Math 319/320 Worksheet 1

Name: **ID:**

The first part of this worksheet is a review of basic logic. Recall the symbols

$$\forall = \text{ for all}, \quad \exists = \text{ there exists}, \quad \wedge = \text{ and}, \quad \vee = \text{ or}, \quad \sim = \text{ not}.$$

Remember also that if P and Q are statements the implication $P \Rightarrow Q$ ("P implies Q") is true if P is false and also if P is true and Q is true. Therefore its negation $\sim(P \Rightarrow Q)$ is logically equivalent to $P \wedge (\sim Q)$, i.e. $\sim(P \Rightarrow Q)$ holds iff P is true and Q is false.

Note: this is a slightly edited version of the sheet that was handed out in class.

Problem 1. *Negate the following statement: "If your glass is half-empty, you are a pessimist or you are thirsty." (Answer in words.)*

Your glass is half-empty but you are an optimist who is not thirsty.

or Although your glass is half-empty, you are neither a pessimist nor thirsty.

Problem 2. *The context in this problem is the set of all human beings. Let $E(x)$ be "x is educated," $F(x)$ be "x is female" and $O(x)$ be "x is older than 30." Then the statement "every uneducated male is older than 30" can be expressed as*

$$\forall x, (\sim E(x) \wedge \sim F(x)) \Longrightarrow O(x)$$

Express the following statements in a similar way.

Since x ranges over all human beings, it is implicit in the question that there are some people who are educated and some who are not, some males and some females and also some people over 30 and some under 30. This comment applies particularly to (i) where the answer given assumes that there are educated people, and that the negative of "over 30" is "under 30". When we do actual mathematics, there won't be this ambiguity.... However, it is still true that the questions have several correct (and equivalent) answers.

(i) *Some educated people are younger than* 30.
$\exists x, E(x) \wedge \sim O(x)$.

(ii) *Every female who is older than* 30 *is educated.*
$\forall x, (F(x) \wedge O(x)) \Longrightarrow E(x)$.
or $\sim \exists x, F(x) \wedge O(x) \wedge (\sim E(x))$.

(iii) *No uneducated person is both female and older than* 30.
logically this is equivalent to (ii). So you could use either of the expressions in (ii) or also
$\forall x, \sim E(x) \Longrightarrow \sim \big(F(x) \wedge O(x)\big)$

Problem 3. *Consider the statement*

"For every natural number n, if n^2 is even, then n is even."

Prove this statement in two different ways: (i) by showing that its contrapositive is true; (ii) by showing that its negation is false.
Note: *the contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.*

(i) Suppose that n is odd. We must show that n^2 is odd. But if n is odd we may write $n = 2k+1$ where $k \in \mathbb{N}, k \geq 0$. Then $n^2 = (2k+1)^2 = 4k^2+4k+1$ is also odd.

(ii) The negation of the implication $P \Rightarrow Q$ is $P \wedge (\sim Q)$. So we must show that it is impossible for n^2 to be even and n to be odd. But if n is odd we may write $n = 2k+1$ where $k \in \mathbb{N}, k \geq 0$. Then $n^2 = (2k+1)^2 = 4k^2+4k+1$ is also odd. So the hypotheses n^2 odd and n even are indeed contradictory.

Note In this simple case, the basic argument is the same in each case. It is just the framing that varies. But with more complicated arguments the choice of framing sometimes makes a difference.

Problem 4. On a bumper sticker, I saw the statement

"For every real number x, there is a real number t such that $t(1-t) > x$."

After some thought I conjectured that this statement must be __________. To prove my conjecture carefully, I found a real number ____ such that for every real number ____ the inequality ____________________ held.

Hint: To understand the behavior of the function $f(t) = t(1-t)$ it might be helpful to draw its graph.

After some thought I conjectured that this statement must be FALSE. *To prove my conjecture carefully, I found a real number x such that for every real number t the inequality $t(1-t) \geq x$ held.*

Note: The graph of $t(1-t)$ is a parabola cupped downwards and takes its maximum at $t = 1/2$, halfway between its zeros at $t = 0$ and $t = 1$. Therefore the maximum value is $1/4$. So you can take x to be anything $\geq 1/4$.