Math 319/320 Review for Midterm I

The exam will cover everything in Chapters 1 and 2 up to the bottom of p 47, i.e. not binary representations and decimals. You are expected to know the basic definitions and how to apply them in straightforward examples and arguments. In detail:

• you should know all the definitions in Ch 1 and Ch 2. In the exam we will ask you to state one or two of these.

• we do not expect you to memorize the 9 properties of a field (2.1.1) or the derivation in 2.1 of all the basic properties of the algebraic operations and order on $\mathbb{R}$.

• you should be able to state the important Properties:
(1.2.1) Well-Ordering Property of $\mathbb{N}$
(2.3.6) Completeness property of $\mathbb{R}$
(2.4.4) Archimedean Property
(2.5.2) Nested Intervals Property

• you should know the statements of the most important theorems and have a good idea of their proofs.

We will not ask you to reproduce any of these proofs, but we will expect you to be able to make short arguments using the basic theorems and properties. There will be five short questions on the exam, each worth 10 points.
Note: We will try to phrase the questions in the exam so that it will be clear what you can assume and what you should prove. If you have any questions of this kind during the exam, please ask one of the proctors for clarification.

Here are some sample problems. Some of the questions are too long for the actual exam, and would be shortened.

1: Let $f : A \to B$ be a function and $C \subset A, D \subset B$.
(i) Show that $f\big(C \cap f^{-1}(D)\big) = f(C) \cap D$.

(ii) Give an example to show that $f\big(C \cup f^{-1}(D)\big)$ need not equal $f(C) \cup D$.

2. (i) Define what it means for a function $D : A \to B$ to be injective.
(ii) Let A be the set of all polynomials of the form $p(x) = ax^2 + bx + c$ and B be the set of all polynomials of the form $q(x) = rx + s$, where a, b, c, r, s are arbitrary real numbers. Define a function $D : A \to B$ by $D(p) = p'$, where p' is the derivative dp/dx. Is D injective? Is it surjective?

3. (i) Show that the following statement is false by giving a counterexample.

$$\forall x \in \mathbb{R}, \exists t \in \mathbb{R}, \quad \frac{t^2}{1-x} > 1.$$

(ii) Prove the following statement by contradiction.

$$\sqrt{2} + \sqrt{3} > \sqrt{5}.$$

Note: here we denote by $\sqrt{x}$ the positive square root of a positive real number x. You may use all the standard results on the order properties of the real numbers without proof.

(iii) Prove the following statement by induction: $3^n \geq 2^n + 3n$ for all $n \geq 3$.

4 Are the following statements true or false? Justify your answer with a brief proof or counterexample.

(i) The intersection $\bigcap_{n=1}^{\infty} [n, +\infty)$ is empty.

(ii) The Completeness Property holds in every ordered field.

(iii) If $f, g : [0, 1] \to \mathbb{R}$ are bounded then

$$\sup\{f(x) + g(x) : x \in [0, 1]\} = \sup\{f(x) : x \in [0, 1]\} + \sup\{g(x) : x \in [0, 1]\}.$$

5 (i) What does it mean for a set to be countable?

(ii) Show from the definition that the set $\mathbb{Q}$ of rational numbers is countable.

(iii) Show from the definition that if S and T are countable then so is $S \cup T$. First consider the case when S and T are disjoint, and then the general case.

7 State the Nested Intervals Property. Give an example to show that it does not hold for intervals in $\mathbb{Q}$.

8 (i) Let S be a subset of $\mathbb{R}$. What is an upper bound of S? What is the supremum of S?

(ii) Let $S \subset \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound for S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.