Math 319 Homework 9

Problem 1. (i) *Let $f : (0, 5) \to \mathbb{R}$ be the function $f(x) = 2x^2+3$. Prove from the definition that $\lim_{x\to 2} f = 11$.*

Suppose given $\epsilon > 0$. We must find $\delta > 0$ so that $|f(x)-L| < \epsilon$ whenever $0 < |x-2| < \delta$.

Now $|f(x)-L| = |2x^2+3-11| = |2x^2-8| = 2|x+2||x-2|$. If $|x-2| < 1$ then $1 < x < 3$ and $|x+2| < 5$. Hence $|f(x) - L| < 10|x-2| < \epsilon$ if, in addition, $|x-2| < \epsilon/10$. Therefore, if we take $\delta = \min(1, \epsilon/10)$ the desired inequality will hold.

(ii) *Find $M, \delta > 0$ so that $|f(x)| \le M$ for all $x \in A \cap (2-\delta, 2+\delta)$.*

$f(x)$ is an increasing and positive function when $x > 0$. Therefore, if $\delta < 2$, the values of f on $(2-\delta, 2+\delta)$ are all $\le M := f(2+\delta)$. For example, we can take $\delta = 1$ and $M = f(3) = 21$.

Problem 2 *Let c be a cluster point of A and $f : A \to \mathbb{R}$. Suppose that $\lim_{x\to c} f = L$. Show that there is a $\delta > 0$ such that f is bounded on the set $A \cap (c-\delta, c+\delta)$.*

Let $\epsilon = 1$. Then there is $\delta_1 > 0$ so that $|f(x)-L| < 1$ for all $x \in A \cap (c-\delta_1, c+\delta_1), x \ne c$. Hence $|f(x)| \le |L| + |f(x)-L| \le |L| + 1$ for these x.

If $c \notin A$, then we may take $M := |L| + 1$ and $\delta := \delta_1$. If $c \in A$ then we should take $M = \max(|L|+1, |f(c)|)$ and again $\delta := \delta_1$.

Problem 3: *Let $A = \{1/n : n \in \mathbb{N}\}$ and let $f : A \to \mathbb{R}$ be the function $f(x) = 1/(1+x)$.*

(i) *Write down the values of f at the points $x = 1, 1/2, 1/3$.*

$f(1) = 1/2, f(1/2) = 2/3, f(1/3) = 3/4$.

(ii) *Does $\lim_{x\to 0} f$ exist? If so evaluate it.*

This limit exists and equals 1. Proof. $|1/(1+x)-1| = |x/(1+x)|$. If $|x| \le 1/2$ then $1+x \ge 1/2$ so $0 < 1/(1+x) < 2$. Hence $|1/(1+x)-1| < 2|x| < \epsilon$ if we also assume that $|x| < \epsilon/2$. Therefore given $\epsilon > 0$ choose $\delta = \min(1, \epsilon/2)$.

(iii) *Does $\lim_{x\to 1/3} f$ exist? If so evaluate it.*

This limit does not exist because $1/3$ is NOT a cluster point of A. (Hence the limit is not defined,– and something that is not defined does not exist!)