DYNAMICS OF POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

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In these notes we will consider the dynamics of polynomial automorphisms of $\mathbb{C}^2$, by which we mean the study of the behavior of the iterates $f^n = f \circ \cdots \circ f$ as $n \to \infty$. An interesting example is given by a (generalized) complex Hénon map, which is a polynomial diffeomorphism of $\mathbb{C}^2$ in the form

$$f(x, y) = (y, p(y) - \delta x) \tag{0.1}$$

where $p(y)$ is a polynomial of degree $d \geq 2$, and $\delta \in \mathbb{C}$ is a nonzero constant. We note that if we conjugate f by a scaling $(x, y) \mapsto (tx, ty)$ for $t \neq 0$, then we can make the leading coefficient in p equal to 1, so that p a monic polynomial. Further, if we conjugate by a translation $(x, y) \mapsto (x + s, y + s)$, then we can bring p into the form

$$p(y) = y^d + a_{d-2} y^{d-2} + \cdots + a_0$$

We may solve to find

$$f^{-1}(x, y) = \left(\frac{p(x) - y}{\delta}, x \right)$$

Thus we see that f^{-1} has the same general form as h in (0.2), except for the small detail that the the polynomial inside f^{-1} is not monic. The differential of f is

$$Df = \begin{pmatrix} f_x & f_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\delta & p'(y) \end{pmatrix}$$

Thus the jacobian determinant is δ, which is a constant.

We note that $f^{\circ n} = (p^{\circ(n-1)}(y) + \cdots, p^{\circ n}(y) + \cdots)$, where '$\cdots$' denotes terms of lower order, so the degree of $f^{\circ n}$ is the same as $(\deg(f))^n$, which is also the same as $(\deg(p))^n$.

Our focus on these maps comes from [9]:

Theorem 0.1 (Friedland-Milnor). *If f is a polynomial diffeomorphism of $\mathbb{C}^2$, then it is conjugate to an element of either $\mathcal{A}$, $\mathcal{E}$, or $\mathcal{H}$, where $\mathcal{A}$ are the affine (linear) transformations, $\mathcal{E}$ are the elementary transformations, and $\mathcal{H}$ consists of maps of the form $f_N \circ \cdots \circ f_1$, where each f_j has the form* (0.1).

The affine and elementary maps have simple dynamics, so in order to study the dynamics of all polynomial diffeomorphisms of $\mathbb{C}^2$, it suffices to study the class $\mathcal{H}$. In fact, for the results we present, we will see that there is no essential loss if we restrict our study to the case of a single Hénon map.

We write $(x_n, y_n) := f^n(x, y)$. With this notation, we see that $x_{n+1} = y_n$, and so the entire orbit $\{f^n(x, y) : n \in \mathbb{Z}\}$ is contained already in either of the infinite sequences

$\ldots, x_{-1}, x_0, x_1, \ldots$ or $\ldots, y_{-1}, y_0, y_1, \ldots$. This means that the map f is equivalent to the infinite recurrence

$$y_{n+1} = p(y_n) - \delta y_{n-1}$$

with $(y_{-1}, y_0) = (x, y)$, and $(x_n, y_n) = (y_{n-1}, y_n)$ for $n \in \mathbb{Z}$. Thus f acts as a shift on the sequence $(y_n)_{n\in\mathbb{Z}}$. We can also write f in a different form by conjugating with the involution $\tau(x, y) = \tau^{-1}(x, y) = (y, x)$:

$$h(x, y) = \tau \circ f \circ \tau = (p(x) - \delta y, x) \tag{0.2}$$

In this case, we could replace the orbit of h by a sequence $(x_n)_{n\in\mathbb{Z}}$, and we would have $h^n(x, y) = (x_n, x_{n-1})$. Both forms (0.1) and (0.2) are equivalent, but we choose to use (0.1) because this form fits more conveniently with the interpretation of the map acting as a shift on the orbit.

Notes. Other surveys of this subject are given by Smillie [18] and the book of Morosawa, Nishimura, Taniguchi, and Ueda [14].

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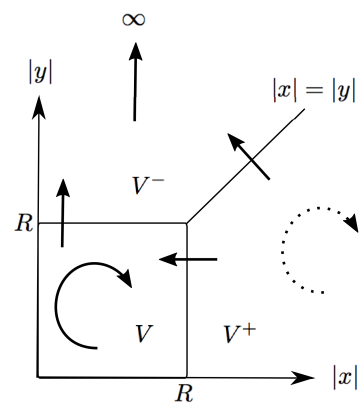

FIGURE 1. *Filtration which shows behavior in the large.*

1. Filtration Properties

This section is devoted to describing the general behavior of a Hénon map f "in the large." The sets of the filtration are pictured in Figure 1, which shows the partition of $\mathbb{C}^2$ into three sets: V, V^+ and V^-. The arrows show the permissible transition behaviors. If a point is in V^-, its forward orbit must stay in V^- and go to infinity as $n \to \infty$. Points in V can go from V can stay in V or go to V^-, but they cannot be mapped to V^-. Points in V^+ can move to V; or points from V^+ can be mapped to V^-; the dotted circle means that points from V^+ might stay in V^+, but a point can stay in V^+ for only finite time before it must leave.

We have seen that f and f^{-1} have the same general form, and this same filtration applies also to f^{-1}, except that we need to flip the locations of all the arrows about the diagonal $x = y$.

We start with an elementary observation about $p(y) = y^d + a_{d-2}y^{d-2} + \cdots + a_0$.

Lemma 1.1. *There are $R, C < \infty$ such that if $|y| \geq R$, then*

$$|p(y) - y^d| \leq C|y|^{d-2}$$

Lemma 1.2. *Write $f(x, y) = (y, z)$. Then for R sufficiently large and $|y| \geq R$, it follows that either $|x| > |y|$ or $|z| > |y|$, or both.*

Proof. If $|x| \leq |y|$, then by Lemmas 1.1 and 1.2, we have

$$|z| \geq |p(y)| - |\delta x| \geq |y^d| - \epsilon|y^{d-1}| - |\delta y| > |y| \qquad \square$$

From this, we conclude:

Corollary 1.3. *If R is sufficiently large, then*

(1) $f(V^-) \subset V^-$
(2) $f(V^- \cup V) \subset V^- \cup V$

Lemma 1.4. *For $\epsilon > 0$, $R = R_\epsilon$ can be chosen sufficiently large that*

$$f(V^-) \subset V^- \cap \{\epsilon|y| > |x|\}$$

and if $(x, y) \in V^-$, then $|y_n| \geq |y_0|/\epsilon^n$.

Proof. We use the notation $f(x, y) = (y, z) = (x_1, y_1)$ and let C and R be as in Lemmas 1.1 and 1.2. With $(x, y) \in V^-$ we have

$$|y_1| = |z| \geq |p(y)| - |\delta x| \geq |p(y)| - |\delta y| \geq |y^d| - C|y^{d-2}| - |\delta y| > |y|(|y|^{d-1} - C|y|^{d-3} - |\delta|)$$

For the first assertion, we increase R if necessary so that $R^{d-1} - CR^{d-3} - |\delta| > \epsilon^{-1}$, and we have $|y_1| > |y_0|/\epsilon$.

Now we iterate the inequality to get $|y_{j+1}| \geq |y_j|/\epsilon$, which gives the second assertion. □

The "bounded/unbounded" dichotomy will be important, so we define

$$\begin{aligned} K^\pm :=& \{q \in \mathbb{C}^2 : \{f^{\pm n}(q) : n \geq 0\} \text{ is bounded}\}, \\ K :=& K^+ \cap K^-, \quad J^\pm := \partial K^\pm, \quad J := J^+ \cap J^- \\ U^\pm :=& \mathbb{C}^2 - K^\pm. \end{aligned}$$

Remark 1.5. Since the coordinates of f are polynomials, it follows that $K^\pm$ and K are polynomially convex. Similarly, any component of the interior of any of these sets is polynomially convex.

We see in Lemma 3 that forward orbits are unbounded, so we have:

Corollary 1.6. *$V^- \cap K^+ = \emptyset$, so $K^+ \subset V \cup V^+$, and $K \subset V$. Thus K has finite volume.*

Lemma 1.7. *We have the following:*

1. *$V^- \subset f^{-1}V^- \subset \cdots$, $\bigcup f^{-n}V^- = U^+$.*
2. *$K^+ = \bigcap_{n\geq 0} f^{-n}(V \cup V^+)$.*
3. *With $V_n := f^n V \cap f^{-n}V$, we have $V_1 \supset V_2 \supset \cdots$, and $\bigcap V_n = K$.*
4. *$W^s(K) = K^+$*

Remark 1.8. Part (1) corresponds to the dotted arrow in Figure 1: it says that an orbit can remain in V^+ for only finite time. Such an orbit must either enter V^-, where it goes to infinity, or the orbit stays in V for $n \geq n_0$ and thus belongs to K^+.

Proof. By Lemma 2 and the Corollary, we see that $V^- \subset f^{-1}V^- \subset \cdots$ and $\bigcup f^{-n}V^- \subset U^+$. To prove (1), we consider an element $(x, y) \in U^+$ and we must show that $(x_n, y_n) \in V^-$ for some $n > 0$. The only possibility is that $(x_n, y_n) \in V^+$ for all n. In this case, we have $x_{n+1} = y_n$, and since $(x_{n+1}, y_{n+1}) \in V^+$, we have $|y_n| \geq |y_{n+1}|$ for all $n \geq n_0$. This sequence is decreasing, the limit $\lim_{n\to\infty} |y_n|$ must exist. On the other hand, this means that $\lim_{n\to\infty} |x_n|$ must exist. Now we apply Lemma 3 to f^{-1}. This says that $|x_n| \geq C|x_{n+1}|$ for points in V^+, which means the limit cannot exist.

Parts (2) and (3) are elementary consequences of (1).

For part (4), we observe that K is compact, so $W^s(K) \subset K^+$ is immediate. So let (x, y) be a point of K^+. By (1), the forward orbit enters V and remains there and is thus bounded. We define the forward limit set $\omega(x, y)$ to be the accumulation points of the forward orbit $\{f^n(x, y) : n \geq 0\}$. It is elementary that $\omega(x, y)$ is also invariant under f^{-1}. This means that it belongs to K. Now, since all accumulation points belong to the compact set K, it follows that $\text{dist}(f^n(x, y), K) \to 0$ as $n \to \infty$. □

We define the *Fatou set* $\mathcal{F}$ of a map f to be the set of points q where the iterates $\{f^n : n \geq 0\}$ are locally normal. By this, we mean that for any subsequence f^{n_j}, there is a further subsequence $f^{n_{j_k}}$ which either diverges to infinity or converges to a (finite) limit, and in either case the convergence is to be uniform on a neighborhood of q. (Equivalently, $\mathcal{F}$ is the set where f is Lyapunov stable, if we extend f to the one point compactification of $\mathbb{C}^2$.) Since our map f is invertible, we have the forward Fatou set $\mathcal{F}^+ := \mathcal{F}(f)$ where the forward iterates are normal (equicontinuous), as well as the backward Fatou set $\mathcal{F}^- := \mathcal{F}(f^{-1})$.

Theorem 1.9. $\mathcal{F}^\pm = \mathbb{C}^2 - J^\pm$

Proof. $\mathbb{C}^2 - J^+ = U^+ \cup \text{int}(K^+)$. If $q \in U^+$, then by Lemma 1.7, the iterates of a neighborhood of q tend uniformly to infinity. If $q \in \text{int}(K^+)$, then also by Lemma 1.7, a neighborhood of q will enter V in finite time and remain inside V for all future time. Thus the forward iterates are uniformly bounded, and by Montel's Theorem they are a normal family.

Conversely, if $q \in J^+$, then the forward orbit of q is bounded. However, every neighborhood of q intersects U^+ and thus contains points that escape to ∞. Thus the iterates of f cannot be normal in any neighborhood of q. □

Notes. We follow Friedland and Milnor [9] and [3] in our treatment of the filtration.

2. Interlude: Fixed Points, Parameter Space

2.1. **Fixed and periodic points.** We say that a point (x, y) is *fixed* if $f(x, y) = (x, y)$, and (x, y) is *periodic* if $f^n(x, y) = (x, y)$ for some integer $n \geq 1$. The minimal value of n is the *period* of (x, y).

Theorem 2.1. *The set of fixed points of $f_N \circ \cdots \circ f_1$ is finite, and the sum of multiplicities is $d_N \cdots d_1$.*

Proof. The condition that (x, y) is a fixed point is that $f_N \circ \cdots \circ f_1(x, y) = (x, y)$. During this proof, we use the notation $(x_1, y_1) = (x, y)$ and $(x_{j+1}, y_{j+1}) = f_j(x_j, y_j)$. Thus a fixed point is defined by the system of equations

$$(x_{j+1}, y_{j+1}) = (y_j, p_j(y_j) - \delta_j x_j), \quad 1 \leq j \leq N$$

Since $x_{j+1} = y_j$, we may drop the x_j's and rewrite our equations: a fixed point corresponds to a solution $(y_1, \ldots, y_N)$ of the system: $y_{j+1} = p_j(y_j) - \delta_j y_{j-1}$, $1 \leq j \leq N$. This has the form

$$\begin{aligned} y_1^{d_1} + \cdots &= 0 \\ y_2^{d_2} + \cdots &= 0 \\ \vdots \qquad &\vdots \\ y_N^{d_N} + \cdots &= 0 \end{aligned}$$

where the $\cdots$ represent terms of lower degree. Now a version of the Bezout Theorem says that this system has $d_1 \cdots d_N$ solutions, counted with multiplicity. $\square$

Corollary 2.2. *For each period N, there are finitely many points of period N.*

2.2. **Quadratic parameter space.** The form of the maps in (0.1) and (0.2) are analogous to the family of one-dimensional polynomials. In the quadratic case, we have the family $p_c(z) = z^2 + c$. If we wish to look at p_c in the neighborhood of a fixed point, we may conjugate by a translation to move the fixed point to the origin. This brings the map to the form $q_\lambda(z) = \lambda z + z^2$. For the analogue in dimension two we consider the family

$$(x, y) \mapsto (p(x) - \delta y, x)$$

which is in the form (0.2). We may re-write this map to center it at a fixed point. The fixed points of this map are solutions of $f(x, y) = (x, y)$, which yields a point (t, t), with t being a solution of the equation $p(t) - \delta t = t$. Thus there are d fixed points, when counted according to multiplicity. Let us choose one of these fixed points $(x, y) = (t_0, t_0)$ and conjugate with the translation $(x, y) \mapsto (x - t_0, y - t_0)$ so that the origin $O = (0, 0)$ is fixed. This means that our map has the form

$$g \ : \ (x, y) \mapsto (a_1 x - \delta y, x) + (q(x), 0)$$

where $q(x)$ is a monic polynomial which vanishes to second order at the origin. Let λ and μ denote he eigenvalues of Dg at the origin. Since a_1 is the trace of Dg and δ is its determinant, we can write our map as

$$(x, y) \mapsto \begin{pmatrix} \lambda + \mu & -\lambda\mu \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q(x) \\ 0 \end{pmatrix} \tag{2.1}$$

This form is convenient if we wish to work at a fixed point and see how the map changes as we change the multipliers at that fixed point. If we conjugate by $\begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$, then we have the form

$$h_{\lambda,\mu}(x) = \begin{pmatrix} \lambda & 1 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ q(\lambda x + y) \end{pmatrix} \tag{2.2}$$

If $\lambda \neq \mu$, we may conjugate $h_{\lambda,\mu}$ by the matrix $\begin{pmatrix} 1 & -(\lambda - \mu)^{-1} \\ 0 & 1 \end{pmatrix}$ to obtain

$$G_{\lambda,\mu} : \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda x \\ \mu y \end{pmatrix} + \frac{q(\lambda x + \mu y)}{\lambda - \mu} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.3}$$

We define $\mathcal{N} := \{(\lambda, \mu) \in \mathcal{P} : \lambda = \mu\}$ (the maps for which the differential at the origin is non-diagonalizable) and $\mathcal{R}_m := \{\mu = \lambda^m\} \cup \{\lambda = \mu^m\}$, $m \geq 2$ (maps with resonance of order m at the origin). Resonances will be discussed at greater length in Section 5.

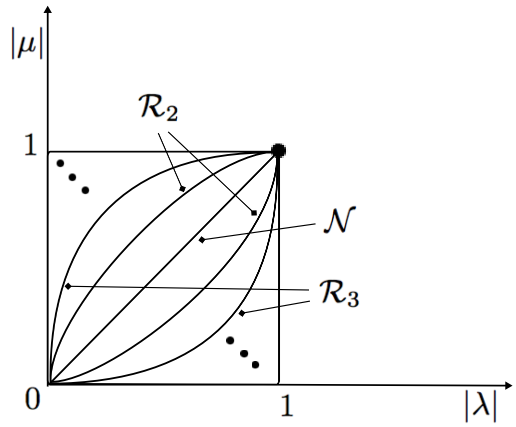

FIGURE 2. *Quadratic parameter locus with resonant curves $\mathcal{R}_m$.*

In case $d = 2$, the map is quadratic, and we have $q(x) = x^2$. The family of quadratic Hénon maps is parametrized by the multipliers at a fixed point. We may consider the parameter space $\mathcal{P} = \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda\mu \neq 0\}$. Strictly speaking, this is actually a covering of the parameter space, since generally there are 4 different choices of (λ, μ) for a given map, since (λ, μ) and (μ, λ) correspond to the same map, and we have two more possilities (λ', μ') by centering at the other fixed point.

Let us consider the region of parameter space $\Delta^2_{\lambda,\mu} := \{(\lambda, \mu) \in \mathcal{P} : |\lambda|, |\mu| < 1\}$. We make an elementary observation:

Theorem 2.3. *If $(\lambda, \mu) \in \Delta^2_{\lambda,\mu}$, then $(0, 0)$ is an attracting fixed point, so $(0, 0) \in \mathrm{int}(K^+)$.*

In the case of quadratic Hénon maps, this is a 2-dimensional analogue of the main cardioid of the Mandelbrot set, which corresponds to the parameters c such that $p_c(z) = z^2 + c$ has an attracting fixed point. However, the 2-dimensional case is more complicated: $\Delta^2_{\lambda,\mu}$ is rich with bifurcations. For instance, attracting cycles of other periods appear in $\Delta^2_{\lambda,\mu}$. The closure of $\bigcup_m \mathcal{R}_m$ contains the topological boundary of $\Delta^2_{\lambda,\mu}$. When $(\lambda, \mu) \in \mathcal{R}_m \cap \Delta^2_{\lambda,\mu}$, f is locally conjugate to a resonant normal form $(z, w) \mapsto (\lambda z + cw^m, \mu w)$ for either $c = 0$ (linearizable) or $c = 1$ (nonlinearizable).

3. Rate of Escape Function; Böttcher coodinate

With the notation $(x_n, y_n) = f^n(x, y)$ we define $G^+_n(x, y) := d^{-n} \log |y_n|$.

Lemma 3.1. *There exists $C < \infty$ such that*

$$|G^+_{n+1}(x, y) - G^+_n(x, y)| \leq \frac{C}{d^{n+1}|y_n|}$$

for all $(x, y) \in V^-$

Proof. For $(x,y)\in V^-$ we have

$$\begin{aligned} G^+_{n+1}-G^+_n &= \frac{1}{d^{n+1}}\left(\log|y_{n+1}|-\log|y_n|^d\right)=\frac{1}{d^{n+1}}\log\left|\frac{y_{n+1}}{y_n^d}\right| \\ &= \frac{1}{d^{n+1}}\log\left|\frac{y_n^d-p(y_n)-\delta y_{n-1}}{y_n^d}\right| = \frac{1}{d^{n+1}}\log\left|1-O\left(\frac{Cy_n^{d-2}}{y_n^d}\right)\right| \\ &= \frac{1}{d^{n+1}}O(y_n^{-2}) \end{aligned}$$

where the last expression on the second line comes from Lemma 1.1. This last expression is uniformly summable on V^-, so the sequence G^+_n is uniformly convergent on V^-. □

Lemma 3.2. *The limit $G^+ := \lim_{n\to\infty} G^+_n$ converges uniformly on V^-, as well as on compact subsets of U^+, and on any $f^{-N}(V^-)$ we have*

$$G^+(x,y)=\log|y|+O(|y|^{-1})$$

Proof. The estimate of Lemma 3.1 gives uniform convergence on V^-. If $S\subset U^+$ is compact then Lemma 1.7, there exists N such that $f^N(S)\subset V^-$. The limit then is uniform on $f^N(S)$, and thus on S itself. Finally, the $O(|y|^{-1})$ estimate is a consequence of Lemma 3.1. □

Corollary 3.3. *G^+ is pluri-harmonic on U^+ and satisfies $G^+\circ f = d\cdot G^+$.*

Theorem 3.4. *We extend G^+ to $\mathbb{C}^2$ by setting it equal to zero on K^+. This extension of G^+ is continuous and pluri-subharmonic on $\mathbb{C}^2$.*

Proof. □

We will use the notation $\log^+|t| = max(\log|t|,0)$.

Theorem 3.5. *For any norm $||\cdot||$ on $\mathbb{C}^2$, we have $G^+ = \lim_{n\to\infty}\frac{1}{d^n}\log^+||f^n||$, and the limit is uniform on compact subsets of $\mathbb{C}^2$.*

Proof. □

Now we define a multiplicative version of G^+, which will provide an analogue of the Böttcher coordinate. We define

$$q(y):=p(y)-y^d, \quad h(x,y):=\frac{q(y)}{y^d}-\frac{\delta x}{y^d} \tag{3.1}$$

so q has degree $d-2$. Thus on V^- we have

$$h(x,y)=O(y^{-2})+O(xy^{-d}) \tag{3.2}$$

so we record this as:

Lemma 3.6. $|h(x,y)|\le C'|y|^{-1}$ *for* $(x,y)\in V^-$.

We have $y_n = y_{n-1}^d(1 + h(x_{n-1}, y_{n-1}))$ so if $|h(x_{n-1}, y_{n-1})| < 1$, we have a well-defined choice of d-th root

$$y_n^{1/d} := y_{n-1}(1 + h(x_{n-1}, y_{n-1}))^{1/d}$$

Iterating this, we have a well-defined d^n-th root:

$$y_n^{1/d^n} := y_0(1 + h(x_{n-1}, y_{n-1}))^{1/d^{n-1}}(1 + h(x_{n-2}, y_{n-2}))^{1/d^{n-2}} \cdots (1 + h(x_0, y_0))$$

Theorem 3.7. *The infinite product*

$$\varphi^+(x, y) := y \prod_{j=0}^{\infty} (1 + h(x_j, y_j)))^{1/d^{j+1}}$$

converges uniformly on V^- and defines a holomorphic function which satisfies $\varphi^+ \circ f = (\varphi^+)^d$. Further $\varphi^+(x, y) = y(1 + O(1/y))$ and $G^+ = \log |\varphi^+|$ on V^-.

Proof. By Lemma 3.6, we may choose R sufficiently large that $|h| < 1/2$ on V^-. We note that for $|\zeta| < 1/2$, $|(1 + \zeta)^{1/d^n} - 1| < c/d^n$, so the infinite product converges uniformly on V^-, and the limit is an analytic function. □

We conclude from this Theorem that in the variable y, φ^+ has a simple pole at infinity, and so it has a Laurent expansion on V^-

$$\varphi^+(x, y) = y + \sum_{n=0}^{\infty} \frac{c_n(x)}{y^n} \tag{3.3}$$

where each $c_n(x)$ is an entire holomorphic function. Let us look more carefully at the coefficients in (3.3). We have $x_n = y^{d^{n-1}} + \cdots$ and $y_n = y^{d^n} + \cdots$, where '$\cdots$' indicates terms of higher degree. If $d = 2$, we have

$$h(x_0, y_0) = h(x, y) = \frac{a_0 - \delta x}{y^2} + \sum_{j=0}^{d-3} \frac{a_j}{y^{d-j}} \tag{3.4}$$

and if $d > 2$,

$$h(x_0, y_0) = \frac{a_{d-2}}{y^2} + \sum_{j=0}^{d-3} \frac{a_j}{y^{d-j}} - \frac{\delta x}{y^d} \tag{3.5}$$

If $n \geq 1$, we have

$$h(x_n, y_n) = \sum_{j \geq 2 \cdot d^n} \frac{A_j(x)}{y^j} + \sum_{j \geq (d^2-1)d^{n-1}} \frac{B_j(x)}{y^j} \tag{3.6}$$

From this we obtain:

Theorem 3.8. *In the expansion* (3.3)*, the coefficients $c_n(x)$ are polynomial in x. The coefficient $c_0(x) = 0$ in* (3.3) *vanishes identically. If $d = 2$, then $c_1(x) = (a_0 - \delta x)/2$, and if $d > 2$, then $c_1(x) = a_{d-2}/2$ is constant.*

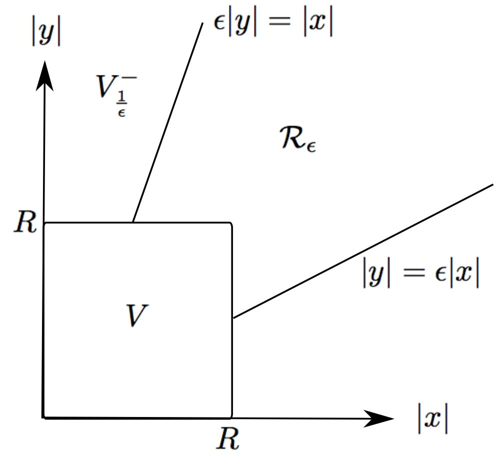

FIGURE 3. *Double trapping region.*

Proof. Consider the factors $(1 + h(x_n, y_n))^{1/d^{n+1}}$ in the product defining φ^+. From (3.6) we see that the nth term is a Laurent series of the form $1 + O(y^{-D_n})$ where the degree D_n tends to infinity as $n \to \infty$. If we fix a degree N, then there are only finitely many terms in the product which can produce something of the form $a(x)y^{-N}$. Since the numerators in (3.6) are polynomials, the coefficient $c_N(x)$ must be a polynomial.

We have seen that the denominators in (3.6) are y^j with $j \geq 2$, so we see that there is no term y^{-1}, so there can be no constant term in φ^+. The coefficient c_1 are given by taking the coefficient of y^{-2} in the root $(1 + h(x_0, y_0))^{1/d}$, so we have c_1 from (3.4) and (3.5). □

4. Structure of U^+ and of $U^+ \cap U^-$

Let τ denote the curve defined by $\theta \mapsto (0, Re^{2i\pi\theta}$ for $0 \leq \theta \leq 1$. We let $\tau \in H_1(U^+; \mathbb{Z})$ denote its homology class.

Proposition 4.1. $\{\tau\} \neq 0$ *in* $H_1(U^+; \mathbb{Z})$.

Proof. We note first that ∂G^+ is a d-closed 1-form on U^+ since $d = \partial + \bar{\partial}$, and G^+ is pluriharmonic. Further, on V^-, we have $2G^+ = \log(\varphi^+ \overline{\varphi^+})$, so $2\partial G^+ = \partial\varphi^+/\varphi^+$. By Theorem 3.7, $\varphi^+ \sim y$ for y large, so

$$\int_\tau \partial G^+ = \frac{1}{2} \int_\tau \frac{\partial \varphi^+}{\varphi^+} = \pi i$$

We conclude that τ defines a nonzero element of $H_1(U^+; \mathbb{Z})$. □

Theorem 4.2. $H_1(U^+; \mathbb{Z}) = \mathbb{Z}[\frac{1}{d}]$. *Specifically, if* $\gamma \in H_1(U^+; \mathbb{Z})$, *then* $\gamma \sim md^{-n}\tau$ *for some* $m, n \in \mathbb{Z}$.

Proof. Topologically, V^- is equivalent to a disk cross and annulus, and the homology of V^- is generated by τ. Thus if σ is a closed curve inside V^-, then $\{\sigma\} \sim m\{\tau\}$ for some $m \in \mathbb{Z}$.

Now if γ is a cycle representing an element of $H_1(U^+;\mathbb{Z})$, then it is compactly supported, so by Lemma 1.7, there exists $n \in \mathbb{Z}$ such that $f_*^n(\gamma)$ is supported in V^-. Thus $f_*^n(\gamma) \sim m\tau$ for some $m \in \mathbb{Z}$. Since f has the form (0.1), we have $f_*\tau \sim d \cdot \tau$. Finally, since f is invertible, f^{-1} maps $H_1(U^+;\mathbb{Z})$ to itself, and we have $\gamma \sim f_*^{-n}(m\tau) \sim d^{-n}m\tau$. ☐

Recall $V = \{|x|, |y| \le R\}$, and for $\mu > 0$ define $V_\mu := \{|y| > \mu|x|\}$. For $\epsilon > 0$, we set

$$\mathcal{R}_\epsilon = V_\epsilon - \left(V \cup V_{1/\epsilon}^-\right)$$

Making small modifications on the previous sections we have:

Proposition 4.3. *For $\epsilon > 0$, R can be made sufficiently large that φ^+ is defined on $V_\epsilon^- - V$, and $f(\mathcal{R}_\epsilon) \subset V_{1/\epsilon}^- - V$.*

For $s \in \mathbb{C}$, we define

$$D_s := \{q \in V_\epsilon^- : \varphi^+(q) = s\}$$

Theorem 4.4. *For $\epsilon > 0$, R may be chosen sufficiently large that for $|s| \ge R$, there is a domain $\Omega_s \subset \mathbb{C}$ such that $D_s = \{y = \psi_s(x) : x \in \Omega_s\}$ is a graph of a holomorphic function ψ_s on Ω. If $d > 2$, then D_s is approximately the horizontal disk $\{|x| < |s|/\epsilon\} \times \{y = s\}$. For $|s|$ large, V_ϵ cuts the parabola $\{2y^2 - 2sy + (a_0 - \delta x)/2 = 0\}$ into two disks D' and D''. Let D' denote the disk that comes close to the point $(0, s)$. If $d = 2$, D_s, is approximately D'.*

Proof. By Theorem 3.7, we have $\partial\varphi^+/\partial y \sim 1$ for large y. If $d > 2$, then by Theorem 3.8, we have $\partial\varphi^+\partial x = O(y^{-2})$. In this case it follows that for y large, D_s is approximately a flat disk, and Ω_s is approximately $\{|x| < |s|/\epsilon\}$.

If $d = 2$, then by Theorem 3.8, we have $\varphi^+ = y + (a_0 - \delta x)/(2y) + O(y^{-2})$ on V_ϵ. Thus the level set $\{\varphi^+ = s\} \cap V_\epsilon$ is approximately contained in the parabola $y^2 - sy + (a_0 - \delta x)/2 = 0$. ☐

Since G^+ is pluriharmonic on U^+, $\omega^+ := \partial G^+$ is a holomorphic 1-form there. Thus ω^+ generates a foliation $\mathcal{G}^+$ on U^+, and the leaves of $\mathcal{G}^+$ are Riemann surfaces. Since $G^+ = \log|\varphi^+|$ on V^-, it follows that if L is a leaf of $\mathcal{G}^+$, then the components of the intersection $L \cap V^-$ are just sets of the form D_s.

Corollary 4.5. *φ^+ cannot be extended to be continuous on $V^- \cup f^{-1}(V^-)$.*

Theorem 4.6. *Let L denote a leaf of the foliation $\mathcal{G}^+$. Then L is dense in the set $\{G^+ = c\}$, and L is uniformized by $\mathbb{C}$.*

Now let us discuss the set $U^+ \cap U^-$, which consists of the points that escape to infinity in both forward and backward time.

Lemma 4.7. *The image of $\iota_*^+ : H_1(U^+ \cap U^-;\mathbb{Z}) \to H_1(U^+;\mathbb{Z})$ is surjective, and thus $H_1(U^+ \cap U^-;\mathbb{Z})$ is not a finitely generated additive group.*

Applying the previous discussion to f^{-1}, we have a foliation $\mathcal{G}^-$ on U^-. Thus there are two foliations, $\mathcal{G}^+$ and $\mathcal{G}^-$ on the intersection $U^+ \cap U^-$, and by construction, we have $\mathcal{R}_\epsilon \subset U^+ \cap U^-$. Since φ^+ and φ^- are both defined on $\mathcal{R}_\epsilon$ and $\varphi^+ \sim y$ and $\varphi^- \sim x$ there, we see that the foliations $\mathcal{G}^+$ and $\mathcal{G}^-$ are transverse at all points of $\mathcal{R}_\epsilon$. We define the critical locus

$$\mathcal{C} := \{\partial G^+ \wedge \partial G^- = 0\} = \{\omega^+ \wedge \omega^- = 0\}$$

We note that there is a holomorphic function $h(x, y)$ on $U^+ \cap U^-$ such that $\omega^+ \wedge \omega^- = h\, dx \wedge dy$, so $\mathcal{C} = \{h = 0\}$ is defined as the zero set of a single analytic function.

Theorem 4.8. $\mathcal{C} \neq \emptyset$

Theorem 4.9. $\overline{\mathcal{C}} \cap J^- \cap U^+ \neq \emptyset$, *and* $\overline{\mathcal{C}} \cap J^+ \cap U^- \neq \emptyset$.

Notes. The foliation $\mathcal{G}^+$, as well as the approach of studying G^+ as a fibration, comes from Hubbard [11]. The topology and biholomorphic type of U^+ are studied Hubbard and Oberste-Vorth in [12]. The results about $\mathcal{C}$ are taken from [5].

5. Interlude: Linearization

We say that f can be *linearized* at a fixed point O if there is a linear map $L : \mathbb{C}^2_{\text{lin}} \to \mathbb{C}^2_{\text{lin}}$ and a local biholomorphism $\Phi : \mathbb{C}^2_{\text{dyn}} \to \mathbb{C}^2_{\text{lin}}$ such that $\Phi \circ L = f \circ \Phi$. We start by discussing *formal linearization*, by which we mean a formal power series

$$\hat{\Phi} = \sum_{m,n \geq 0} \hat{\Phi}(m, n) x^m y^n$$

such that $\hat{\Phi} \circ L = f \circ \hat{\Phi}$ holds in the sense of formal power series. A formal power series is not assumed to converge, but all the power series coefficients in the composition $f \circ \hat{\Phi}$ are well defined, and are equal to the power series coefficients of $\hat{\Phi} \circ L$.

A *resonance* between numbers λ and μ is a relation of the form $\lambda = \lambda^m \mu^n$ or $\mu = \lambda^m \mu^n$ where m and n nonnegative integers with $m + n \geq 2$. We suppose that $f = L + f_2 + O_3$, where $L(x, y) = (\lambda x, \mu y)$, $f_2 = \sum_{m,n \geq 0} f(m, n) x^m y^n$ is homogeneous of degree 2, and O_3 represents terms of degree 3 and higher. Let us find

$$\Phi_2 = id + (\varphi_2^1, \varphi_2^2) = (x, y) + \sum_{m+n=2} (\varphi_2^1(m, n), \varphi_2^2(m, n)) x^m y^n$$

such that $\Phi_2 \circ L = F \circ \Phi + O_3$. This gives

$$L + \varphi_2(\lambda x, \mu y) = L + (\lambda \varphi_2^1, \mu \varphi_2^2) + f_2(x, y) + O_3. \tag{5.1}$$

Solving for the coefficient of $x^m y^n$ inside (5.1) with $m + n = 2$, we find

$$\varphi_2^1(m, n)(\lambda^m \mu^n - \lambda) = f_2^1, \quad \varphi_2^2(m, n)(\lambda^m \mu^n - \mu) = f_2^2, \tag{5.2}$$

which we may solve if there is no resonance.

Now we define $F_3 := \Phi_2^{-1} \circ f \circ \Phi_2 = L + f_3 + O_4$. Here f_3 is homogeneous of degree 3, but it is only defined after we have found Φ_2. Now we solve for $\Phi_3 = id + \varphi_3$ by finding the coefficients of $x^m y^n$ with $m + n = 3$. This leads to

$$\varphi_3^1(m, n)(\lambda^m \mu^n - \lambda) = f_3^1, \quad \varphi_3^2(m, n)(\lambda^m \mu^n - \mu) = f_3^2, \tag{5.3}$$

which we may solve if there is no resonance. We continue in this fashion and find successively Φ_2, Φ_3, ..., and at each stage we have

$$F_{n+1} := \Phi_n^{-1} \circ \cdots \circ \Phi_2^{-1} \circ f \circ \Phi_2 \circ \cdots \circ \Phi_n = L + f_{n+1} + O_{n+2}$$

Theorem 5.1. *If the pair (λ, μ) is non-resonant, then there is a formal mapping $\hat{\Phi} := \lim_{n\to\infty} \Phi_2 \circ \cdots \circ \Phi_n$ such that $\hat{\Phi} \circ L = f \circ \hat{\Phi}$ holds in the sense of formal power series.*

Proof. We have found Φ_n, and $\Phi_n = id + O_n$. It follows that up to degree n, the coefficients of $\Phi_2 \circ \cdots \circ \Phi_n$ and $\Phi_2 \circ \cdots \circ \Phi_n \circ \Phi_{n+1}$ are the same. All the coefficients of $\Phi_2 \circ \cdots \circ \Phi_n$ are eventually constant as $n \to \infty$. Thus the limit defining $\hat{\Phi}$ exists and gives the desired formal conjugacy. □

When $(\lambda, \mu) \in \mathcal{R}_m \cap \Delta^2_{\lambda,\mu}$, f is locally conjugate to a resonant normal form $(z, w) \mapsto (z + cw^m, w)$ for either $c = 0$ (linearizable) or $c = 1$ (nonlinearizable).

Theorem 5.2. *If $(\lambda, \mu) \in \Delta^2_{\lambda,\mu} \cap \mathcal{R}_2$, then the map $h_{\lambda,\mu}$ in (2.2) cannot be linearized at $(0, 0)$.*

For generic $(\lambda, \mu) \in \mathcal{R}_m$, $h_{\lambda,\mu}$ can not be linearized, which is to say that generically a resonance produces an obstruction to linearization. But does it always produce an obstruction in the case of polynomial automorphisms? When $m \geq 3$ we do not know whether there is a value of $(\lambda, \mu) \in \mathcal{R}_m$ for which it is linearizable.

Question 5.1. Suppose that $f = f_N \circ \cdots \circ f_1$ is real, in the sense that it preserves $\mathbb{R}^2$. Suppose, too, that it preserves volume, so we may assume $\delta = 1$. If $O \in \mathbb{R}^2$ is a real fixed point, then the multipliers are λ and $\mu = \lambda^{-1}$. Then there are infinitely many resonances $\lambda^m \mu^n = \lambda$, with $m = n + 1 > 0$. It it *never* the case that f can be linearized at O?

Up to this point we have discussed the algebraic (formal) aspect of linearization. Once we have a formal series which gives a formal linearization, we need to address the question of convergence. In solving for Φ_2 and Φ_3, we needed to divide by the denominators like $(\lambda^m \mu^n - \lambda)$ in (5.2) and (5.3). Since these quantities may be small, we encounter a "small divisor" problem. One well known condition is (5.4), which gives a lower bound on the size of these divisors (non-resonances):

$$\min\left(|\lambda^m \mu^n - \lambda|, |\lambda^m \mu^n - \mu|\right) \geq c(m + n)^{-N} \tag{5.4}$$

for some $c > 0$ and $N < \infty$ and all $m, n \geq 0$, $m + n \geq 2$. There are more general sufficient conditions, but (5.4) is sufficient for our purposes here. For instance, (5.4) clearly holds if $0 < |\lambda|, |\mu| < 1$. We also note (reference???):

1. If $|\mu| < 1$, the set of λ, for which (5.4) holds is full measure in $\{|\lambda| = 1\}$,
2. If $p, q \geq 1$ are integers, then $\lambda = \alpha^p$, $\mu = \alpha^q$ satisfies (5.4) for almost every $|\alpha| = 1$,
3. The set of λ, μ for which (5.4) holds is full measure in $\{|\lambda| = |\mu| = 1\}$.

One of the results of Small Divisor Theory is that condition (5.4) is sufficient to guarantee convergence of the formal linearizing coordinate Φ (see Pöschel [17]):

Theorem 5.3. *Let $f : U \to \mathbb{C}^2$ be a holomorphic map of a neighborhood U of $O = (0,0) \in \mathbb{C}^2$. If $f(O) = O$ is a fixed point, and if the multipliers λ, μ of $D_O f$ satisfy* (5.4)*, then the formal linearization Φ is convergent in a neighborhood of O.*

6. Fatou Components: Conservative Maps

We say that f is *conservative* if it preserves volume, which is equivalent to the condition that $|\delta| = 1$ for mappings of the form (0.1) or (0.2) . We know that U^+ is a component of the Fatou set, and by Theorem 1.9 all other components of the Fatou set must belong to $\mathrm{int}(K^+)$. Here we discuss the possibilities.

Theorem 6.1. *If f is conservative, then* $\mathrm{Vol}(K^\pm - K) = 0$*. Thus*

$$\mathrm{int}(K^+) = \mathrm{int}(K^-) = \mathrm{int}(K)$$

Proof. By Lemma 1.7, $f^{-1}(K^+ \cap V) \supset K^+ \cap V$. Since f preserves volume, it follows that $f^{-1}(K^+ \cap V) - (K^+ \cap V)$ must have zero volume. Further, by Lemma 1.7, we conclude that $K^+ - V$ has zero volume. Finally, by Lemma 1.7, we conclude that the volume of $K^+ - K$ is also zero. This is the first assertion of the Theorem. Since a set of measure zero can have no interior, the second assertion follows. □

Corollary 6.2. *If f is conservative, then every Fatou component is periodic. That is, conservative maps have no wandering Fatou components.*

Proof. Let Ω denote a component of the interior of K. By the invariance of the interior of K and the invertibility of f, we know that either Ω and $f^n\Omega$ are disjoint, or they coincide. Since $K \subset V$, it has finite volume. Thus the sets Ω, $f(\Omega)$, $\ldots$, $f^n(\Omega)$ can remain disjoint for at most finitely many n, and then we have $\Omega = f^{n+1}(\Omega)$. □

Question 6.1. It would be interesting to gain some basic topological information about Fatou components Ω for a conservative map f. Can a Fatou component of f be homeomorphic to a cell of real dimension 4? Can it be homeomorphic to something else?

Theorem 6.3. *Let f be conservative, and let Ω be a bounded Fatou component with $f(\Omega) = \Omega$. $O \in \Omega$ be a fixed point, and let $A := D_O f$. Then we may diagonalize $A \sim \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $|\lambda| = |\mu| = 1$. Further, there is a linear map $\Psi : \Omega \to \mathbb{C}^2_{\mathrm{lin}}$ such that $\Psi(O) = (0,0) \in \mathbb{C}^2_{\mathrm{lin}}$, and $\Psi \circ f = A \circ \Psi$.*

Proof. Since $\Omega \subset V$, it follows that the iterates $f^n|_\Omega$ are bounded by R. By Cauchy's estimates, $D_O(f^n) = (D_O f)^n$ is also bounded, independently of $n \in \mathbb{Z}$. It follows that the eigenvalues λ and μ both must have modulus one. If $\lambda \neq \mu$, then A is diagonalizable. If $\lambda = \mu$, and A is non-diagonalizable, then it must have a nontrivial Jordan canonical form. But in this case, the powers A^n are not bounded. Thus A is diagonalizable in every case.

If A is diagonal, the maps $A^{-n} f^n|_\Omega$ are bounded by R. Thus the averages

$$\Phi_n := \frac{1}{n+1}\left(\mathrm{id} + A^{-1} \circ f + \cdots + A^{-n} \circ f^n\right)$$

are also bounded by R. Thus Φ_n is a normal family, and there are convergent subsequences Φ_{n_j}. We let Ψ denote any limit of a sequence Φ_{n_j}. We see that for any n, we have

$$A^{-1}\Phi_n \circ f - \Phi_n = \frac{1}{n+1}\left(A^{-n-1} \circ f^{n+1} - \mathrm{id}\right)$$

so the right hand side tends to zero as $n \to \infty$. This shows that $A^{-1}\Psi \circ f = \Psi$. □

Corollary 6.4. *In Theorem 6.3, neither λ nor μ can be a root of unity.*

Proof. If $\lambda^N = 1$, then A^N will be the identity on the x-axis $\{y = 0\}$. Thus f^N will have a curve of fixed points, which is not possible by Theorem 2.1. □

Question 6.2. Let f be conservative. Does every Fatou component contain a fixed point? In other words, does every Fatou component arise from linearizing a fixed point, as in Theorem 6.3?

For an invariant Fatou component Ω, we define $\mathfrak{G} = \mathfrak{G}(\Omega)$ to be the set of all normal limits $h := \lim_{j\to\infty} f^{n_j}$. A priori, each limit gives a holomorphic map $h : \Omega \to \overline{\Omega} \subset \mathbb{C}^2$. However, h must preserve volume, so it is an open map $h : \Omega \to \Omega$, which by a Theorem of H. Cartan must be an automorphism (see Narasimhan [15] for Theorems of H. Cartan). Since $\mathfrak{G}$ is the set of limits of maps of a bounded set, it is compact subset of the automorphism group $Aut(\Omega)$. Another Theorem of H. Cartan says that $Aut(\Omega)$ is a Lie group. Since $\mathfrak{G}$ is generated by the iterates of a single map, it is Abelian, so we have:

Theorem 6.5. *Let f be conservative, and let Ω be an invariant Fatou component. Then $\mathfrak{G}$ is a compact, Abelian Lie group.*

We let $\mathfrak{G}_0$ denote the connected component of the identity. Since this is a Lie group, it is equivalent to a torus $\mathbb{T}^\rho$ for some nonnegative integer ρ. We say that ρ is the rank of Ω.

Theorem 6.6. *The rank of Ω is either one or two.*

Proof. $\mathfrak{G}$ contains all the iterates $f^n|_\Omega$, so it is infinite. Since it is also compact, we cannot have $\rho = 0$. Now consider the action of $\mathfrak{G}$ on Ω. By Theorem 2.1, the fixed points are discrete, so a generic orbit will have real dimension ρ. Now suppose that $\rho = 3$. We may choose $q \in \Omega$ so that the orbit $M := \mathbb{T}^3 \cdot q$ is a smooth 3-manifold at q. Let $H_q := T_q(M) \cap iT_q(M)$ be the (unique) $\mathbb{C}$-linear subspace of the tangent space $T_q(M)$. Let $L \subset \mathbb{T}^3$ be the 2-dimensional linear subspace (not necessarily closed in $\mathbb{T}^3$) such that the tangent to the orbit $L \cdot q$ at q is H_q. It follows that the tangent space to $L \cdot q$ is complex at each point. Thus $L \cdot q$ is a Riemann surface. If we pull back the coordinate functions x and y from $\mathbb{C}^2$ to L, they are holomorphic and bounded on L. However, L is equivalent to $\mathbb{C}$, which is a contradiction. It is clear that we cannot have $\rho = 4$. □

Global linear model: (p, q)-action. Let us now discuss how linearization gives us the existence of conservative maps such that K has nonempty interior. We may choose a pair of multipliers λ, μ with $|\lambda| = |\mu| = 1$ and a polynomial $q(x)$ which vanishes to order at least 2 at the origin. Let $G_{\lambda,\mu}$ be as in (2.3). If we choose the pair λ, μ to satisfy (5.4), then by

Theorem 5.3 it follows that $G_{\lambda,\mu}$ can be linearized in a neighborhood of the origin. Since the linear map $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ is unitary, it follows that K contains a neighborhood of the origin.

For instance, we may choose $\lambda = \alpha^p$ and $\mu = \alpha^q$, $p, q \geq 1$, satisfying (5.4) for some $|\alpha| = 1$. In this case, L induces the $\mathbb{T}^1$-action, where $\mathbb{T}^1$ acts on a point $(z, w) \in \mathbb{C}^2_{\text{lin}}$ according to: $\theta \mapsto (e^{ip\theta}z, e^{iq\theta}w)$. Thus the L-orbits are closed curves, and the varieties $z^q = cw^p$ are L-invariant. The Fatou component Ω containing O as an example of a rank 1 rotation domain.

Question 6.3. Suppose that Ω is a rank 1 rotation domain, not necessarily having a fixed point. Is the $\mathfrak{G}$-action on Ω conjugate to a (p, q)-action?

Question 6.4. Is it possible for a conservative map to induce a (p, q)-action as above, but with $p > 0 > q$? The invariant varieties for L are now of the form $z^{|q|}w^p = c$. We note that if such a component Ω has a fixed point, then the multipliers λ, μ at the fixed point will have a resonance. (This is because we may assume that p and $|q|$ are relatively prime, so there will exist positive integers m, n such $mp + nq = 1$, and so $\lambda^m\mu^n = \alpha$.) If there is no fixed point, it is not clear that this resonance causes a problem.

Global linear model: $\mathbb{T}^2$-action. Another possibility is that we choose λ, μ to satisfy (5.4), and in addition we can choose λ and μ to be *multiplicatively independent*, which means that $\lambda^m\mu^n \neq 1$ for all $m, n \in \mathbb{Z}$ with $(m, n) \neq (0, 0)$. In this case, L induces a $\mathbb{T}^2$-action on $\mathbb{C}^2_{\text{lin}}$ which acts on a point $(z_1, z_2) \in \mathbb{C}^2_{\text{lin}}$ as

$$\mathbb{T}^2 \ni (\theta_1, \theta_2) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2). \tag{6.1}$$

The generic orbit of this point is the 2-torus

$$T_{c_1,c_2} = \{|z_1| = c_1, |z_2| = c_2\} \subset \mathbb{C}^2_{\text{lin}}.$$

Even if there is no fixed point in Ω, we can think of the $\mathfrak{G}$-action on Ω as an abstract 2-torus action, and we refer to (6.1) as the linear action. It was shown in [1] that such an abstract torus action is equivariantly equivalent to a Reinhardt action (6.1). Thus we may identify the Fatou component Ω with a Reinhardt domain $\Omega_0 \subset \mathbb{C}^2_{\text{lin}}$.

A Reinhardt domain $\Omega_0 \subset \mathbb{C}^2_{\text{lin}}$ is uniquely defined by its real profile $\omega_0 = \Omega_0 \cap \mathbb{R}^2_+$. We define the logarithmic image $\log(\omega_0) := \{(\xi_1, \xi_2) : (e^{\xi_1}, e^{\xi_2}) \in \omega_0\}$. It is clear that Ω must be polynomially convex, and thus $\log(\omega_0)$ is a convex subset of $\mathbb{R}^2$.

Question 6.5. What is the biholomorphic type of Ω?

(1) More specifically, can Ω be biholomorphically equivalent to something familiar such as the unit ball $\mathbb{B}^2$?
(2) Or the bidisk Δ^2?
(3) Or more generally, can you say anything about the convex set $\log(\omega_0)$?

Question 6.6. Let f be conservative. What are the possibilities for the number of components of $\text{int}(K)$?

(1) Can $\text{int}(K)$ be connected? Equivalently, can there be just one Fatou component?

(2) Can $\mathrm{int}(K)$ have infinitely many components?

Question 6.7. Can there be a Fatou component Ω such that $\overline{\Omega} = K$? Is it possible for $\mathrm{int}(K)$ to be dense in K?

Notes. Theorem 6.3 is taken from Herman [10]. Many of the other results in this section come from [4].

7. Fatou Components: Dissipative Maps

If a map f is not conservative, then $|\delta| \neq 1$. Here we discuss the components of $int(K^+)$ in the non-conservative case. Because of the following Proposition, we will restrict our attention to the *dissipative* case: $|\delta| < 1$.

Proposition 7.1. *If* $|\delta| > 1$, *then* K^+ *has zero volume, and thus* $\mathrm{int}(K^+) = \emptyset$.

Proof. Let V be the central bidisk in the filtration. By the filtration properties, $K^+ \cap V \subset f^{-1}(K^+ \cap V)$, and $K^+ = \bigcup f^{-n}(K^+ \cap V)$. We have $Vol(K^+ \cap V) < \infty$, and

$$Vol(f^{-1}(K^+ \cap V)) = |\delta|^{-1}|Vol(K^+ \cap V),$$

so we conclude that $Vol(K^+) = 0$. □

In contrast to the conservative case, it is not known whether Fatou components can be wandering for dissipative maps:

Question 7.1. Can a dissipative Hénon map have a wandering Fatou component?

Let us consider a component Ω of $int(K^+)$ which satisfies $\Omega = f(\Omega)$. Given a dissipative map f, we define the set of all normal (uniform on compacts) limits of sequences of iterates:

$$\mathfrak{H} = \mathfrak{H}(\Omega) := \{\text{all normal limits } h = \lim_{j\to\infty} f^{n_j} : \Omega \to \overline{\Omega} \subset \mathbb{C}^2\}$$

If the limit $h = \lim_{j\to\infty} f^{n_j}$ exists, then so does the limit $\lim_{j\to\infty} f^{n_j+1}$, and we have

$$h \circ f = \lim_{j\to\infty} f^{n_j} \circ f = \lim_{j\to\infty} f \circ f^{n_j} = f \circ h$$

on Ω. It follows that if $\Sigma := f(\Omega)$, then $f(\Sigma) = \Sigma$ is invariant. We define the *rank* of $h \in \mathfrak{H}$ to be the maximum rank of $D_z h$ for $z \in \Omega$. Since $|\delta| < 1$, we see that the Jacobian determinant of h is 0. Thus the rank of each $h \in \mathfrak{H}$ is either 0 or 1. If the rank is 0, then $\Sigma = h(\Omega) = z_0$ is a fixed point. We will say that Ω *has rank 0* if every $h \in \mathfrak{H}(\Omega)$ has rank 0. We start by discussing the case of rank 0.

Proposition 7.2. *Suppose that every* $h \in \mathfrak{H}$ *has rank 0. Then there is a fixed point* $z_0 \in \overline{\Omega}$ *such that* $\lim_{n\to\infty} f^n(z) = z_0$ *for all* $z \in \Omega$. *In other words, if* Ω *has rank 0, then it is attracted to a unique fixed point* $z_0 \in \overline{\Omega}$.

Proof. We have seen that if $h_0 \in \mathfrak{H}$ has rank 0, then $h_0(\Omega) = z_0$ is a fixed point. We will show that the constant function h_0 is the only function in $\mathfrak{H}$. Suppose there is some other $h_1 \in \mathfrak{H}$ with $h_1(\Omega) = z_1$. The number of fixed points of f is finite. Let us write them $\{z_0, \ldots, z_N\}$. We may find neighborhoods V_j of z_j such that $(V_j \cup f(V_j)) \cap V_k = \emptyset$ for all $1 \le j, k \le N$. Since h_0 and h_1 are both in $\mathfrak{H}$, there are sequences $\{n_j\}$ and $\{m_j\}$, both tending to infinity, and with $n_j < m_j < n_{j+1}$ such that for some fixed $w' \in \Omega$, we have $f^{n_j}(w') \in V_0$ and $f^{m_j}(w') \in V_1 \cup \cdots \cup V_N$. Let p_j denote the first value of $n_j < p$ such that $f^p(w') \notin (V_0 \cup \cdots \cup V_N)$. Since $\{f^{p_j}\}$ is a normal family, we may extract a subsequence which converges to a mapping $\hat{h} \in \mathfrak{H}$ with $\hat{h}(w') \in \overline{\Omega} - \bigcup_{j=0}^{N} V_j$. Thus $\hat{h}(\Omega)$ is not a fixed point, which is a contradiction. □

Now let $z_0 \in \overline{\Omega}$ be the fixed point of Proposition 7.2, and let λ and μ be the eigenvalues of $D_{z_0} f$. Since $\lambda\mu = \delta$ has modulus < 1, we may assume that $|\mu| < 1$. If $|\lambda| > 1$, then z_0 is a saddle point, and $W^s(z_0)$ is a manifold of real dimension 2. This is not possible, since by Proposition 7.2, the whole open set Ω is contained in $W^s(z_0)$. Thus $|\lambda| \le 1$. When the multipliers of a fixed point z_0 are $|\lambda|, |\mu| < 1$, z_0 is *attracting.* If λ is a root of unity and $|\mu| < 1$, we say that z_0 is *semi-parabolic/semi-attracting.* A semi-attracting fixed point has an interesting structure, which will be described in Section 16. By the following two Theorems, we see that the fixed point of a rank 0 domain is either attracting or semi-attracting.

Theorem 7.3. *Suppose that Ω has rank zero, and let z_0 be its fixed point as in Proposition 7.2. If $z_0 \in \Omega$, then z_0 is an attracting fixed point, and Ω is its basin of attraction.*

Proof. We know that the multipliers of $D_{z_0} f$ are $|\lambda| = 1$ and $|\mu| < 1$. If $z_0 \in \Omega$, then the iterates $\{f^n\}$ are converge normally to the constant z_0 in a neighborhood of z_0. Thus the derivative of the iterates $D_{z_0} f^n$ also converges to $D_{z_0} h = 0$. Thus we must have $|\lambda| < 1$. Thus z_0 is an attracting fixed point, and by Proposition 7.2, Ω is contained in the basin of attraction of z_0. Since Ω is also a component of the interior of K^+, it follows that Ω must contain the whole basin. □

Theorem 7.4. *Suppose that Ω has rank zero, and let z_0 be its fixed point as in Proposition 7.2. If $z_0 \in \partial\Omega$, then z_0 is semi-parabolic/semi-attracting, and Ω is its basin.*

Proof. □

Let $\mathcal{R} \subset \mathbb{C}$ be either the disk $\{|\zeta| < 1\}$ or the annulus $\{r < |\zeta| < R\}$. If there is an imbedding $\chi : \mathcal{R} \to \hat{\mathcal{R}} := \chi(\mathcal{R}) \subset \mathbb{C}^2$ such that $f(\chi(\zeta)) = \chi(e^{i\kappa\pi}\zeta)$, then we will say that $\hat{\mathcal{R}}$ is a *rotational disk/annulus.* We will always assume that a rotational disk/annulus is maximal: there is no strictly larger rotational disk/annulus which contains it. Since the fixed points of f are discrete, we see that we must have $\kappa \notin \mathbb{Q}$, which means that f induces an irrational rotation on $\hat{\mathcal{R}}$. It is clear from the Section on Linearization that for suitable $|\lambda| = 1$ and any $0 < |\mu| < 1$, we can linearize fixed points with multipliers λ, μ and obtain a rotational disk. However, a rotational annulus has no fixed point, so we cannot simply construct one by linearizing a fixed point. This raises the question:

Question 7.2. Can a Hénon map f have a rotational annulus?

The following Theorem shows that a rotational disk/annulus is contained in a Fatou component where f can be globally linearized.

Theorem 7.5. *If $\hat{\mathcal{R}}$ be rotational, then there is a Fatou component Ω such that $\chi : \mathcal{R} \to \hat{\mathcal{R}} \subset \Omega$ is a proper imbedding. Further, there is a biholomorphic map $\Phi : \mathcal{R} \times \mathbb{C} \to \Omega$ such that $\Phi \circ f = L \circ \Phi$, where $L(z, w) = (e^{i\kappa\pi} z, \delta e^{-i\kappa\pi} w)$, and $\Phi(\zeta, 0) = \chi(\zeta)$.*

Proof. □

Let Ω be a component of $int(K^+)$ such that $f(\Omega) = \Omega$. We say that Ω is *recurrent* if there is a point $z' \in \Omega$ such that the sequence $f^n(z')$ does not converge to the boundary. In other words, there is a point $z' \in \Omega$, and there is a subsequence $f^{n_j}(z')$ which converges to a point $\hat{z} \in \Omega$. The condition that Ω is not recurrent is equivalent to the statement that $h(\Omega) \subset \partial\Omega$ for all $h \in \mathfrak{H}$.

Theorem 7.6. *Let Ω be a periodic Fatou component which is recurrent. Then Ω is the basin of either an attracting fixed point or a rotational disk or annulus.*

Proof. □

A point z is *recurrent* if it is contained in its ω-limit set, which means that there is a sequence $f^{n_j}(z) \to z$ as $n_j \to \infty$.

Corollary 7.7. *A periodic Fatou component is recurrent if and only if it contains a recurrent point.*

Theorem 7.5 classifies the recurrent Fatou components, although it leaves open the existence/nonexistence or rotational annuli. For the non-recurrent case, we ask:

Question 7.3. Suppose that Ω is a periodic Fatou component which is not recurrent. Is Ω necessarily a semi-parabolic basin? Lyubich and Peters [13] have shown that the answer to this question is "yes" if f has sufficient dissipation, i.e., if $|\delta| < 1/d^2$.

8. Interlude: Pictures

Computer graphics have been effective in showing Julia sets for polynomial and rational maps of $\mathbb{C}$. Here we describe the *unstable slice* pictures for complex Hénon maps. These were introduced by Hubbard in the 1980s and have proved to be very useful, especially in the study of dissipative maps. In addition, their validity has been established by subsequent theoretical work.

Let p be a saddle point. The *unstable slice* is simply the intersection $W^u(p) \cap K^+$. Since p is a saddle point of some period N, the multipliers of $D_p f^N$ are $|\mu| < 1 < |\lambda|$. Without loss of generality, we assume $N = 1$. The unstable manifold is uniformized by an entire function

$$\xi : \mathbb{C} \to W^u(p) \subset J^- \subset \mathbb{C}^2$$

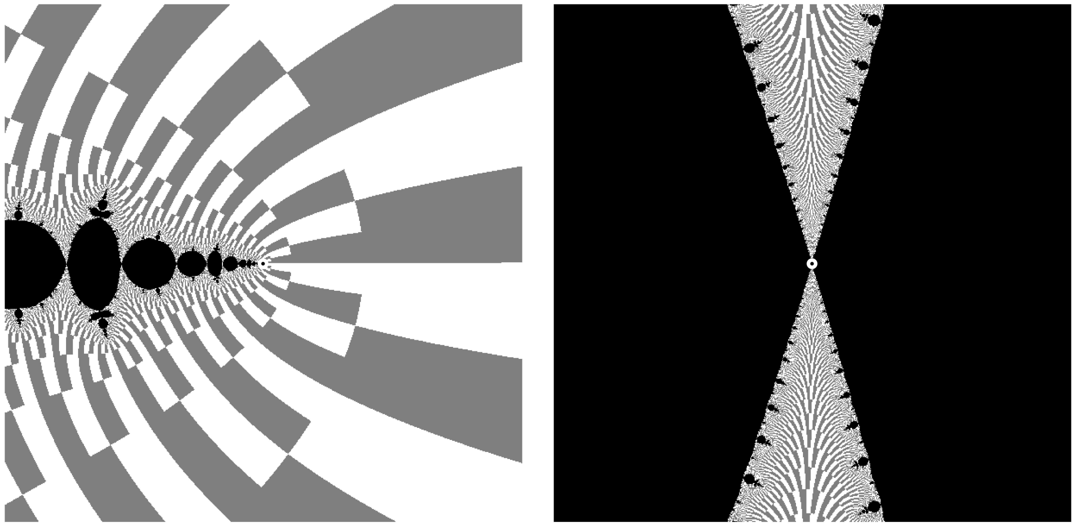

FIGURE 4. *Unstable slices $W^u(p) \cap K^+$ for the map $f(x,y) = (x^2 + c - \delta y, x)$, $c = -1.1$ and $\delta = .15$. The black region is $W^u(p) \cap K^+$, and the white/gray region corresponds to the binary digits in the argument of φ^+. Here the points p are taken to be the two fixed points of f. On the left, the multiplier is ~ 3.49931; on the right it is ~ -1.10663.*

with the property that $\xi(0) = p$ and $\xi(\lambda\zeta) = f(\xi(\zeta))$ for all $\zeta \in \mathbb{C}$. Let E^u be the λ-eigenvector for D_pf, so the line $\zeta \mapsto p + \zeta E^u$ is tangent to $W^u(p)$ at p. The uniformization may be computed as

$$\xi(\zeta) := \lim_{n\to\infty} f^n(p + \lambda^n \zeta E^u)$$

In order to show $K^+ \cap W^u(p)$, we plot $\xi^{-1}(K^+)$ inside $\mathbb{C}$. This may be done by plotting the level sets of $G^+ \sim d^{-k}\log|y_k|$, where k is chosen so that y_k is sufficiently large. At this value of k, we say that the orbit has escaped. We may color the level sets, as in Figures 4 and 6, in white/gray, depending on whether $y_k \in \mathbb{C}$ is in the upper/lower half plane when it escapes. When φ^+ exists, its modulus is $|\varphi^+| = e^{G^+}$, and the white/gray coloring corresponds to the digits 0/1 in the kth place of the binary expansion of the argument of φ^+. While we are showing the sets on which the argument of φ^+ is constant and equal to $2^{-k}\pi$, the level sets of G^+ are implicitly visible.

We will illustrate unstable slices with the map $f(x,y) = (x^2 + c - \delta y, x)$, $c = -1.1$ and $\delta = .15$. This map has an attracting 2-cycle. The points of $K^+ \cap W^u(p)$ are colored black, and $U^+ \cap W^u(p)$ are white/gray, and the fixed point $\zeta = 0$ is a small black dot in the center. By construction, the unstable slice picture shows a self-similarity, since it is invariant under the scaling $\zeta \mapsto \lambda\zeta$. Since the multiplier on the right hand of Figure 4 is negative, the two large black regions, which are in the basin of the attracting 2-cycle, are interchanged. The multiplier for the left hand image in Figure 4 is much larger than the multiplier for the right hand image, and this explains why the level sets of G^+ on the right hand image are much narrower.

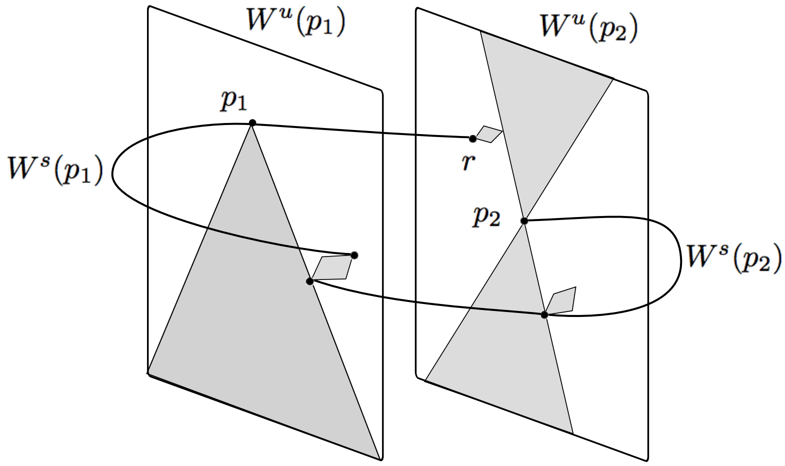

FIGURE 5. *This is a highly stylized representation of how the stable manifolds of saddle points connect visual features ("tip points" and "cut points" in this case) within a given unstable slice, as well as the connections between different slices. By [2], the transverse intersections of $W^u(p_j)$ and $W^s(p_k)$, $j,k = 1,2$ are dense in the boundary of the unstable slice $W^u(p_j) \cap K^+$.*

In fact, there are infinitely many saddle points, so there are infinitely many unstable slice pictures. Figure 4 gives the unstable slice pictures based at the two fixed (saddle) points p_1 and p_2. In the left hand image, "shape" of the slice $W^u(p_1) \cap K^+$ at p_1 is like a "tip", whereas p_2 is a "cut point" for the right hand image. However, in some sense, both of these unstable slice pictures contain the same information. The unstable slices in Figure 4 are connected by the stable manifolds $W^s(p_1)$ and $W^s(p_2)$, as is shown schematically in Figure 5. Namely, by [2] there are points $r \in W^s(p_1) \cap W^u(p_2) \cap K^+$ such that $W^s(p_1)$ and $W^u(p_2)$ intersect transversally at r. Let $r \in D \subset W^u(p_2)$ be a small disk. As we map D forward, the Lambda Lemma says that $f^n(D)$ approaches $W^u(p_1)$ in C^1 topology. Let λ_1 be the unstable multiplier of $D_{p_1} f$. It follows from the Lambda Lemma that the local dilations of $D \cap K^+$ by factor λ_1^n, centered at r, will converge to the unstable slice picture $W^u(p_1) \cap K^+$. By [2], such points r are dense in $\partial(W^u(p_2) \cap K^+)$, so the left hand side image appears at infinitely small scale in a dense subset of the boundary of the right hand image.

By looking at Figure 4, we can "see" that "tip" points in the shape of the slice through p_1 are dense in the boundaries of both images. Similarly we see the set of "cut" points appearing densely. Two more unstable slice pictures are given in Figure 6. These do not have such simply described shapes as "tip" or "cut" point, but these are shapes that appear densely at infinitely small scale. This is discussed with more rigor in [7].

A lot of information, for instance connectivity, can be obtained from unstable slices. In the following result from [6], we uniformize $\xi : \mathbb{C} \to W^u(p)$ and consider the preimage $\xi^{-1}(W^u(p) \cap K^+)$ in $\mathbb{C}$. The statement "$W^u(p) \cap K^+$ is connected" means that this preimage is connected if we take its closure by adding the point at infinity to $\mathbb{C}$.:

Theorem 8.1. *Suppose that $|\delta| < 1$. The following are equivalent:*

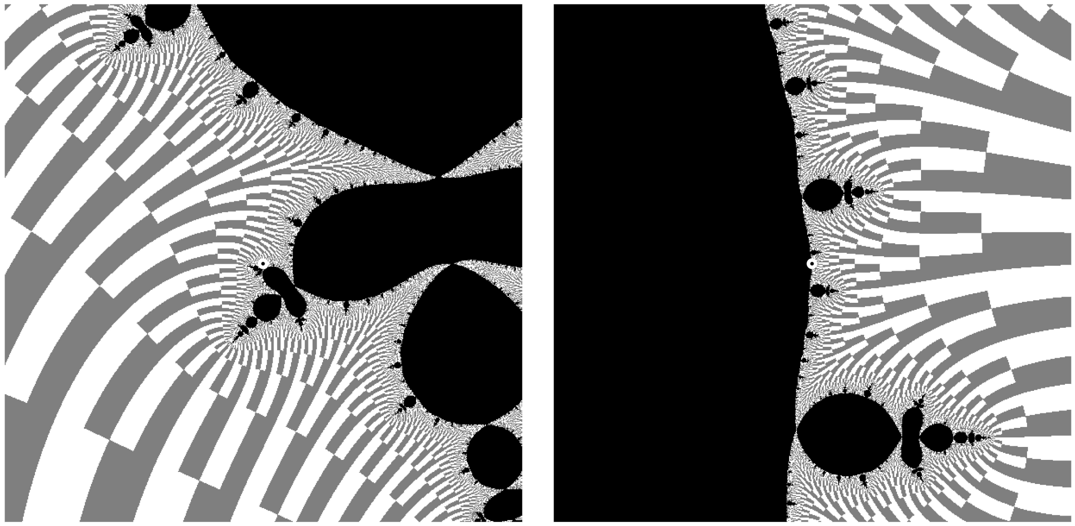

FIGURE 6. *More unstable slices for the map of Figure 4. Image on the left: the saddle point has period 3 and multiplier* $\sim 2.44918 + 4.43005i$. *Since this multiplier is non-real, we see that the slice* $W^u(p) \cap K^+$ *spirals towards* p. *Image on the right: the saddle point has period 4 and multiplier* ~ 6.26274.

(1) J *is connected.*
(2) K *is connected.*
(3) $W^u(p) \cap K^+$ *is connected for some saddle point* p.
(4) $W^u(p) \cap K^+$ *is connected for every saddle point* p.

Of course, we can also consider the *stable slices* $W^s(p) \cap K^-$. It seems that these sets are always totally disconnected in the dissipative case, so we ask:

Question 8.1. Suppose that $|\delta| < 1$. Is the stable slice $W^s(p) \cap K^- = W^s(p) \cap J^-$ always totally disconnected? Dujardin and Lyubich [8] have showed that this is indeed the case if f is sufficiently dissipative: if $|\delta| < d^{-2}$.

Notes. The unstable slices were introduced and used extensively by Hubbard in the 1980s. The Thesis of R. Oliva [16] gives a number of computer pictures of bifurcations of maps which have attracting 2-cycles.

There are also very interesting graphic approaches by S. Ushiki, which may be seen at: `http://www.math.h.kyoto-u.ac.jp/~ushiki/` . Some of this software visualizes the periodic points of f as a "cloud" in 4-space.

9. Interlude: Currents

10. Uniqueness of Currents

11. Julia set J^+

Theorem 11.1. *If* p *is a saddle point, then* $W^s(p)$ *is a dense subset of* J^+.

Corollary 11.2. *J^+ is connected.*

Theorem 11.3. *If Ω is a periodic Fatou component which is either recurrent or a semi-parabolic basin, then $\partial\Omega = J^+$.*

12. Interlude: Model maps

12.1. **Complex solenoid.**

12.2. **Complex horseshoes.**

13. Julia set $J = J^+ \cap J^-$

14. Julia set J^*

15. Hyperbolicity and Quasi-hyperbolicity

Question 15.1. Can a quasi-hyperbolic map have a wandering Fatou component?

16. Semi-parabolic fixed points

17. Parallels between Dimensions 1 and 2

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