Math 319 Solutions to Homework 7

Problem 1. *Suppose that* (x_n) *is a sequence with limit* 3*. Prove carefully from the definition that* $\lim \frac{1}{x_n} = \frac{1}{3}$.

Since $\lim x_n = 3$ there is K such that $x_n \in (2,4)$ for $n \geq K$. Then $1/|x_n| \leq 1/2$ for $n \geq K$. Hence when $n \geq K$

$$|\frac{1}{x_n} - \frac{1}{3}| = \frac{|3 - x_n|}{3|x_n|} \leq \frac{2|3 - x_n|}{3} < \epsilon,$$

if $|3 - x_n| < 3\epsilon/2$. Since $\lim x_n = 3$ there is K_1 so that $|3 - x_n| < 3\epsilon/2$ for all $n \geq K_1$. Hence if $K_2 = \max(K, K_1)$, $|\frac{1}{x_n} - \frac{1}{3}| < \epsilon$ for all $n \geq K_2$.

Problem 2. *Suppose that the sequences* (x_n) *and* (z_n) *both converge to* w *and that* $x_n \leq y_n \leq z_n$ *for all* n*.*
(i) *Use Theorem 3.1.10 to show that the sequence* $(y_n - x_n)$ *converges to* 0*.*

We know $0 \leq y_n - x_n \leq z_n - x_n$ for all n. Hence $|y_n - x_n| \leq |z_n - x_n|$ for all n. But $\lim z_n - x_n = w - w = 0$ by Theorem 3.2.3(a). Hence, applying Theorem 3.1.10 with $c = 1$, we find that $\lim(y_n - x_n) = 0$.
(ii) *Use (i) and the sum theorem (3.2.3(a)) to conclude that* (y_n) *converges to* w*.*

Applying Thm 3.2.3(a) to the sum $y_n = (y_n - x_n) + x_n$m, we see that $\lim y_n = 0 + w = w$.

Problem 3. *Find the limits, justifying your answer carefully.*
(i) $x_n = \frac{4n^2-n+1}{n^2-3n}$.

Assume that $n > 3$ so that $n^2 - 3n \neq 0$. Then

$$\lim \frac{4n^2 - n + 1}{n^2 - 3n} = \lim \frac{4 - 1/n + 1/n^2}{1 - 3/n} = \frac{4 - \lim 1/n + \lim 1/n^2}{\lim(1 - 3/n)} = 4.$$

Here we applied the sum and quotient rules in Thm 3.2.3, using the fact that $\lim(1-3/n) = 1 \neq 0$.

(ii) $(-1)^n \frac{n+1}{n^2+2}$.

$|x_n| = (n+1)/(n^2+2) \leq (n+1)/n^2 \leq 2n/n^2 = 2/n$. Now use the fact that $\lim 1/n = 0$ together with Thm 3.1.10 with $c = 2$ to conclude that $\lim x_n = 0$.

Problem 4. (i) *Show that the sequence* $x_n = (n^2 - n)/(n+1)$ *is monotonic.*

$x_n \leq x_{n+1}$ iff $(n^2 - n)/(n+1) \leq ((n+1)^2 - n - 1)/(n+2)$ which is true iff $(n+2)(n^2 - n) \leq (n^2 + n)(n+1)$. This holds iff $n^3 + n^2 - 2n \leq n^3 + 2n^2 + n$, which is TRUE for all n. Hence x_n is monotonic increasing.
(ii) *Define* x_n *inductively by the relation* $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$*. Assume that* $x_n \geq x_{n+1} > 0$ *for all* n*. Then* (x_n) *converges by Theorem 3.3.2. What is its limit?*

If the $\lim x_n = x$ then (because $\lim x_{n+1} = x$ also) x must satisfy the identity $x = \frac{1}{2}(x + 5/x)$. Simplifying we find $x^2 = 5$. Since x_n is monotonic increasaing and always > 0 by hypothesis, the limit is positive and hence $x = \sqrt{5}$.

Problem 5. *Consider the sequence defined inductively by:* $x_{n+1} = 4 - 3/x_n$.
(i) *Show that if* $1 < x_1 < 3$ *then this is monotonic increasing. What is the limit?*

$x_{n+1} \geq x_n$ iff $4 - 3/x_n \geq x_n$. We now want to multiply through by x_n and so need to know whether $x_n \geq 0$ or not.

So, let's do a preliminary calculation, showing by induction that $x_n > 1$ for all n. This is true for $n = 1$ by hypothesis. If it holds for x_k then $3/x_k < 3$; hence $x_{k+1} = 4 - 3/x_k > 1$. Hence it holds for x_{k+1}, and hence for all x_n by induction.

Therefore we find that $x_{n+1} \geq x_n$ iff $4x_n - 3 \geq x_n^2$ iff $0 \geq x_n^2 - 4x_n + 3 = (x_n - 3)(x_n - 1)$. We saw above that $x_n - 1 > 0$ for all n. Hence to finish we need to see that $x_n - 3 \leq 0$ for all n. Again this is true for $n = 1$ by hypothesis. Suppose it holds for $n = k$. Then $0 < x_k \leq 3$ so $3/x_k \geq 1$. Hence $x_{k+1} = 4 - 3/x_k \leq 4 - 1 = 3$. Therefore, by induction, $1 < x_n < 3$ for all n. Hence $(x_n - 3)(x_n - 1) < 0$ and so $x_{n+1} > x_n$ for all n. Therefore the sequence is monotonic increasing and the limit is $\sup\{x_n : n \geq 1\}$ and so is > 1.

But its limit x must satisfy the equation $x^2 - 4x + 3 = 0$ and hence must be 1 or 3. This $x = 3$.

(ii) *Show that if* $x_1 > 3$ *then this is monotonic decreasing. What is the limit?*

Let us first show by induction that $x_n > 3$ for all n. This is true for $n = 1$ by hypothesis. And if it holds for $n = k$ then it holds for $n = k+1$ since $3/x_k < 1$ so that $x_{k+1} = 4-3/x_k > 3$.

Since $x_k > 0$ always we can apply the previous reasoning to see that $x_{n+1} \leq x_n$ iff $0 \leq (x_n - 3)(x_n - 1)$. This holds since both factors are positive. The limit must then be 3.

(iii) *Is there a value of* x_1 *that gives a sequence with limit* 1*?*

If we take $x_1 = 1$ then we get $x_n = 1$ for all n and so the sequence does have limit 1. If $x_1 < 1$ the sequence will not have limit 1. I think what happens is thatr it decreases until $x_k \leq 0$ and then $x_{k+1} \geq 4$ and we are in case (ii). (1 is what is called an **unstable** fixed point: such things are discussed in MAT 351.) I think the proof of this fact would be a nice project – I don't want to get into it right now.