Math 319/320 Homework 3

Due Thursday, September 22, 2005
revised version

Problem 1. *Show that*

$$\left[\frac{1}{2}(a+b)\right]^2 \leq \frac{1}{2}(a^2+b^2)$$

for all $a, b \in \mathbb{R}$. *Show that equality holds if and only if* $a = b$.

We must show that $(a^2 + 2ab + b^2)/4 \leq (a^2 + b^2)/2$. Multiplying both sides by 4, we see this is equivalent to $a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$ and hence to $0 \leq a^2 + b^2 - 2ab$. But this last inequality holds since $a^2 + b^2 - 2ab = (a - b)^2$ and $x^2 \geq 0$ for all x. (We proved last on the last HW.) This shows that the inequality holds.

Also it shows that we have equality if and only if $0 = a^2 + b^2 - 2ab$, i.e. $(a-b)^2 = 0$. But this holds iff $a = b$. (Again this was proved in last HW.)

Problem 2. *Assume that* $a < x < b$ *and* $a < y < b$. *Show that* $|x - y| \leq b - a$. *Find a geometric explanation for the obtained inequality.*

First proof If $x = y$ the inequality is obvious since $b - a > 0$ by hypothesis. Now assume that $x < y$. We know that $x < b$. Also $a < y$ implies $-y < -a$. Adding we get

$$|x - y| = x - y = x + (-y) < b + (-a) = b - a.$$

Similarly, if $y < x$ we may reverse the roles of x and y to find: $|x - y| = y - x < b - a$. Hence in all cases $|x - y| \leq b - a$. (in fact we have $<$ here.)

Second proof (which I learnt as I was correcting your HW; it's essentially the same but slicker.)

By hypothesis $a < x < b$ and $a < y < b$. Multiply the second inequality by -1 to get $-b < -y < -a$. Then add this to the first inequality to get $a - b < x - y < b - a$. This has the form $-C < Z < C$ where $Z = x - y$ and $C = b - a > 0$. Hence it is equivalent to $|Z| < C$, i.e. $|x - y| < b - a$.

$|x - y|$ is the distance between x and y. So geometrically we are saying that the distance between any two points in the interval (a, b) is at most $b - a$.

Problem 3. *Let* $a, b \in \mathbb{R}$ *and* $a \neq b$. *Show that there exist* ϵ*-neighborhood* $U_\epsilon(a)$ *of* a *and* ϵ*-neighborhood* $V_\epsilon(b)$ *of* b *such that* $U_\epsilon(a) \cap V_\epsilon(b) \neq \emptyset$.

The problem here is to show you can choose ϵ large enough that these sets do intersect. So you must specify ϵ.

Proof 1 By renaming a, b we may suppose that $a < b$. Choose $\epsilon = 2(b - a)$, twice the distance between a and b. Then $b \in U_\epsilon(a) = (a - \epsilon, a + \epsilon)$. This is geometrically obvious, since $U_\epsilon(a)$ contains all points whose distance from a is $< \epsilon$ and b has distance $b - a < 2(b-a) = \epsilon$ from a. Since $b \in U_\epsilon(b)$ for any ϵ, b is in the intersection.

But to show it in formulas, note that

$$U_\epsilon(a) = (a - 2b + 2a, a + 2b - 2a) = (3a - 2b, 2b - a).$$

We need to see that $3a - 2b < b < 2b - a$. But $3a < 3b$ implies $3a - 2b < b$; while $b < 2b - a = b + (b - a)$ since $b - a > 0$.

As some of you noticed, any $\epsilon > |b - a|/2$ will do, since then the average $(a+b)/2$ will lie in the intersection. Here is a nice argument to show this:

Proof 2: As above, we may suppose that $a < b$. Note that $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$ and $V_\epsilon(b) = (b - \epsilon, b + \epsilon)$. Since $a < b$, $a - \epsilon < b_\epsilon$ and the only way to have an overlap of these intervals is for $a + \epsilon > b - \epsilon$. (You draw this.) i.e. we need $2\epsilon > b - a$ or $\epsilon > (b-a)/2$. If ϵ satisfies this inequality then the average $(a+b)/2$ is in both intervals.

Problem 4. *Let $S := \{x \in \mathbb{R} : x \geq 0\}$. Show that S has lower bounds, but no upper bounds. Show that* $\inf\ S = 0$.

Clearly 0 is a lower bound for S. Moreover if $y > 0$ then y is not a lower bound for S since y is *not* $\geq$ the element $0 \in S$. Hence every lower bound for S is ≤ 0. Hence 0 is the greatest lower bound, i.e. $0 = \inf S$.

To show S has no upper bounds: Proof 1

Since every $n \in \mathbb{N}$ is positive and so > 0, then $\mathbb{N} \subset S$. If y were an upper bound for S, we would have $y \geq n$ for all $n \in \mathbb{N}$, in contradiction to Archimedes' Principle. Hence S has no upper bounds.

Proof 2: Suppose that u is an upper bound for S. Then $u \in \mathbb{R}$ and $u \geq 0$, since $0 \in S$. Also $u + 1 > u$ (this is true for all real numbers.) Hence $u + 1 \in \mathbb{R}$ and $u + 1 > u \geq 0$. Hence $u + 1 \in S$. Hence $u \geq u + 1$. But this is impossible, by the trichotomy rule. (we cannot have both $u + 1 > u$ and $u \geq u + 1$.) Hence there is no upper bound.

Some of you combined the two arguments above, but it is simpler (and hence better) to use one OR the other.

Problem 5. *If $S \subset \mathbb{R}$ contains one of its upper bounds, then this upper bound is the supremum of S.*

Let y be an upper bound for S and suppose that $y \in S$. We must show that no $z < y$ is an upper bound for S. But if z is an upper bound for S, then $z \geq y$ since $y \in S$ and z is $\geq$ *every* element in S. Therefore z cannot also be $< y$. Hence y is the least upper bound for S, i.e. it is the supremum of S.

Note: this argument is almost the same as the proof in ex. 4 that $0 = \inf S$.