Math 319 Solutions to Homework 6

Problem 1. (i) *Let* $x_n = \dfrac{1}{4n-3}, n \geq 1$. *Find an integer* K *such that* $x_n < \frac{1}{35}$ *for all* $n \geq K$. *Explain your reasoning.*

$1/4n-3 < 1/35$ iff $35 < 4n-3$ iff $38/4 < n$. This holds for any $n \geq 10$. Therefore we may take $K = 10$ (or any integer ≥ 10.)

(ii) *Same as (i) with* $x_n = 1/4^n$.

Now we want $35 < 4^n$. But $4^3 = 64$ is already large enough. Therefore any $k \geq 3$ will do.

(You might have wanted to use the hint if 1/35 was replaced with w=something like 1/10000.)

Problem 2. *Use the definition of limit to prove that:* $\lim \dfrac{n}{2n-1} = \dfrac{1}{2}$.

$$|\frac{1}{2} - \frac{n}{2n-1}| = |\frac{2n-1-2n}{2(2n-1)}| = \frac{1}{4n-2} \leq \frac{1}{n},$$

where the last inequality holds because $n < 4n-2$ for all $n \geq 1$. Therefore given $\epsilon > 0$ choose $K > 1/\epsilon$. Then

$$|\frac{1}{2} - \frac{n}{2n-1}| \leq \frac{1}{n} \leq \frac{1}{K} < \epsilon, \quad \text{for all } n \geq K.$$

(ii) $\lim \dfrac{\sqrt{n}}{n+1} = 0$.

Note that

$$\frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \epsilon$$

if $1/\epsilon < \sqrt{n}$ or $1/\epsilon^2 < n$. Therefore, given $\epsilon > 0$ choose K so that $1/\epsilon^2 < K$. Then $|\frac{\sqrt{n}}{n+1} - 0| < \epsilon$ for all $n \geq K$.

Problem 3. *Prove Bernoulli's inequality: if* $x+1 > 0$ *then* $(x+1)^n \geq 1+nx$ *for all* $n \geq 1$.

Base case: if $n = 1$ this says $x+1 \geq x+1$, which is true (for all x.)

Now suppose the statement holds for $n = k$ and suppose that $n = k+1$. We must estimate $(x+1)^{k+1}$. But

$$(x+1)^{k+1} = (x+1)(x+1)^k \geq (x+1)(1+kx)$$

by the inductive hypothesis. (And here we need the factor $x+1$ to be nonnegative.) Also

$$(x+1)(1+kx) = 1+(k+1)x+kx^2 \geq 1+(k+1)x.$$

Hence the statement holds for $n = k+1$ and therefore holds for all n by induction.

Problem 4. *Show that the sequence* $n^2 - \sin n$ *has no limit.*

This sequence is not bounded above. For if there were an upper bound M then $n^2 - \sin n < M$ which implies $n < n^2 < M + \sin n < M + 1$ for all n. But the integers are not bounded above (By Archimedes).

Since convergent sequences are bounded by Thm 3.2.2, this sequence must diverge.

Problem 5. (i) *Suppose that* $\lim z_n = z$ *where* $z \neq 0$. *Show that there is* $K \in \mathbb{N}$ *such that* $z_n \neq 0$ *for all* $n \geq K$.

Choose $\epsilon = |z|/2$. Then the interval $V_\epsilon(z) = (z - \epsilon, z + \epsilon)$ does NOT contain 0. Since $\epsilon > 0$ there is K so that $z_n \in V_\epsilon(z)$ for all $n \geq K$. Hence $z_n \neq 0$ for all $n \geq K$.

(ii) *Suppose that* $\lim z_n = z$ *where* $z \neq 0$ *and* $z_n \neq 0$ *for all* n. *Show there is* $\delta > 0$ *such that* $|z_n| > \delta$ *for all* n.

Choose K as in (i). Then $|z_n| > |z|/2$ for all $n \geq K$. Hence take

$$\delta = \min\{|z_1|/2, \ldots, |z_{K-1}|/2, |z|/2.$$

Since δ is the minimum of a finite set of positive numbers, $\delta > 0$. To see that $|z_n| > \delta$ for all n we divide in two cases. If $n < K$, then $|z_n| > |z_n|/2 \geq \delta$ by definition of δ. While if $n \geq K$, we use the inequalities $|z_n| > |z|/2 \geq \delta$.