Math 319 Midterm I – Comments on the solutions
September 29, 2005

Problem 1. *(i) Show that if $f : A \to B$ is injective and $E \subset A$ then $f^{-1}(f(E)) = E$.*

A lot of you made heavy weather with this, though it should be easy. Perhaps one problem was that you didn't enough space on the page for your answer. Anyway, you should simply prove first that $E \subset f^{-1}(f(E))$ and second that $f^{-1}(f(E)) \subset E$. Your argument must be sufficiently clear that one can see where you use the fact that f is injective. You must also be very clear that both E and $f^{-1}(f(E))$ are subsets of A, not B.

Here is a hint for the second step. Suppose that $x \in f^{-1}(f(E))$. Then (by definition) $f(x) \in f(E)$. Since every point in $f(E)$ has the form $f(e)$ for some $e \in E$, this means that $f(x) = f(e)$ for some $e \in E$. If f were an arbitrary function there is nothing more you can say. But since f is injective..... Remember that the conclusion fo this argument is meant to be that $x \in E$!

(ii) *Give an example to show that equality need not hold if f is not injective.* Here you must specify the function $f : A \to B$ and the set E and then verify that $E \neq f^{-1}(f(E))$. Thus you must give A, B, f and E. It is not good enough to say: let $f(x) = x^2$... without mentioning A, B (since this function is injective if the domain is $[0, \infty)$...)

Problem 2. *Prove the following statement by induction:*

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n, \quad \text{for all } n \in \mathbb{N}.$$

Most of you did this okay. But you must explain the logic on the inductive step properly. Some of you just hurriedly wrote down a sequence of equalities without explaining what you were doing. eg say:

assume that $P(k)$ holds, ie that $3 + \cdots + (8k - 5) = 4k^2 - k$. Consider $P(k + 1)$. Then, *by the inductive hypothesis,*

$$3 + 11 + \cdots + (8k - 5) + (8(k + 1) - 5) = 4k^2 - k + (8(k + 1) - 5)....$$

Now do some algebra....

Problem 3. *(i) State the Archimedean property of the real numbers.*

For every $x \in \mathbb{R}$ there is a positive integer n such that $n > x$.

Use it to prove that if $x > 0$ is any real number then there is a rational number r such that $x > r > 0$.

This confused some of you. You are given x and must find r. It is easiest to take r of the form $1/n$, since then the statement is an almost immediate consequence of (a).

Problem 4. *(i) Let S be a subset of $\mathbb{R}$. Give the definitions of a lower bound of S and of* $\inf S$.

x is a lower bound of S if $x \leq a$ for all $a \in S$.

$x = \inf S$ if it is a lower bound of S and if for every other lower bound y of S, $x \geq y$.

(There are many acceptable ways to phrase these definitions, but you must get the quantifiers right.)

(ii) *If possible, give examples of sets S with the following properties. If there is no such example, give a brief explanation of why.*

(a) *a set $S \subset \mathbb{R}$ with no lower bound.*

take $S = (-\infty, 0)$. If x were a lower bound for S then $x < -n$ for all $n \in \mathbb{N}$. Hence $-x > n$ for all $n \in \mathbb{N}$, contradicting Archimedes' principle.

(b) *a set $S \subset \mathbb{R}$ with a lower bound but no infimum.*

There is no such example. Any set that is bounded below has an infimum by the completeness axiom. (The completeness axiom axoim says that any set that is bounded above has a supremum, but S is bounded below iff $-S := \{-x : x \in S\}$ is bounded above. And, multiplying by -1 also takes a supremum to an infinum, and conversely.)

Problem 5. *Let us say that a set S has property F if every injection $f : S \to S$ is surjective.*

(i) *Show that $\mathbb{N}$ does not have property F.*

This question confused a lot of you, because property F is defined in terms of the *set of functions* $S \to S$ rather than the *points of* S. Hence it makes no sense to talk about points of S having property F.

If S has property F, EVERY injection $f : S \to S$ is surjective. Hence if a set does NOT have property F THERE IS an injection $f : S \to S$ that is not surjective. Hence to do this bit, you just have to define one function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective.

(ii) *Show that if S has property F and $T \subset S$ then T has property F.*

Here you must show that every injection $g : T \to T$ is surjective (using the fact that every injection $f : S \to S$ is surjective.) The easiest way to do this is by contradiction. Suppose that there is an injection $g : T \to T$ that is NOT surjective. Then use this to construct $f : S \to S$ that is injective, but not surjective. This contradicts the fact that S has property F.

To construct f you just need to extend g. ie define $f(x) = g(x)$ for $x \in T$. Then think of a way to define $f(x)$ for $x \in S \setminus T$ so that it remains injective but is still not surjective. I might help to do a specific example; eg take $T = \mathbb{N}$ and $S = \mathbb{Z}$.

Here I am using a property of functions that might be a little unfamiliar. Suppose that $T \subset S$ and that $g : T \to B$ is any function. Then $f : S \to B$ EXTENDS g if $f(x) = g(x)$ for all $x \in T$.