Math 319/320 Worksheet 4 solutions

Problem 1. *Find all $x \in \mathbb{R}$ that satisfy the following inequality:*
(a) $|x| + |x+1| < 2$

Method 1: draw a graph.
Method 2: Divide into cases, depending on the signs of x and $x+1$.
Suppose $x \geq 0$. Then we have $2x+1 < 2$, i.e. $x < 1/2$.
Suppose $-1 \leq x \leq 0$. Then we have $-x+x+1 < 2$ – always true.
Suppose $x < -1$. Then we have $-x-x-1 < 2$, i.e. $x > -3/2$.
So answer: $x \in (-3/2, 1/2)$.

(b) $4 < |x+2| + |x-1| < 5$
Case 1: $x > 1$. Then we have $4 < x+2+x-1 = 2x+1 < 5$. That is, $3 < 2x < 4$ or $x \in (3/2, 2)$.
Case 2: $-2 \leq x \leq 1$. Then we have $4 < x+2+1-x = 3 < 5$. This is impossible.
Case 3: $x < -2$. Then we have $4 < -x-2-x+1 = -2x-1 < 5$ or $5 < -2x < 6$ i.e. $-3 < x < -5/2$. So $x \in (-3, -5/2)$.
Thus the answer: $x \in (-3, -5/2) \cup (3/2, 2)$.

Problem 2. *Assume that A and B are bounded subsets of $\mathbb{R}$.*
(a) *Show that $A \cup B$ is a bounded subset of $\mathbb{R}$.*

Since A is bounded there is $K \in \mathbb{R}$ such that $|x| \leq K$ for all $x \in A$. Similarly there is $M \in \mathbb{R}$ such that $|y| \leq M$ for all $y \in B$. Therefore if $z \in A \cup B$ we have

$$|z| \leq K \leq \max(K, M), \text{ if } z \in A, \quad \text{and } |z| \leq M \leq \max(K, M), \text{ if } z \in B.$$

Thus $\max(K, M)$ is a bound for $A \cup B$.

(b) *Show that* $\sup(A \cup B) = \sup\{\sup\ A, \sup\ B\}$.

Let $K := \sup A$ and $M := \sup B$. The argument in (a) shows that $\max(K, M)$ (which is precisely the same thing as $\sup\{\sup\ A, \sup\ B\}$) is an upper bound for $A \cup B$.

To see that $\max(K, M)$ is the supremum of $A \cup B$ it suffices to show that no number p strictly less than $\max(K, M)$ is an upper bound for $A \cup B$.

Without loss of generality we may assume that $K \leq M$. (If not, we can simply rename the sets A, B to make this true.) Then $\max(K, M) = M$. Hence if $p < \max(K, M)$, $p < M$ and so p is NOT an upper bound for B. Thus there is $b \in B$ such that $b > p$. Since $b \in A \cup B$ this means that p is not an upper bound for $A \cup B$. Thus $\max(K, M)$ is the supremum of $A \cup B$.

Problem 3. *Show that nonempty finite subset S always has a supremum and that it contains it. (Hint: Use Mathematical Induction.)*

We will prove this by induction of $n := |S|$, the number of elements in S.

Base case: $n = 1$. Then $S = \{a\}$. Hence $a = \sup S$. Also $a \in S$. Thus the statement holds.

Now suppose that $|S| = k + 1$ and that the statement holds for all sets with k elements. Choose any $x \in S$ and let $T := S \setminus \{x\}$. Then $|T| = k$. Therefore, by the inductive hypothesis, $\sup T$ exists and equals some element $y \in T$.

Case (i): $y \geq x$. Then y is an upper bound for S. And no number smaller than y can be an upper bound for S, since it would also have to be an upper bound for $T \subset S$ and $y = \sup T$. Hence $y = \sup S$. Since $y \in T \subset S$, the statement holds for S.

Case (ii): $y < x$. In this case, x is an upper bound for S since by transitivity of order it is $>$ all the elements in $S \setminus \{x\}$. Any other upper bound u for S must satisfy $u \geq x$ (since $x \in S$.) Hence $x = \sup S$. Thus the statement holds in this case too.

Thus the inductive step is proven. Hence the statement holds by Mathematical Induction.

Problem 4. *Let S be a subset of $\mathbb{R}$ that is bounded below. Show that* $\inf S = -\sup\{-s | s \in S\}$.

Define $T := -S := \{-s : s \in S\}$.

Note that

$$(*) \qquad x \leq s,\ \forall s \in S \iff -x \geq -s := t,\ \forall t \in T.$$

Thus x is a lower bound for S iff $-x$ is an upper bound for T.

Suppose now that $x := \inf S$. Then, $-x$ is an upper bound for T and we just need to see that it is the least upper bound. But if not, there is an upper bound u for T such that $u < -x$. Then $-u > x$ and, by (*), $-u$ is a lower bound for S. But this is impossible; because $x := \inf S$, x must be $\geq$ every lower bound for S, in particular we must have $x \geq -u$.