there exists an Oka image D_1 of an X_1, $X_1 = \{z \in G \mid |f_j(z)| < 1-\varepsilon,\ j=1,\ldots,r\}$ such that $K \subset\subset X_1 \subset\subset X_0$ and $\check{\mathcal{F}}(D_1)$ is globally finitely generated, say by $t_1,\ldots,t_k$. The restrictions of the t_i to X_1 generate $(\mathcal{F}(X_1))_x$ at every $x \in X_1$.

Proof of B. It is enough to show that $H^q(D_1,\check{\mathcal{F}}) = 0$ for all $q > 0$. For, consider any covering U of X_1, think of it as a covering of D_1. A cochain on $U \mid X_1$ is an assignment of a section of $\mathcal{F}$, and hence by the trivial extension, it can be considered on U and is an assignment of a section of $\check{\mathcal{F}}$: hence a cochain on U. A cocycle on $U \mid X_1$ is a cochain satisfying a certain relation. By the trivial extension, it is a cocycle on U. Hence if $H^q(D_1,\check{\mathcal{F}}) = 0$ then $H^q(X_1,\mathcal{F}) = 0$ for $q > 0$. By Theorem 52B, $H^q(D_1,\check{\mathcal{F}}) = 0$ for all $q > 0$.

Chapter 17. Coherent Sheaves in Regions of Holomorphy

§1. Statement of the Fundamental Theorems

Theorem 55A. Let X be a region of holomorphy and $\mathcal{F}$ a coherent sheaf over X. Then global sections of $\mathcal{F}$ generate $\mathcal{F}_x$ at every $x \in X$.

Theorem 55B. Under the same hypothesis as in Theorem 55A, $H^q(X,\mathcal{F}) = 0$ for all $q > 0$.

These theorems have numerous applications which will be given later on.

§2. Preparations for the proof

Notation. $(z_1,\ldots,z_{n-1}) = Z$, $z_n = z$ so that $(z_1,\ldots,z_n) = (Z,z)$.

Theorem 56. (Cartan) Let $P(Z,z)$ be a Weierstrass polynomial of degree s, $P(Z,z) = z^s + a_1(Z)z^{s-1} + \ldots + a_s(Z)$; the a_j are holomorphic in a neighborhood of the origin and $a_j(0) = 0$. Let $r_1,\ldots,r_{n-1},r_n = r$ be > 0 and such that each a_j is holomorphic for $|z_j| \le r_j$ and let $P(Z,z) \neq 0$ for $\{(Z,z) \mid |z_j| \le r_j,\ j = 1,\ldots,n-1$ and $|z| = r\}$. Let $f(Z,z)$ be holomorphic in $D = \{(Z,z) \mid |z_j| \le r_j$ for all $j\}$, and let $|f| \le 1$ on D. Then

(1) $f = QP + R$ where Q is holomorphic in $\operatorname{int} D$ and R is a polynomial of degree $s-1$ in z with holomorphic coefficients in $\operatorname{int} D$, $R = \sum_{j=0}^{s-1} b_j(Z)z^j$, and

(2) $|Q(Z,z)| \le K$ and $|b_j(Z)| \le K$ where K does not depend on f.

Proof. (This proof is independent of the Division Theorem, Theorem 47, and hence gives a new proof of it.)

Let $Q(Z,z) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(Z,\zeta)\,d\zeta}{P(Z,\zeta)(\zeta-z)}$; Q is holomorphic in $\operatorname{int} D$. Then

$R(Z,z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(Z,\zeta)}{P(Z,\zeta)} \left[\frac{P(Z,\zeta)-P(Z,z)}{\zeta-z}\right] d\zeta$. But

$\frac{P(Z,\zeta)-P(Z,z)}{\zeta-z} = \sum_{j=0}^{s-1} a_j(Z) \frac{\zeta^{s-j} - z^{s-j}}{\zeta - z}$, $a_0 = 1$. Carrying out the division gives $R(Z,z) = \sum_{j=0}^{s-1} b_j(Z)\, z^j$, where

$$b_j(Z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(Z,\zeta)}{P(Z,\zeta)} (\zeta^{s-j-1} + a_1\zeta^{s-j-2} + a_2\zeta^{s-j-3} + \ldots$$

$$+ a_{s-j-1})\, d\zeta.$$

Hence R is of the required form.

Now to estimate Q and the coefficients of R. For $|\zeta| = r$ and $|z_j| \leq r_j$, $j = 1,\ldots,n-1$, $|f| \leq 1$ and $|P|$ has a lower bound $b > 0$ and $|a_j| < c$, a constant, for all j. Hence $|b_j(Z)| \leq K_1$, a constant independent of f. To estimate Q write $Q = (f-R)/P$. Then

$$|Q| \leq \frac{1+K_1(1+r+\ldots+r^{s-1})}{b} = K_2, \text{ a constant independent of } f.$$

Take $K = \max(K_1,K_2)$.

Theorem 57. Let M_j be a submodule of $\mathcal{O}_0^{q_j}$ (vector-valued holomorphic functions in $\mathbb{C}^n$ near the origin) of dimension q_j, $j = 1,\ldots,L$; i.e. each M_j is a set of columns $\left\{\begin{pmatrix}\phi_1\\ \vdots \\ \phi_{q_j}\end{pmatrix}\right\}$, ϕ_i holomorphic near zero, such that this set is closed under addition, and under multiplication by holomorphic functions near the origin. Then

(1) Each M_j has a finite basis B_j, i.e. for every M_j there exist a finite number N_j of elements of M_j such that any other element of M_j is a linear combination of these with holomorphic coefficients.

(2) After a linear change of variables, we can find a sequence of polydiscs D_ν about the origin $D_1 \supset D_2 \supset \ldots$, $\bigcap D_\nu = \{0\}$, and the finite bases B_j are defined in D_1, such that if $\vec{\phi} \in M_j$ and is holomorphic in some D_ν, and $\|\vec{\phi}(z)\| = \sup_{\substack{i \\ z\in D_\nu}} |\phi_i(z)| \leq 1$ then $\vec{\phi} = \sum_{\ell=1}^{N_j} \psi_\ell \vec{\phi}_\ell$,

where $(\vec{\phi}_1,\ldots,\vec{\phi}_{N_j}) = B_j$ and the ψ_ℓ are holomorphic in D_ν and $\|\psi_\ell\| \leq K_\nu$ (a constant depending on ν).

Proof. We use a double induction. First we show that if for a fixed n this theorem holds for ideals (1-dimensional modules), it holds in general. Then we use induction on n.

Consider M_j, j fixed. Let $\begin{pmatrix}\phi_1\\ \vdots\\ \phi_{q_j}\end{pmatrix} \varepsilon M_j$. Consider the ideal I_1 of all those ϕ which can occur as a ϕ_1; then the ideal I_2 of all those ϕ which can occur as a ϕ_2 for $\phi_1 = 0$; etc. till I_{q_j} of ϕ_{q_j}'s for $\phi_1 = \phi_2 = \ldots = \phi_{q_j-1} = 0$. By hypothesis each I_k has a finite basis $\hat{B}_k = \{\Phi^k_1,\ldots,\Phi^k_{r_k}\}$. Then the basis B_j for M_j is

$$\left\{\begin{pmatrix}\Phi^1_{i_1}\\ 0\\ \vdots\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \Phi^2_{i_2}\\ 0\\ \vdots\\ 0\end{pmatrix}, \ldots, \begin{pmatrix}0\\ \vdots\\ 0\\ \Phi^{q_j}_{i_{q_j}}\end{pmatrix}, \quad i_j = 1,\ldots,r_j\right\}$$

Part (2) follows similarly.

Now, for $n = 0$ the M_j are the finite (q_j) dimensional vector spaces of all q_j-tuples of constants, and the statements are obvious.

Assume the theorem for $n-1$ and every finite number of modules. We must prove it for n and any finite number of ideals. Assume that none of the ideals I_j is identically zero, so that we may pick a non-identically zero element ϕ_j from each. Make a linear transformation such that these elements are normalized with respect to the variable z_n. Consider any one of the ideals I with element $\phi \not\equiv 0$. Then $\phi = \phi_0 p$; where ϕ_0 is a unit and p is a Weierstrass polynomial, say of degree s, $p = z^s + a_1(Z)z^{s-1} + \ldots + a_s(Z)$, $a_j(0) = 0$. Assume, without loss of generality, that ϕ_0 doesn't appear, then $p \varepsilon I$. Let ψ be any element of I. ψ is holomorphic in some closed neighborhood N of the

origin. In N, $|\psi| \leq c$, a constant. Hence $\psi/c = \chi \varepsilon I$, is holomorphic in N, and $|\chi| \leq 1$ there. In perhaps a smaller closed neighborhood D of the origin we may apply Theorem 56. Then $\chi = qp + r$; q is holomorphic in int D, $|q| \leq K$, and $r = b_0(Z) + b_1(Z)z + \dots + b_{s-1}(Z)z^{s-1}$, where the b_j are holomorphic in int D and $|b_j| \leq K$. Since χ and $qp \varepsilon I$ so does r. Consider all s-tuples
$\begin{pmatrix} b_0(Z) \\ \vdots \\ b_{s-1}(Z) \end{pmatrix}$ such that $b_0 + b_1 z + \dots + b_{s-1}z^{s-1} \varepsilon I$. They form a module in the n-1 variables Z. By hypothesis, this module has a finite basis $\begin{pmatrix} B_0^\nu(Z) \\ \vdots \\ B_{s-1}^\nu(Z) \end{pmatrix}$, $\nu = 1,\dots,m$;

so that $b_j(Z) = \sum_{\nu=1}^{m} \alpha_\nu(Z)\, B_j^\nu(Z)$; (*) α_ν holomorphic in int D, $j = 0,1,\dots,s-1$. Then $r = \sum_{j=0}^{s-1} b_j(Z)z^j = \sum_{j=0}^{s-1} \left(\sum_{\nu=1}^{m} \alpha_\nu(Z)B_j^\nu(Z)\right) z^j$

and therefore

$$\begin{aligned} \frac{\psi}{c} = \chi &= qp + \sum \left(\sum \alpha_\nu B_j^\nu\right) z^j \\ &= qp + \sum_\nu \alpha_\nu\left(\sum_j B_j^\nu z^j \right), \qquad (**) \end{aligned}$$

Hence $\{p$ and $\sum_{j=0}^{s-1} B_j^\nu z^j$, $\nu = 1,\dots,m\}$ is a finite basis for I. Do this for all the ideals I_j.

Now for each module L_j consisting of all s-tuples $\begin{pmatrix} b_0^j(Z) \\ \vdots \\ b_{s-1}^j(Z) \end{pmatrix}$ such that $\sum_{k=0}^{s-1} b_k^j z^k \varepsilon I_j$, we find polydiscs $D_\nu \subset \mathbb{C}^{n-1}$ with the required property (2). Take a sequence of polydiscs D_ν' in $\mathbb{C}^n$ such that every hyperplane z = constant $\cap D_\nu'$ is contained in D_ν, $D_{\nu+1}' \subset D_\nu'$, $\cap D_\nu' = \{0\}$, and the bases for the I_j are defined in D_1'. Consider any $\psi \varepsilon I_j$, say given by (**). Then by our induction hypothesis, in (*) the $\|\alpha_j\| \leq K_\nu$ and we already have $|q| \leq K$. Therefore q and α_j, the coefficients in the basis expansion of ψ, do have norms bounded by a constant depending on ν.

Theorem 58. Let M be a submodule of $\mathcal{O}_0^q$. Let $\{\vec{\phi}_\nu\}$ be a sequence of elements of M which are all holomorphic in a fixed neighborhood N of the origin. If the $\vec{\phi}_\nu$ converge uniformly in N to $\vec{\phi}$, then $\vec{\phi} \in M$.

Proof. The $\vec{\phi}_\nu$ converge componentwise; $\phi_\nu^j \to \phi^j$ uniformly in N, $j = 1,\ldots,q$. In any compact subset of N the ϕ_ν^j are uniformly bounded; assume $|\phi_\nu^j| \leq 1$. By Theorem 57, in perhaps a smaller polydisc D, $\phi_\nu^j = \sum_{\ell=1}^{N} \psi_\ell^\nu \Phi_\ell^j$ where the $\{\vec{\Phi}_\ell\}$ are the basis vectors and the ψ_ℓ^ν are holomorphic in D with $\|\psi_\ell\| \leq K_D$. Hence for j fixed and for each fixed ℓ we have a sequence $\{\psi_\ell^\nu\}$ of uniformly bounded holomorphic functions in D. Thus $\{\psi_\ell^\nu\}$ contains a subsequence which converges normally, say to $\{\Psi_\ell\}$. Therefore for each j there is a subsequence of $\{\phi_\nu^j\}$ which converges normally on D to $\sum_{\ell=1}^{N} \Psi_\ell \Phi_\ell^j$. Then $\phi^j = \sum_{\ell=1}^{N} \Psi_\ell \Phi_\ell^j$ on D, where Ψ_ℓ is holomorphic in D. Hence $\vec{\phi} \in M$.

§3. Proof of Theorem A

Now we are ready to prove Theorem A. Exhaust X by analytic polyhedra, $X_j \subset\subset X_{j+1}$, $\bigcup X_j = X$. Let $K_j = \mathrm{cl}\, X_j$. Apply Theorem 54A to each $K_j \subset X_{j+1}$. Then for each j there exists an analytic polyhedron N_j with $K_j \subset N_j \subset\subset X_{j+1}$ and a finite number of sections of $\mathcal{F}(N_j)$ generate $\mathcal{F}_x$ at every $x \in N_j$; call them $s_{j1}, s_{j2}, \ldots, s_{jL_j}$. Consider, for fixed j, all sections of the following form, $s = \sum_{\nu=1}^{L_j} \phi_\nu s_{j\nu}$,

where the ϕ_ν are holomorphic and bounded in X_j. The $\vec{\phi}_j = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{L_j} \end{pmatrix}$ form a Banach space A_j of vector-valued holomorphic functions under the norm, $\|\vec{\phi}\|_j = \max_{\nu, X_j} |\phi_\nu(z)|$;

A_j is complete by Theorem 58. Let A_j^o denote the subspace of A_j of relations, i.e. $\vec{\phi} \in A_j^o$ if and only if $s = \sum \phi_\nu s_{j\nu} = 0$. A_j^o is a closed linear subspace of A_j, by Theorem 58. Hence we may form $A_j/A_j^o = B_j$. Denote an element of B_j, represented by $\vec{\phi}$, by $[\vec{\phi}]$. B_j is a Banach space with norm $\|[\vec{\phi}]\|_j = \inf_{\vec{\phi} \in [\vec{\phi}]} \max_{\nu, X_j} |\phi_\nu|$.

Proposition. (Oka-Weil Theorem for sections). Let σ be a section of $\underline{\mathcal{J}}$ over X_j such that $\sigma = \sum_\nu \phi_\nu s_{j\nu}$ and $[\vec{\phi}] \in B_j$. Let $\varepsilon > 0$ be given. Then there exists a section τ of $\underline{\mathcal{J}}$ over X such that $\|\sigma - \tau\|_j < \varepsilon$, i.e. $\|\sigma-\tau\|_j = \|[\vec{\Phi}]\|_j < \varepsilon$, where $(\sigma - \tau) \mid X_j = \sum_\nu \Phi_\nu s_{j\nu}$.

Proof. We claim that every section $s_{jk} = \sum_{\ell=1}^{L_{j+1}} \chi_\ell^{(k)} s_{j+1,\ell}$, where the $\chi_\ell^{(k)}$ are holomorphic in a neighborhood of K_j; $k = 1,\dots,L_j$. For, consider the mapping $f : (\underline{\mathcal{O}}^{L_{j+1}}(N_j))_x \to (\underline{\mathcal{J}}(N_j))_x$, $x \in N_j$, given by

$$\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{L_{j+1}} \end{pmatrix} \to \sum_{\ell=1}^{L_{j+1}} \chi_\ell (s_{j+1,\ell})_x .$$

f is a homomorphism of $\underline{\mathcal{O}}^{L_{j+1}}(N_j)$ into $\underline{\mathcal{J}}(N_j)$ and is onto because the $s_{j+1,\ell}$ generate $(\underline{\mathcal{J}}(N_j))_x$ at every $x \in N_j$. Hence $\underline{0} \to \underline{G} \to \underline{\mathcal{O}}^{L_{j+1}}(N_j) \to \underline{\mathcal{J}}(N_j) \to \underline{0}$, where $\underline{G} = \ker f$, is an exact sequence of coherent sheaves. By Theorem 54B there is an analytic polyhedron $\hat{N}_j$ such that $K_j \subset \hat{N}_j \subset\subset N_j$ and $H^q(\hat{N}_j, \underline{G}) = 0$ for all $q > 0$. Therefore $H^0(\hat{N}_j, \underline{\mathcal{O}}^{L_{j+1}}) \to H^0(\hat{N}_j, \underline{\mathcal{J}}) \to 0$ is an exact sequence, implying that the mapping is onto. Hence every section of $\underline{\mathcal{J}}(\hat{N}_j)$ is a $\sum \chi_\ell s_{j+1,\ell}$ where the χ_ℓ are holomorphic in $\hat{N}_j$; therefore so is each s_{jk}.

Now, $\sigma = \sum \phi_\nu s_{j\nu}$ and $s_{j\nu} = \sum \chi_\ell^{(\nu)} s_{j+1,\ell}$, where ϕ_ν and $\chi_\ell^{(\nu)}$ are holomorphic in a neighborhood of K_j. Thus

$\sigma = \sum \psi_\ell s_{j+1,\ell}$; $\psi_\ell = \sum_\nu \phi_\nu \chi_\ell^{(\nu)}$ is holomorphic in a neighborhood of K_j. On X_j, the ψ_ℓ can be approximated by functions $\psi_\ell^{(1)}$ holomorphic in a neighborhood of K_{j+1}, $|\psi_\ell - \psi_\ell^{(1)}| < \varepsilon/2$; by Theorem 53. Hence $\sigma_1 = \sum_\ell \psi_\ell^{(1)} \cdot s_{j+1,\ell}$ is a section of $\underline{\mathcal{F}}$ over a neighborhood of K_{j+1}, and $\|\sigma - \sigma_1\|_j < \varepsilon/2$. Similarly $\sigma_1 = \sum_\ell \psi_\ell^{(1)} s_{j+1,\ell} = \sum_\ell \psi_\ell^{(1)} (\sum_{m=1}^{L_{j+2}} \chi_m^{(\ell)} s_{j+2,m}) = \sum_m (\sum_\ell \psi_\ell^{(1)} \chi_m^{(\ell)} s_{j+2,m}$, where the $\sum_\ell \psi_\ell^{(1)} \chi_m^{(\ell)}$ are holomorphic in a neighborhood of K_{j+1}, and therefore can be approximated on X_{j+1} by holomorphic functions $\psi_\ell^{(2)}$ in a neighborhood of K_{j+2}, $|\psi_\ell^{(2)} - \sum \psi_\ell^{(1)} \chi_m^{(\ell)}| < \varepsilon/4$. Then $\sigma_2 = \sum \psi_\ell^{(2)} s_{j+2,\ell}$ is a section over a neighborhood of K_{j+2} and $\|\sigma_2 - \sigma_1\|_{j+1} < \varepsilon/4$, etc. obtaining $\sigma_3, \sigma_4, \ldots$ such that $\|\sigma_i - \sigma_{i+1}\|_{j+i} < \varepsilon/2^{i+1}$. Since the B_j are complete, on each X_k, $k = j, j+1, j+2, \ldots$, there is a section τ_k of $\underline{\mathcal{F}}$ such that $\|\tau_k - \sigma_i\|_p < \varepsilon$ for i large enough and $p \leq k$. But then for each k, τ_k must be the restriction to X_k of τ_{k+1}. Hence there exists a section τ of $\underline{\mathcal{F}}$ over X such that $\|\tau - \sigma_i\|_p < \varepsilon$ for all p and sufficiently large i. In particular $\|\tau - \sigma\|_j < 2\varepsilon$.

Note that if s is any section of $\underline{\mathcal{F}}$ over X_{j+1}, then $\|s\|_j \leq c_j \|s\|_{j+1}$, where c_j is a constant depending only on j. For $\|s\|_j = \inf_{[\vec{\phi}]} \max_{\nu, X_j} |\phi_\nu|$ for $\sum \phi_\nu s_{j\nu} = s$ and $\|s\|_{j+1} = \inf_{[\vec{\psi}]} \max_{\nu, X_{j+1}} |\psi_\nu|$ for $\sum_\nu \psi_\nu s_{j+1,\nu} = s$. Take any representation of the $s_{j+1,\nu}$ in terms of the $s_{j\nu}$; $s_{j+1,\nu} = \sum_\mu \chi_\mu^{(\nu)} s_{j\mu}$. Then

$s = \sum_\mu (\sum_\nu \psi_\nu \chi_\mu^{(\nu)}) s_{j\mu}$ on X_j. Hence $\|s\|_j = \inf \max |\phi_\nu|$

$\leq \inf_{[\vec{\psi}]} \max_{\mu, X_j} |\sum_\nu \psi_\nu \chi_\mu^{(\nu)}|$, where the inf is taken

over all representatives $\vec{\psi} \in [\vec{\psi}]$, $s = \sum \psi_\nu s_{j+1,\nu}$.

Since ν depends only on j, say $\nu = 1,\dots,L_{j+1}$;
$\|s\|_j \leq L_{j+1} \inf \max_{\mu,\nu,X_j} |\psi_\nu| |\mathfrak{X}_\mu^{(\nu)}|$. On X_j, $|\mathfrak{X}_\mu^{(\nu)}|$ is bounded by a constant b_j depending only on j. Therefore $|s|_j \leq b_j L_{j+1} \inf_{[\psi]} \max_{\nu,X_j} |\psi_\nu| \leq b_j L_{j+1} \|s\|_{j+1}$. This proves the proposition.

To complete the proof of Theorem A, we must show that every stalk $\mathcal{F}_x$, $x \in X$, can be generated by global sections. So, let $x \in X$. Then x lies in some X_j. The sections $s_{j1},\dots,s_{jL_j}$ of X_j generate the stalk at every point of X_j. By the proposition, there are global sections $t_{j1},\dots,t_{jL_j}$ such that on X_j

$$(*) \qquad t_{jk} = \sum_{\ell=1}^{L_j} (\delta_{k\ell} + \psi_{k\ell}) s_{j\ell} ,$$

where the $\psi_{k\ell}$ are holomorphic in X_j and $|\psi_{k\ell}| < \varepsilon$ there. For ε sufficiently small, the transformation (*) is nonsingular, so that we may solve for the $s_{j\ell}$ in terms of the t_{jk}. Hence the t_{jk} generate $\mathcal{F}_x$.

§4. Proof of Theorem B

First we define the tensor product sheaf $\mathcal{F} \otimes \Omega^{0,q}$ over X, where $\Omega^{0,q}$ is the sheaf of germs of differential forms of type $(0,q)$. This sheaf is called the sheaf of germs of differential forms of type $(0,q)$ with values in the sheaf $\mathcal{F}$. Let $x \in X$. Both $\mathcal{F}_x$ and $\Omega_x^{0,q}$ are $\mathcal{O}_x$ modules. Consider all finite sums $\sum (\phi_j f_j \otimes \psi_j \omega_j)$, for $\phi_j, \psi_j \in \mathcal{O}_x$ and $f_j \in \mathcal{F}_x$, $\omega_j \in \Omega_x^{0,q}$. Define addition of two sums in the natural way,

$$\sum_1^N \alpha_j + \sum_1^M \alpha_j' = \alpha_1 + \dots + \alpha_N + \alpha_1' + \dots + \alpha_M'.$$ Allow interchanges of the order of terms of a sum and drop any term with ϕ_j, f_j, ψ_j or ω_j equal to zero. Identify the terms $(\phi_j f_j \otimes \psi_j \omega_j)$ and $(f_j \otimes \phi_j \psi_j \omega_j)$. Then these finite

sums modulo the identification form an Abelian group $G_x = \mathcal{F}_x \otimes \Omega_x^{0,q}$. Under the following topology, we obtain the sheaf $\underline{\mathcal{F}} \otimes \Omega^{0,q}$: Take a representative of any element $g \in G_x$, $\sum(\phi_j f_j \otimes \psi_j \omega_j)$. In a sufficiently small neighborhood N_x of x, the ϕ_j and ψ_j are holomorphic functions, the f_j are sections of $\underline{\mathcal{F}}$ over N_x, the ω_j are differential forms, and the projection maps of the sheaves $\underline{\mathcal{F}}$ and $\Omega^{0,q}$ are homeomorphisms. Then, for each $y \in N_x$, assign that class in G_y, $[\sum (\phi_j' f_j' \otimes \psi_j' \omega_j')]$, for which ϕ_j' and ψ_j' are the direct analytic continuations of ϕ_j and ψ_j, and the f_j' and ω_j' are sections of $\underline{\mathcal{F}}$ and $\Omega^{0,q}$ through f_j and ω_j, respectively. We define the collection of all these classes to be an open set; and these open sets are to form a basis for the topology.

Now define $\bar{\partial} (\sum (\phi_j f_j \otimes \psi_j \omega_j)) = \sum (\phi_j f_j \otimes \psi_j \bar{\partial}\omega_j)$. Then $\bar{\partial}$: $\underline{\mathcal{F}} \otimes \Omega^{0,q} \to \underline{\mathcal{F}} \otimes \Omega^{0,q+1}$ is a homomorphism of the sheaves; and $\bar{\partial}^2 = 0$. Since $H^q(X,\underline{\mathcal{O}}) = 0$ for all $q > 0$, (cf. Theorem 36, p. 117), by Dolbeault's theorem (Theorem 26B, p. 95) we have the Poincaré lemma with respect to $\bar{\partial}$ in X. Hence the sequence $\underline{0} \to \underline{\mathcal{F}} \xrightarrow{1} \underline{\mathcal{F}} \otimes \Omega^{0,0} \xrightarrow{\bar{\partial}} \underline{\mathcal{F}} \otimes \Omega^{0,1} \xrightarrow{\bar{\partial}} \underline{\mathcal{F}} \otimes \Omega^{0,2} \to \dots$ is exact. For, at $\underline{\mathcal{F}} \otimes \Omega^{0,0}$ ker $\bar{\partial}$ is $\underline{\mathcal{F}} \otimes \underline{\mathcal{O}}$, and elsewhere exactness follows from the Poincaré lemma. This sequence is a resolution of $\underline{\mathcal{F}}$. Indeed, $\underline{\mathcal{F}} \otimes \Omega^{0,p}$, $p \geq 0$ are fine sheaves. For, define multiplication by a C^{∞} function; if $\alpha \in C^{\infty}$ then $\alpha(\sum (\phi_j f_j \otimes \psi_j \omega_j)) = \sum (\phi_j f_j \otimes \psi_j (\alpha\omega_j))$; and then proceed as in the example on p. 152. Then by the Abstract de Rham Theorem (p. 153), $H^q(X,\underline{\mathcal{F}}) \simeq$ ($\bar{\partial}$ closed (0,q) forms with values in $\underline{\mathcal{F}}$)/($\bar{\partial}$ exact (0,q) forms with values in $\underline{\mathcal{F}}$). Now exhaust X by analytic polyhedra X_j. Then for every j, $H^q(X_j,\underline{\mathcal{F}}) = 0$ for all $q > 0$ by Theorem 54B. As in the proof that $H^q(X,\underline{\mathcal{O}}) = 0$ for all $q > 0$ (Theorem 36), we obtain a lemma β for the sheaf $\underline{\mathcal{F}}$, and hence $H^q(X,\underline{\mathcal{F}}) = 0$ for all $q>0$.

§5. Applications of the Fundamental Theorems

The following results are all obtained relatively easily from the fundamental theorems A and B. Some of these results have been obtained previously, with more effort.

Note. In the following, X is a region of Holomorphy and $\mathcal{F}$ a coherent analytic sheaf over X.

A. Theorem 59a. Let V be an analytic set in X; i.e. $V \subset X$, and every point $p \in V$ has a neighborhood N_p such that $X \cap N_p$ is the set of common zeroes of a finite number of functions defined and holomorphic in N_p. Then $V = \{x \in X \mid f_i(x) = 0$ for every $i \in I\}$ where the f_i are functions holomorphic in X and I is some index set.

We cannot prove this theorem, since it relies on the fact that the sheaf $\mathcal{J}_V(X)$ is coherent (Theorem 50). However, we can establish:

Theorem 59. If V is a regularly imbedded analytic subvariety in X, then there are functions f_i, $i \in I$, holomorphic in X, such that $V = \{x \in X \mid f_i(x) = 0$ for every $i \in I\}$.

Proof. $\mathcal{J}_V(X)$ is coherent by Theorem 51. Hence, by Theorem 55A, the global sections of $\mathcal{J}_V(X)$ generate the stalk at every point. For any point $p \in X - V$, the stalk $(\mathcal{J}_V(X))_p$ contains the germ "1"; hence "1" is a linear combination of functions holomorphic on X and vanishing on V (with appropriate coefficients). Hence at least one function does not vanish at p; hence there is a function holomorphic in X, $= 0$ on V and $\neq 0$ at p.

Theorem 60. Oka's Fundamental Lemma; a general form.

Let V be a regularly imbedded subvariety in X; P the closure of a polynomial polyhedron in X. Then $\mathrm{cl}\,(V \cap P) = (\mathrm{cl}\,(V \cap P))^*$, its polynomial hull.

Proof. $\mathrm{cl}\,(V \cap P) \subset P$ which is defined by polynomial inequalities. Hence there exists a polynomial polyhedron P' such that $P \subset\subset P' \subset\subset X$ and $\mathcal{J}_V(P')$ is globally finitely generated (applying Theorem A to the coherent sheaf $\mathcal{J}_V(X)$). At any point $p \in P - V$, $(\mathcal{J}_V(P'))_p$ contains the germ "1"; which can be expressed as a linear combination, with $\mathcal{O}_p$ coefficients, of global sections of $\mathcal{J}_V(P')$ at p. But these global sections are holomorphic functions on P' vanishing on V. Hence, using a sufficiently high partial sum, we obtain a polynomial which is close to 1 at p, and close to 0 on $\mathrm{cl}\ V \cap P$. Hence $p \notin (\mathrm{cl}\ (V \cap P))^*$, as desired.

Theorem 61. Cousin I is solvable in X.

Proof. It suffices to show that $H^1(X, \underline{\mathcal{O}}) = 0$. But $\underline{\mathcal{O}}$ is coherent, so Theorem B applies.

Theorem 62. In X, every $\bar{\partial}$-closed form is $\bar{\partial}$-exact.

Proof. Appeal to the Dolbeault isomorphism theorem and Theorem B.

Theorem 63. Suppose there exist finitely many local sections $s_1, \ldots, s_r$ generating all the $\mathcal{F}_x$, $x \in X$. Then every global section s is of the form $s = \sum \phi_j s_j$ where the ϕ_j are holomorphic in X.

Proof. Consider the sheaf homomorphism $\underline{\mathcal{O}}^r(X) \to \underline{\mathcal{F}}$ defined by: $\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}_x \to \sum \phi_i (s_i)_x$. This map is onto by hypothesis; hence we may form the exact sequence:

$$\underline{0} \to \underline{G} \to \underline{\mathcal{O}}^r \to \underline{\mathcal{F}} \to \underline{0}\ .$$

G is also coherent; hence we obtain the exact cohomology sequence:

$$H^0(X, \underline{\mathcal{O}}^r) \to H^0(x, \underline{\mathcal{F}}) \to H^1(X, \underline{G})\ .$$

Now $H^1(X, \underline{G}) = 0$ by Theorem B; and since $H^0(X, \underline{\mathcal{O}}^r)$, $H^0(X, \underline{\mathcal{F}})$ are the global sections in $\underline{\mathcal{O}}^r$, $\underline{\mathcal{F}}$ respectively; Theorem 63 is complete.

Corollary 1. Let $U \subset\subset X$, U open. Then there exist finitely many global sections $s_1, \ldots, s_r$ of $\underline{\mathcal{F}}$ such that every section s of $\underline{\mathcal{F}}(U)$ is of the form $s = \sum \phi_j s_j$, where the ϕ_j are holomorphic in U.

Proof. By Theorem 63, it is enough to show that there exist a finite number of global sections of $\mathcal{F}$ generating the stalks of $\mathcal{F}(U)$ at every point. But this is Theorem A.

Corollary 2. Let $D \subset \mathbb{C}^n$, D open. Then the following are equivalent:

i) D is a region of holomorphy

ii) Whenever $\phi_1,\ldots,\phi_r$ are holomorphic functions in D without common zeroes, there exist holomorphic functions $\psi_1,\ldots,\psi_r$ in D such that $\sum \phi_j\psi_j \equiv 1$.

Proof. i) implies ii). View the ϕ_i as global sections of the sheaf $\mathcal{O}(D)$. It is thus enough to show that they generate the stalks $\mathcal{O}_x$ at every point $x \in D$, as "1" is a global section and Theorem 63 applies. But this is just the hypothesis of ii).

ii) implies i). If D has no boundary points, $D = \mathbb{C}^n$ and so is a region of holomorphy. Hence, assume D has boundary points; we shall show that every such point is essential. Let "a" ε bdry D; $a = (a_1,\ldots,a_n)$. Consider the n holomorphic functions $\phi_j = z_j - a_j$. They have a common zero at the point a, only; hence they have no common zero in D. Hence there exist ψ_j, holomorphic in D, such that $\sum \psi_j(z)(z_j-a_j) \equiv 1$ in D. If the ψ_j are all holomorphic in a neighborhood "a", $(\sum \psi_j(z)(z_j-a_j))_a = 1$. But this is clearly a contradiction, so at least one of the ψ_j is singular at "a".

B. Recall that, in a region of holomorphy, we have an extension theorem for functions defined on regularly imbedded, globally presented hypersurfaces. This theorem extends as follows (X still denotes a region of holomorphy):

Theorem 64. Let $Y \subset X$ be a regularly imbedded subvariety. Then every function holomorphic on Y is the restriction of a function holomorphic on X.

Proof. Consider the sheaf $\mathcal{J}_Y(X)$. We may form the exact sequence:

$$\underline{0} \to \mathcal{J}_Y(X) \to \underline{\mathcal{O}}(X) \to \underline{\mathcal{O}}/\mathcal{J}_Y \to \underline{0},$$

where $\underline{\mathcal{O}}/\mathcal{J}_Y = \underline{\mathcal{O}}(Y)$ (cf. Remark p. 174). We therefore have the exact cohomology sequence:

$$H^0(X,\underline{\mathcal{O}}) \to H^0(Y,\underline{\mathcal{O}}) \to H^1(X,\mathcal{J}_Y)$$

and $H^1(X,\mathcal{J}_Y) = 0$ by Theorem B.

Theorem 65. Let points $z_j \in X$, $\{z_j\}$ discrete, be given together with numbers a_j. Then there exists a function ϕ, holomorphic in X, such that $\phi(z_j) = a_j$.

Proof. $\{z_j\}$ is a regularly imbedded subvariety, of dimension zero.

Theorem 66. With the z_j as above, let polynomials $P_j(z)$, of degree N_j, be given. Then there exists a function ϕ, holomorphic in X, such that in some neighborhood of z_j, $\phi(z) = P_j(z) + O(\|z\|^{N_j+1})$; i.e. ϕ has any given Taylor expansion up to any given order.

Proof. Consider the sheaf $\underline{\mathcal{J}}$, defined by its stalks as follows. If for $x \in X$, $x \neq z_j$ set $\mathcal{J}_x = \mathcal{O}_x$. If for $x \in X$, $x = z_j$ set $\mathcal{J}_x = \{$germs of functions whose Taylor expansions about z_j have no terms of order $\leq N_j$; i.e. which vanish at z_j of order at least $N_j+1\}$.
$\underline{\mathcal{J}}$ is an open subsheaf of $\underline{\mathcal{O}}(X)$. For coherence, we must show that $\underline{\mathcal{J}}$ is locally finitely generated. But, for points $x \neq z_j$, this is clear; and at z_j the stalks are generated by the polynomials in $z-z_j$ of degree N_j+1. Hence, we may form the exact sequence:

$$\underline{0} \to \underline{\mathcal{J}} \to \underline{\mathcal{O}} \to \underline{\mathcal{O}}/\underline{\mathcal{J}} \to \underline{0}$$

and therefore the exact cohomology sequence:

$$H^0(X,\underline{\mathcal{O}}) \to H^0(X, \underline{\mathcal{O}}/\underline{\mathcal{J}}) \to 0$$

by Theorem B. But

$$(\mathcal{O}/\mathcal{J})_x = \begin{cases} 0 & \text{if } x \neq z_j \\ \text{germs of polynomials of degree} \leq N_j & \text{if } x = z_j \end{cases}.$$

C. Recall our attempt to solve the Poincaré problem in the strong sense. This is not possible in an arbitrary region of holomorphy; but is possible in its weak sense:

Theorem 67. Given a function g, meromorphic in X, then $g = h/f$ where h and f are holomorphic in X.

Proof. Locally, we can find holomorphic functions f such that fg is holomorphic. Hence, define the sheaf $\mathcal{J}$ as follows: $\mathcal{J}_x \equiv$ those germs $f_x \in \mathcal{O}_x$ such that $f_x g_x$ is holomorphic. $\mathcal{J}$ is a subsheaf of $\mathcal{O}$. For coherence, it suffices to show that $\mathcal{J}$ is locally finitely generated. Recall that the Poincaré problem is solvable in the strong sense in any polydisc: $g = h_1/f_1$; h_1, f_1 are coprime and the representation is unique up to units. We claim f_1 generates the stalks at every point (in the disc): If $(f_1)_x \neq 0$, this is clear. If $(f_1)_x = 0$, the only functions regularizing g are then multiples of f_1.

Hence, $\mathcal{J}$ is a coherent nontrivial sheaf. By Theorem A, there exists some global section f. But fg is then holomorphic at every point; set $fg = h$.

Chapter 18. Stein Manifolds (Holomorphically Complete Manifolds)

Stein manifolds were designed to generalize the more characteristic properties of regions of holomorphy.

§1. Definition and examples

Definition 77. A complex manifold X is called a Stein manifold if:

Condition 0: It is the union of countably many compact sets.

Condition 1: X is holomorphically convex, i.e. for every $K \subset\subset X$, $\hat{K} \subset\subset X$, where $\hat{K}$ denotes the hull with respect to functions holomorphic on the manifold X.

Condition 2: The holomorphic functions separate points; i.e. for every distinct $p, q \in X$ there exists a function g holomorphic on X such that $g(p) \neq g(q)$.

Condition 3: The collection of functions holomorphic on X contain for each point a set of local coordinates at that point.

Examples. i) Any region of holomorphy.

ii) Regularly imbedded n-dimensional subvariety X of $\mathbb{C}^N$.

We note first that X, being regularly imbedded, is closed. Condition 1 is established by observing that if $K \subset\subset X$ but $\hat{K} \not\subset\subset X$, then there is a discrete sequence $(z_n) \in \hat{K}$. Hence by Theorem 65, there is a holomorphic function ϕ on $\mathbb{C}^n$ with $|\phi(z_n)| \to \infty$; contradicting $(z_n) \in \hat{K}$.
Condition 2 is trivial, as is Condition 3 once it is observed that every point of X has local coordinates such that $z_{n+1} = \dots = z_N = 0$ describe X.

iii) If X is a Stein manifold, and f is holomorphic on X, then the set $\{x \mid f(x) \neq 0\}$ is also a Stein manifold.

iv) The product of two Stein manifolds is also one.

We leave it to the reader to verify that iii) and iv) are Stein manifolds, while stating the following theorems (without proof).

Theorem 68. (Stein) Every universal covering space of a Stein manifold is again a Stein manifold.

Theorem 69. (Behnke-Stein) Every open Riemann surface is a Stein manifold.

We remark that there exist manifolds which are not Stein manifolds; that conditions 2 and 3 can be replaced by a "K-completeness" condition. (A complex manifold X is K-complete if for every $x \varepsilon X$ there exist finitely many functions $f_1, \ldots, f_K$ holomorphic on X such that x is an isolated point of the set $\{y \varepsilon X \mid f_1 y = f_1 x, \ldots, f_K y = f_K x\}$.), and that:

Theorem 70. (Grauert) Conditions 1, 2, and 3 imply condition 0.

§2. An approximation theorem

Definition 78. An analytic polyhedron Y in a complex manifold X is defined as follows: $Y \subset\subset X$, such that there exist a set X_0 and functions $f_1, \ldots, f_r$ holomorphic in X such that:

$$Y \subset\subset X_0 \subset\subset X \quad \text{and} \quad Y = \{z \mid z \varepsilon X_0 \ , \ |f_j(z)| < 1\}.$$

Theorem 71. Let Y be an analytic polyhedron in the complex manifold X, as above, and let g be a function defined and holomorphic in Y. Then g can be expanded in a normally convergent series of functions of the z_j and the coordinate functions f_i, holomorphic in X.

Proof. By adding functions f_j, we may assume that: $Y = \{z \mid z \varepsilon X_0; \ |f_j(z)| < 1, \ j = 1, \ldots, N\}$ and that the Oka map $(z_1, \ldots, z_n) \to (f_1(z), \ldots, f_N(z))$ is one to one, of maximal rank, of Y into $\{|\zeta_j| < 1, \ j = 1, \ldots, N\}$. The image of Y in the disc is a regularly imbedded analytic subvariety of the disc. Therefore by Theorem 64, g can be extended to a function G holomorphic in the disc. Hence in the disc $G = \sum c_{i_1 \ldots i_N} \zeta_1^{\,1} \ldots \zeta_N^{\,N}$ and this series converges normally. But, setting $\zeta_i = f_i(z_1, \ldots, z_n)$, we obtain the desired normally convergent expansion.

§3. The fundamental theorems for Stein manifolds

Theorem 72. Theorems A and B hold for Stein manifolds.

Corollary. All consequences of these theorems, except Corollary 2, hold also. In particular, we have the complex de Rham theorem:

$$H^q(X,\mathbb{C}) \simeq \frac{\text{closed holomorphic q-forms}}{\text{exact holomorphic q-forms}} .$$

Note that this result shows also that the cohomology of differential forms on any Stein manifold is trivial.

These statements need no proof!

§4. Characterization of Stein manifolds

Theorem 73. Let X be a manifold satisfying condition 0. Then the following are equivalent:

i) X is Stein.

ii) $H^1(X,\mathcal{J}) = 0$ for every coherent sheaf of ideals $\mathcal{J}$; i.e. for every coherent subsheaf of $\mathcal{O}$.

Proof. i) implies ii): Theorem B.

ii) implies i): Recall the corollaries of theorems A and B:

Given a discrete sequence of points, there exists a function taking prescribed values. This implies holomorphic convexity and separation of points.

At every point there exists a function with a prescribed expansion in terms of local coordinates. This implies the existence of local coordinates which are holomorphic functions.

Now recall that the proof of these corollaries required only Theorem B in the form of ii).

Theorem 74. (Grauert-Narasimhan) Let X be a complex manifold satisfying condition 0. Then the following are equivalent:

i) X is Stein.

ii) There exists a strongly plurisubharmonic real-valued function ϕ on X such that $\{\phi < \alpha\} \subset\subset X$, for every α .

Proof of this theorem is essentially that of the solution to the Levi problem, and will not be given here.

Theorem 75. (Bishop; Narasimhan) Let X be a complex manifold of dimension n. Then the following are equivalent:

i) X is holomorphically equivalent to a regularly imbedded subvariety of $\mathbb{C}^{2n+1}$.

ii) X is Stein.

Note that this gives an imbedding theorem for regions of holomorphy.

Proof. i) implies ii). Clear by the examples.

ii) implies i) will not be proved here. One must find $2n+1$ functions such that the mapping defined by them is one to one, of maximal rank, and "proper" in that the inverse image of a compact set is compact. We do not establish this, but make the following remarks: This mapping is not unique. However, in the space of all holomorphic maps $X \to \mathbb{C}^{2n+1}$, under the topology of normal convergence, the functions of i) are dense.

Appendix

This appendix is concerned with proving the theorem of L. Schwartz appearing in chapter 13 on page 139. We actually prove a weaker theorem than is stated there, but one which nonetheless suffices for our purposes.

We assume that E and F are vector spaces, each having a nested sequence of norms defined on it;

$$\| \ \|_n \leq \| \ \|_{n+1} \ , \ n = 1, 2, \dots .$$

Furthermore, E and F are metric spaces with metric

$$d(x_1,x_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x_1-x_2\|_n}{1+\|x_1-x_2\|_n} \quad .$$

Under the topology induced by each metric, we assume that E and F are complete and separable, and that E has the added property that for each n, $\{x \in E \mid \|x\|_{n+1} \leq 1\}$ is totally bounded (relatively compact) with respect to the norm $\| \ \|_n$.

The theorem we are going to prove is the following:

<u>Theorem</u> (L. Schwartz). If A and B are continuous linear mappings of E into F and A is onto and B is compact, then F/(A+B)E is finite dimensional.

Before proceeding with the proof, we note that if $E = C^0(D,U',\mathcal{O}) \oplus Z^1(D_0,U'',\mathcal{O})$ and $F = Z^1(D,U',\mathcal{O})$, then E and F have the properties assumed above, for:

The nested sequence of norms on each space is defined as follows. First, in each $u_i' \in U'$ take a sequence of subsets $K_{ij}' \subset\subset K_{i,j+1}' \nearrow u_i'$. In each $u_j'' \cap u_k'' \neq \emptyset$ of sets of U" take a sequence of subsets $K_{jk\ell}'' \subset\subset K_{jk,\ell+1}'' \nearrow (u_j'' \cap u_k'')$; and in each $u_i' \cap u_j' \neq \emptyset$ of sets of U' take a sequence of subsets $K_{ijk}' \subset\subset K_{ij,k+1}' \nearrow (u_i' \cap u_j')$. Then for $f \in E$; i.e. $f = g + h, g \in C^0$ assigns the holomorphic function g_i to u_i' and $h \in Z^1$ assigns the holomorphic function h_{jk} to $u_j'' \cap u_k'' \neq \emptyset$; define

$$\|f\|_n = \max_{i;z \in K_{in}} |g_i(z)| + \max_{j,k;z \in K_{jkn}''} |h_{jk}(z)| \quad ;$$

and for $k \in F$; i.e. $k \in Z^1(D,U',\mathcal{O})$ assigns k_{ij} to $u_i' \cap u_j'$; define

$$\|k\|_n = \max_{i,j;z \in K_{ijn}'} |k_{ij}(z)| \quad .$$

Clearly these are norms and $\| \ \|_n \leq \| \ \|_{n+1}$ for all n = 1, 2, ... The separability of E and F is obvious.

Finally, the totally boundedness of $\{x \in E \mid \|x\|_{n+1} \leq 1\}$ in the norm $\| \ \|_n$ follows from the fact that a uniformly bounded sequence of holomorphic functions contains a normally convergent subsequence.

I. Preliminaries.

Henceforth x and y shall denote elements of E and F, respectively. (e and f shall denote elements of the dual spaces). Let

$$S_n = \{x \in E \mid \|x\|_n \leq 1\}, \quad \Sigma_n = \{y \in F \mid \|y\|_n \leq 1\},$$
$$\partial S_n = \{x \in E \mid \|x\|_n = 1\}, \quad \partial\Sigma_n = \{y \in F \mid \|y\|_n = 1\},$$
$$S_n^o = \{x \in E \mid \|x\|_n < 1\}, \quad \Sigma_n^o = \{y \in F \mid \|y\|_n < 1\}.$$

Note. 1. $\mathcal{O} \subset E$ (or F) is open if and only if for every point $\hat{x} \in \mathcal{O}$ there is an n and a $k > 0$ such that $\hat{x} + kS_n \subset \mathcal{O}$.

2. $x_i \longrightarrow x$ in E (or F) means $\|x_i - x\|_n \underset{i}{\longrightarrow} 0$ for all n.

E^* denotes the space of continuous linear functionals on E. The following remarks, although stated for E^*, are true as well for F^*.

For $e \in E^*$ define $\|e\|_n^* = \sup_{x \in S_n} |e(x)|$. These are not really norms because for some n, $\|e\|_n^*$ may be infinite. However for each fixed n, the elements of E^* with finite norm $\| \ \|_n^*$ form a Banach space.

1. For each $\hat{e} \in E^*$ there is an n such that $\|\hat{e}\|_n^* \leq \infty$.

Proof. Since $\hat{e}$ is continuous, $\{x \in E \mid |\hat{e}(x)| < 1\}$ is an open set containing 0, and therefore contains a set kS_n for some n and $k > 0$. Then $|\hat{e}(x)| \leq 1$ for all $x \in kS_n$, and by linearity $|\hat{e}(x)| \leq \frac{1}{k}$ for all $x \in S_n$, implying $\|\hat{e}\|_n^* \leq \frac{1}{k}$.

2. $\| \ \|_{n+1}^* \leq \| \ \|_n^*$.

3. Given $e \in E^*$, if there is an n for which $\|e\|_n^* = 0$, then $e = 0$.

Proof. If $e \neq 0$ then there is an $x \in E$ and $e(x) = a \neq 0$. Since $\|x\|_n = N < \infty$ for some N, $\frac{x}{N} \in S_n$ and $e(\frac{x}{N}) = \frac{a}{N} \neq 0$, contradicting $\|e\|_n^* = 0$.

4. Since E is separable, the usual argument based on the Cantor diagonal process shows that from any sequence $e_i \in E^*$ with $\|e_i\|_n^* \leq C$ for some fixed n, we may extract a subsequence e_{i_j} converging at every $x \in E$. Set $e(x) = \lim_j e_{i_j}(x)$, then $e \in E^*$ and $\|e\|_n^* \leq C$. Let

$$S_n^* = \{e \in E^* \mid \|e\|_n^* \leq 1\} \text{ and } \Sigma_n^* = \{f \in F^* \mid \|f\|_n^* \leq 1\}.$$

Define mappings $A^*, B^*: F^* \to E^*$ by $(A^*f)x = f(Ax)$ and $(B^*f)x = f(Bx)$ for $f \in F^*$ and $x \in E$.

1. Since A is onto, A^* is 1-1.

2. B compact means that there is an n such that $B(S_n)$ is totally bounded in F.

3. A^* and B^* are continuous linear maps.

II. Easy Results.

Proposition A. Given n, there exist m_n and C_n such that for every $f \in F^*$, $\|f\|^*_{m_n} \leq C_n \|A^*f\|^*_n$.

Proof. Since A is onto, $F = \bigcup_{\ell=1}^{\infty} A(\ell S_n) = \bigcup_{\ell=1}^{\infty} \overline{A(\ell S_n)}$, (the bar denoting closure). By the Baire Category Theorem, one of the closed sets $\overline{A(\ell S_n)}$ contains an open set, therefore a set of the form $y + k\sum_{m_n}$, $y \in F$ and $k > 0$. If we make ℓ even larger, we can get $\overline{A(\ell S_n)} \supset \sum_{m_n}$. For, $y \in \overline{A(\ell S_n)}$ implies $y = Ax$, $x_i \to x$, $x_i \in \ell S_n$, so that $A(x+\ell S_n) \supset k\sum_{m_n}$, implying $A(\frac{x}{k}+\frac{\ell}{k}S_n) \supset \sum_{m_n}$. Let $f \in F^*$. For $y \in A(\ell S_n)$, $y = Ax$, $\|x\|_n \leq \ell$ and $|f(y)| = |f(Ax)| = |(A^*f)x| \leq \|x\|_n \|A^*f\|^*_n \leq \ell \|A^*f\|^*_n$. By the continuity of f, this inequality is valid for $y \in \overline{A(\ell S_n)}$ and hence for $y \in \sum_{m_n}$. Take $C_n = \ell$.

Proposition B. There is a fixed positive integer p such that for every m, $B^*(\sum^*_m)$ is totally bounded in the norm $\|\ \|^*_p$.

Proof. Choose n so that $B(S_n)$ is totally bounded in F. (p will be n + 1). Then $\overline{B(S_n)}$ is compact in F. Given any m, $\overline{B(S_n)} \subset F = \bigcup_{\ell=1}^{\infty} \ell \Sigma_m^o$. By compactness, a finite number of the $\ell \Sigma_m^o$ cover $\overline{B(S_n)}$, and since they are nested, choosing the largest ℓ gives $B(S_n) \subset \overline{B(S_n)} \subset \ell \Sigma_m^o$. Let $f \in \Sigma_m^*$ and $x \in S_n$, then $|(B^*f)x| = |f(Bx)| \leq \|f\|_m^* \|Bx\|_m \leq \ell$. Therefore, $B^*(\Sigma_m^*)$ is a set of functionals which are uniformly bounded on S_n, and since the functionals are linear, this set is equicontinuous with respect to the norm $\| \ \|_n$. Furthermore, $S_{n+1} \subset S_n$ is totally bounded in the norm $\| \ \|_n$. Hence by the Arzelà-Ascoli theorem, from any sequence in $B^*(\Sigma_m^*)$ we may extract a subsequence which converges uniformly on S_{n+1}, i.e. $B^*(\Sigma_m^*)$ is totally bounded with respect to the norm $\| \ \|_{n+1}^*$.

Corollary 1. If $f \in F^*$, then $\|B^*f\|_p^* < \infty$. (For, $f \in F^*$ implies $f \in K\Sigma_m^*$ for some m, $K < \infty$.)

Corollary 2. If $\{f_i\} \in F^*$ are uniformly bounded in some norm, then there is a subsequence $\{f_{i_j}\}$ such that $\|B^*(f_{i_j} - f_{i_k})\|_p^* \xrightarrow[jk]{} 0$.

Theorem 1. $N = \{f \in F^* \mid (A^* + B^*)f = 0\}$ is finite dimensional.

Proof. Since A^* is 1-1, if suffices to show that $A^*(N)$ is finite dimensional. Let p be as in Proposition B. Then $A^*(N)$ is a Banach space under the norm $\| \ \|_p^*$, because for every $e \in A^*(N)$, $\|e\|_p^* < \infty$ since $e = A^*f$, $f \in N$, implies $\dot{e} = -B^*f$ which has finite norm $\| \ \|_p^*$ by Corollary 1.

Suppose $A^*(N)$ is infinite dimensional. Then there is a sequence $\alpha_1, \alpha_2, \ldots$ of linearly independent elements of $A^*(N)$. We claim that there exists $\{e_k\} \in A^*(N)$ satisfying $\|e_k\|_p^* = 1$ and $\|e_k - e_j\|_p^* \geq \frac{1}{2}$ for $k \neq j$. The proof is based on the following.

<u>Lemma</u>. If E is a Banach space and G is a closed, linear, proper subset of a linear set $D \subset E$, then there is an $x_0 \in D$ with $\|x_0\| = 1$ and $\|x_0 - G\| \geq \frac{1}{2}$.

<u>Proof</u>. Take $x' \in D - G$ and let d = distance of x' to G. Then there is a $y' \in G$ with $d \leq \|x' - y'\| \leq 2d$. Set $x_0 = \dfrac{x' - y'}{\|x' - y'\|}$.

Now, let $G_1 = \{\alpha_1\}$, i.e. the linear space spanned by α_1, and let $D_{12} = \{\alpha_1, \alpha_2\}$. Since every finite dimensional linear subspace of a Banach space is closed, we may apply the above lemma to G_1 and D_{12} as subsets of $A^*(N)$ with norm $\| \ \|_p^*$. Hence, there exists $e_1 \in D_{12}$ with $\|e_1\|_p^* = 1$ and $\|e_1 - e\|_p^* \geq \frac{1}{2}$ for all $e \in G_1$. Next, apply the lemma to $G_{12} = \{\alpha_1, \alpha_2\}$ and $D_{123} = \{\alpha_1, \alpha_2, \alpha_3\}$ and get $e_2 \in D_{123}$ with $\|e_2\|_p^* = 1$ and $\|e_2 - e\|_p^* \geq \frac{1}{2}$ for all $e \in G_{12}$. Continue this process, obtaining a sequence $\{e_k\} \in A^*(N)$ with $\|e_k\|_p^* = 1$ and $\|e_k - e\|_p^* \geq \frac{1}{2}$ for all e belonging to the space spanned by $\alpha_1, \ldots, \alpha_k$; but then $\|e_k - e_j\|_p^* \geq \frac{1}{2}$ for $k \neq j$. On the other hand, since $\{e_k\} \in A^*(N)$, $e_k = A^*f_k$, $f_k \in N$.

By Proposition A, there are constants m_p and C_p such that $||f_k||^*_{m_p} \leq C_p$. Then by Corollary 2, (B^*f_k) contains a subsequence which is Cauchy in the norm $|| \;\; ||^*_p$. However, $B^*f_k = -A^*f_k = -e_k$, and $\{e_k\}$ can have no Cauchy subsequence in the norm $|| \;\; ||^*_p$ by construction.

Theorem 2. Given n, there exist m_n and $K_n < \infty$ such that if $e \in (A^*+B^*)F^*$ and $||e||^*_n < \infty$, then there is an $f \in F^*$ with $e = (A^*+B^*)f$ and $||f||^*_{m_n} \leq K_n||e||^*_n$.

Proof. Let p be as in Proposition B, and let m_n be given by Proposition A. It suffices to consider only $n \geq p$; since once we have established the theorem for $n = p$ we have it at once for all $n < p$ by choosing for such n, the constants $m_n = m_p$ and $K_n = K_p$, and recalling that $|| \;\; ||^*_{n+1} \leq || \;\; ||^*_n$. Let N denote the nullspace of (A^*+B^*), as before. We claim that it is sufficient to prove that there is a $K_n < \infty$ such that for all $f \in F^*$ with $||(A^*+B^*)f||^*_n < \infty$, $||f - N||^*_{m_n} \leq \frac{K_n}{2}||(A^*+B^*)f||^*_n$. Indeed, if $(A^*+B^*)f = e$, then there will exist a $\phi \in f - N$ satisfying $||\phi||^*_{m_n} \leq K_n||(A^*+B^*)f||^*_n = K_n||e||^*_n$ and $(A^*+B^*)\phi = (A^*+B^*)f = e$.

Suppose such a K_n does not exist. Then there is a sequence $f_i \in F^*$ such that $||f_i - N||^*_{m_n} \xrightarrow[i]{} \infty$ and $||(A^*+B^*)f_i||^*_n = 1$. Take $\phi_i \in f_i - N$ so that $||f_i - N||^*_{m_n} \leq ||\phi_i||^*_{m_n} \leq 2||f_i - N||^*_{m_n}$, and set $\psi_i = \frac{\phi_i}{||\phi_i||^*_{m_n}}$. Then $\frac{1}{2} \leq ||\psi_i - N||^*_{m_n}$, $||\psi_i||^*_{m_n} = 1$ and $||(A^*+B^*)\psi_i||^*_n = \frac{1}{||\phi_i||^*_{m_n}} \xrightarrow[i]{} 0$.

By Proposition B, a subsequence of the ψ_i, call them again $\{\psi_i\}$, satisfies $\|B^*(\psi_i-\psi_j)\|_p^* \xrightarrow[i,j]{} 0$, and therefore $\|B^*(\psi_i-\psi_j)\|_n^* \xrightarrow[i,j]{} 0$. Then by the triangle inequality, $\|A^*(\psi_i-\psi_j)\|_n^* \xrightarrow[i,j]{} 0$, implying by Proposition A that $\|\psi_i - \psi_j\|_{m_n}^* \xrightarrow[i,j]{} 0$. Hence there is a $\psi \in F^*$ with $\|\psi_i - \psi\|_{m_n}^* \xrightarrow[i]{} 0$, and $\|\psi - N\|_{m_n}^* \geq \frac{1}{2}$. Then $\psi \notin N$, while
$((A^*+B^*)\psi)x = \psi((A+B)x) = \lim_{i\to\infty} \psi_i((A+B)x) = \lim_{i\to\infty} ((A^*+B^*)\psi_i)x = 0$
for all $x \in E$ means that $\psi \in N$, a contradiction.

III. Main Results.

Let $y_0 \in \overline{(A+B)E}$, and let $M = \{f \in F^* \mid f(y_0) = 0\}$. Call $(A^*+B^*)M = L$.

Lemma 1. Suppose $e_i \in L$, $\|e_i\|_n^* \leq C$ for some n and $C < \infty$, and suppose $e_i(x) \xrightarrow[i]{} e(x)$ for all $x \in E$. Then $e \in L$.

Proof. By Theorem 2, we can find $f_i \in F^*$ with $e_i = (A^*+B^*)f_i$ and $\|f_i\|_{m_n}^* \leq K$. We claim that $f_i \in M$. Indeed, there are $\phi_i \in M$ for which $e_i = (A^*+B^*)\phi_i$. Set $\psi_i = f_i - \phi_i$, then $(A^*+B^*)\psi_i = 0$. Since $y_0 \in \overline{(A+B)E}$, there is a sequence $\{x_\alpha\} \in E$ such that $(A+B)x_\alpha \xrightarrow[\alpha]{} y_0$. Hence $\psi_i(y_0) = \lim_{\alpha\to\infty} \psi_i((A+B)x_\alpha) = \lim_{\alpha\to\infty} ((A^*+B^*)\psi_i)x_\alpha = 0$, implying that $\psi_i \in M$, but then $f_i = \psi_i + \phi_i \in M$.

Now, because the $\{f_i\}$ is uniformly bounded in the norm $\|\ \|_{m_n}^*$, it follows that a subsequence of the $\{f_i\}$, call them again $\{f_i\}$, converges at every point y of F, $f_i(y) \to f(y)$, $f \in F^*$ and $f(y_0) = 0$ so that $f \in M$. But then, $((A^*+B^*)f)x = f((A+B)x) = \lim_{i\to\infty} f_i((A+B)x)$
$= \lim_{i\to\infty} ((A^*+B^*)f_i)x = \lim_{i\to\infty} e_i(x) = e(x)$ for all $x \in E$. Thus
$e = (A^*+B^*)f \in L$.

Before proceeding, we recall some notions from functional analysis. The Hahn-Banach Theorem states that if S is a linear subspace of T, a Hausdorff locally convex topological vector space, and if $x \in \mathcal{O}$, an open convex subset of T such that $\mathcal{O} \cap S = \phi$, then there is a closed hyperplane $H \supset S$ with $H \cap \mathcal{O} = \phi$. From this theorem it follows that since $T = H + \{x\}$, i.e. H + the linear space spanned by x, if we define for $t \in T$, $\phi(t) = \lambda$ where $t = h + \lambda x$, then ϕ is a continuous linear functional on T whose nullspace is H. Hence there exists a continuous linear functional ϕ on T satisfying $\phi(x) = 1$ and $\phi(S) = 0$.

Theorem 3. If $e \in E^*$ but $e \notin L$, then there is an $x_0 \in E$ such that $e(x_0) = 1$ while $g(x_0) = 0$ for every $g \in L$.

Proof. (1) Since $e \in E^*$, there is an n for which $\|e\|_n^* < \infty$. If $n = 1$, i.e. $\|e\|_1^* < \infty$, then, since $\| \;\|_2^* \leq \| \;\|_1^*$, $\|e\|_2^* < \infty$. Therefore $\|e\|_n^* < \infty$ for some $n \geq 2$. In order to simplify notation we will assume $\|e\|_2^* < \infty$: the proof in the general case is (essentially) the same.

(2) There is an $\eta > 0$ such that if $g \in L$ and $\|g - e\|_2^* \leq 1$, then $\|g - e\|_1^* > \eta$.

Proof. Assume that no such η exists. Then there is a sequence $(g_j) \in L$ satisfying $\|g_j - e\|_2^* \leq 1$ and $\|g_j - e\|_1^* \xrightarrow[j]{} 0$, i.e. for every $x \in E$, $|g_j(x) - e(x)| \xrightarrow[j]{} 0$. But $\|g_j - e\|_2^* \leq 1$ implies $\|g_j\|_2^* \leq 1 + \|e\|_2^* < \infty$ so that by Lemma 1, $e \in L$; a contradiction.

(3) Take η smaller so that $2\eta \leq 1$, then if $g \in L$ and $\|g - e\|_2^* \leq 2\eta$, $\|g - e\|_1^* > \eta$.

(4) Let $\{x_i^{(1)}\}$ be a dense sequence in ∂S_1. Such a sequence exists since $\partial S_1 \subset E$ a separable metric space. Then there exists an integer $N_1 > 0$ such that if $g \in L$ and $\|g - e\|_2^* \leq 2\eta$ the following inequalities cannot hold simultaneously

$$|g(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta; \; i = 1, \ldots, N_1.$$

Proof. Assume the contradiction, then there is a sequence $(g_N) \in L$ with $\|g_N - e\|_2^* \leq 2\eta$ and $|g_N(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta$ for $i = 1, \ldots, N$. $\|g_N - e\|_2^* \leq 2\eta$ implies that $\|g_N\|_2^* \leq 2\eta + \|e\|_2^* < \infty$ so that there is a subsequence, call it again (g_N), such that at every $x \in E$, $g_N(x) \underset{N}{\Rightarrow} g(x)$, $g \in L$ by Lemma 1, and $\|g - e\|_2^* \leq 2\eta$, $|g(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta$ for $i = 1, 2, \ldots$. Hence $|g(x) - e(x)| \leq \eta$ for $x \in \partial S_1$, which implies that $\|g - e\|_1^* \leq \eta$, contradicting (3).

(5) Let $\{x_i^{(2)}\}$ be a dense sequence in ∂S_2. Then there exists an N_2 such that if $g \in L$ and $\|g - e\|_3^* \leq 3y$ the following inequalities cannot hold simultaneously

$$|g(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta \; ; \quad i = 1, \ldots, N_1$$
$$|g(x_i^{(2)}) - e(x_i^{(2)})| \leq 2\eta; \quad i = 1, \ldots, N_2$$

Proof: Assume the contradiction, then there is a sequence $(g_N) \in L$ with $\|g_N - e\|_3^* \leq 3\eta$ and $|g_N(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta$ for $i = 1, \ldots, N_1$; $|g_N(x_i^{(2)}) - e(x_i^{(2)})| \leq 2\eta$ for $i = 1, \ldots, N_2$.

$||g_N - e||_3^* \leq 3\eta$ implies that $||g_N||_3^* \leq 3\eta + ||e||_3^* \leq 3\eta + ||e||_2^* < \infty$ so that there is a subsequence of the (g_N) converging at every point of E to a $g \in L$ satisfying $|g(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta$ for $i = 1, \ldots, N_1$, and $|g(x_i^{(2)}) - e(x_i^{(2)})| \leq 2\eta$ for $i = 1, \ldots, N_2$. Hence $|g(x) - e(x)| \leq 2\eta$ for $x \in \partial S_2$, so that $||g - e||_2^* \leq 2\eta$; but $g \in L$, and $|g(x_i^{(1)}) - e(x_i^{(1)})| \leq \eta$ for $i = 1, \ldots, N_1$; contradicting (4).

(6) Continue this process: hence if $g \in L$ and $||g - e||_k^* \leq k\eta$ then the following inequalities <u>cannot</u> hold simultaneously

$$|g(x_i^{(s)}) - e(x_i^{(s)})| \leq s\eta, \quad i = 1, \ldots, N_s \; ; \; s = 1, \ldots, k - 1.$$

(7) Let $\{\alpha_n\}$ denote the sequence $\left\{\frac{x_1^{(1)}}{\eta}, \frac{x_2^{(1)}}{\eta}, \ldots, \frac{x_{N_1}^{(1)}}{\eta}, \frac{x_1^{(2)}}{2\eta}, \frac{x_2^{(2)}}{2\eta}, \ldots, \frac{x_{N_2}^{(2)}}{2\eta}, \frac{x_1^{(3)}}{3\eta}, \ldots\right\} \in E$. Since $\left\|\frac{x_i^{(k)}}{k\eta}\right\|_k = \frac{1}{k\eta}$, and $|| \; ||_n \leq || \; ||_{n+1}$, for any fixed k, $||\alpha_n||_k \xrightarrow[n]{} 0$, i.e. $\alpha_n \xrightarrow[n]{} 0$ in E.

(8) If $g \in L$ then for some k, $||g - e||_k^* < \infty$. Since $|| \; ||_{k+1}^* \leq || \; ||_k^*$, for k sufficiently large $||g - e||_k^* \leq k\eta$. By (6), at least one of the inequalities $|g(x_i^{(s)}) - e(x_i^{(s)})| \leq s\eta$ for $i = 1, \ldots, N_s$; $s = 1, \ldots, k - 1$; is invalid. This means that there is an n, depending on g, such that $|g(\alpha_n) - e(\alpha_n)| > 1$. Then for all $g \in L$, $\sup_n |g(\alpha_n) - e(\alpha_n)| > 1$.

(9) Set $P = \{(g(\alpha_1), g(\alpha_2), \ldots, g(\alpha_n), \ldots) | g \in L\}$ and $q = (e(\alpha_1), e(\alpha_2), \ldots, e(\alpha_n), \ldots)$. Since g and e are continuous on E, $g(\alpha_n) \xrightarrow[n]{} 0$ and $e(\alpha_n) \xrightarrow[n]{} 0$, by (7). P, $q \subset \ell^0$, the Banach space of sequences of complex numbers tending to zero under the norm,

$\|v\| = \sup_i |v_i|$ where $v \in \ell^0$, $v = (v_1, v_2, \ldots)$. P is a linear subspace of ℓ^0. $\|P - q\| = \inf_{g \in P} \sup_i |g(a_i) - e(a_i)| \geq 1$ by (8). Therefore there is a continuous linear functional on ℓ^0 which is 0 on P and 1 at q. This means that there is a sequence of complex numbers a_k with $\sum |a_k| < \infty$ such that if $g \in L$ then $\sum a_k g(\alpha_k) = 0$ and $\sum a_k e(\alpha_k) = 1$.

(10) From (7) and $\sum |a_k| < \infty$, we see that the sequence of partial sums of $\sum a_k \alpha_k$ is Cauchy in E and therefore converges to an $x_0 \in E$. Then $e(x_0) = 1$, while $g(x_0) = 0$ for every $g \in L$; completing the proof.

Lemma 2. $(A+B)E$ is closed.

Proof. Let $y_0 \in \overline{(A+B)E}$. If $y_0 = 0$ then we know $y_0 \in (A+B)E$, so we may as well assume $y_0 \neq 0$. Then there is an $f_0 \in F^*$ such that $f_0(y_0) = 1$, so that $f_0 \notin M$. Set $e_0 = (A^*+B^*)f_0$, $e_0 \notin L$: for if $e_0 \in L$, then $e_0 = (A^*+B^*)\phi_0$, $\phi_0 \in M$, so that if $\psi_0 = f_0 - \phi_0$, then $(A^*+B^*)\psi_0 = 0$, but $\psi_0(y_0) = \lim_\alpha \psi_0((A+B)x_\alpha) = \lim_\alpha ((A^*+B^*)\psi_0)x_\alpha = 0$ implies $\psi_0 \in L$ and hence $f_0 \in L$. Since $e_0 \in E^*$ but $e_0 \notin L$, by Theorem 3 there exists an $x_0 \in E$ such that $e_0(x_0) = 1$ and $g(x_0) = 0$ for every $g \in L$. Let $f \in F^*$, then $f - f(y_0)f_0 \in M$. Therefore if $e = (A^*+B^*)f$, then $e - f(y_0)e_0 \in L$, which means that $e(x_0) - f(y_0)e_0(x_0) = 0$, or $e(x_0) = f(y_0)$, i.e. $((A^*+B^*)f)x_0 = f((A+B)x_0) = f(y_0)$ for all $f \in F^*$. Hence $y_0 = (A+B)x_0 \in (A+B)E$.

<u>Theorem 4</u>. $F/(A+B)E$ is finite dimensional.

<u>Proof</u>. Let $K = (A+B)E$. By Lemma 2, K is a closed subspace of F. If F/K is infinite dimensional, then there is a sequence of linearly independent $y_i \in F$, so that $K_0 = K, K_1 = K_0 + \{y_1\}, K_2 = K_1 + \{y_2\}, \ldots$ are closed subspaces of F satisfying K_i properly contained in K_{i+1}. For each i, then, there is a p_i such that $\| K_i - y_{i+1} \| p_i > 0$. Hence we can find continuous linear functionals $f_i \in F^*$ with $f_i(K_i) = 0$ and $f_i(y_{i+1}) = 1$. The f_i are linearly independent and for every i, $f_i(K) = 0$. Therefore for every $x \in E$, $((A^* + B^*)f_i)x = f_i((A+B)x) = 0$, so that $f_i \in N$, and N therefore is infinite dimensional. This contradicts Theorem 1 and completes the proof.