Math 319/320 Worksheet 2

Problem 1. *Fill in the blanks in the following proof that* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

If $x \in A \cup (B \cap C)$ then either $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ and so $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$ then $x \in B$ and $x \in C$. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Now suppose that $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$.
If $x \in A$ then: $x \in A \cup (B \cap C)$.
On the other hand if $x \notin A$ then $x \in B$ (because $x \in A \cup B$) and $x \in C$ (because $x \in A \cup C$). So $x \in B \cap C$.
Therefore x is either in A or in $B \cap C$, i.e. $x \in A \cup (B \cap C)$.

Problem 2. *It is possible to take intersections and unions of many sets* $A_i, i \in I$, *not just two. We define*

$$\cup_{i \in I} A_i := \{x : \exists i \in I \text{ such that } x \in A_i\}, \qquad \cap_{i \in I} A_i := \{x : x \in A_i \forall i \in I\}.$$

The set I *is called the indexing set. Often it is the set of the first* n *integers* $\{1, \ldots, n\}$, *but sometimes it is the infinite set* $\mathbb{N}$ *of all positive integers.*

(i) *Find three subsets* A_1, A_2, A_3 *of the plane* $\mathbb{R}^2$ *such that each double intersection* $A_i \cap A_j$ *is nonempty but the triple intersection* $A_1 \cap A_2 \cap A_3$ *is empty.*

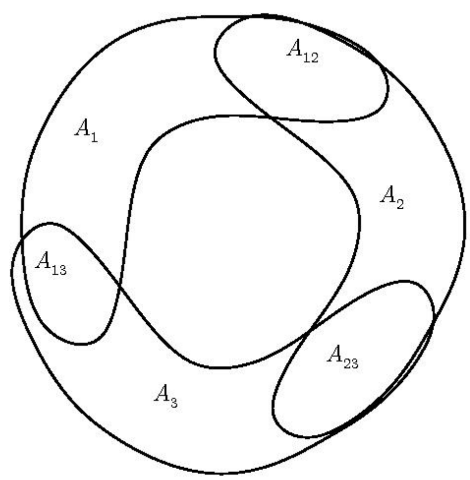

FIGURE 1. Here I wrote A_{12} to mean $A_1 \cap A_2$, etc.

(ii) *Find open intervals* $A_i = (a_i, b_i) \subset \mathbb{R}$ *such that each finite intersection* $\cap_{1 \le i \le n} A_i$ *is nonempty but the infinite intersection* $\cap_{i \in \mathbb{N}} A_i$ *is empty.*

Take for example, $A_i = (0, 1/i)$.

Problem 3. *Let $f : A \to B$ be a function and $C \subset A, D \subset B$. Show that $C \subseteq f^{-1}(f(C))$ and $f(f^{-1}D) \subseteq D$.*

$f^{-1}(f(C)) = \{a \in A : f(a) \in f(C)\}$. If $x \in C$ then $f(x) \in f(C)$ by defn of $f(C)$, hence x satisfies the condition to be in $f^{-1}(f(C))$.

(In words: $f^{-1}(f(C))$ is the set of all points whose image is contained in the the image $f(C)$ of C. But obviously the points in C have image in $f(V)$.)

If $x \in f^{-1}(D)$ then $f(x) \in D$ by defn of the inverse image. Hence $f(f^{-1}D) \subset D$.

If f is injective, do either of these inclusions become equalities?

f is injective iff $f(x) = f(y)$ implies $x = y$. To say $f(x) \in f(C)$, means that there is $c \in C$ such that $f(x) = f(c)$ (by defn of the set $f(C)$.) Hence if f is injective x must equal c. Since this holds for all x such that $f(x) \in f(C)$, $f^{-1}(f(C)) = C$.

But the second statement about D won't hold unless the image of f contains D, and you can only be sure of this when f is surjective.

(eg take $f : [0, \infty) \to \mathbb{R}, x \mapsto x$ and $D = (-2, -1)$.)

What if f is surjective? Now the first statement need not hold, but the second will. You should find examples here on your own.

Problem 4. *Let A, B be subsets of a universal set U. Simplify the following expressions. You can draw Venn diagrams to help you. (i) $(A \cap B) \cup (U \setminus A)$ and (ii) $A \cup [B \cap (U \setminus A)]$.*

(i) $(A \cap B) \cup (U \setminus A) = B \cup (U \setminus A)$.

Proof: Since $A \cap B \subset B$, $(A \cap B) \cup (U \setminus A) = B \cup (U \setminus A)$.

Now suppose $x \in B \cup (U \setminus A)$. If $x \in U \setminus A$ then $x \in (A \cap B) \cup (U \setminus A)$, as required. So we need to consider the case when $x \notin U \setminus A$. This means that $x \in A$. Since $x \in B \cup (U \setminus A)$, in this case x must be in B. Hence $x \in A \cap B$. So x does lie in $(A \cap B) \cup (U \setminus A)$.

(ii) $A \cup [B \cap (U \setminus A)] = A \cup B$. Here again it is obvious that LHS $\subseteq$ RHS (where LHS = left hand side means the set $A \cup [B \cap (U \setminus A)]$ and RHS = right hand side means $A \cup B$. To show RHS $\subseteq$ LHS we only need to consider the case $x \notin A$ (since if $x \in A$ it is obvious.) But if $x \notin A$ and $x \in B$ then $x \in U \setminus A$ and $x \in B$, i.e. $x \in B \cap (U \setminus A) \subseteq$ LHS.