Math 319/320 Solutions to Homework 2

Problem 1. *Show that the set S_{odd} of odd (positive and negative) integers is denumberable by (a) enumerating them and (b) giving an explicit formula for the corresponding bijection $f : S_{odd} \to \mathbb{N}$.*

An emumeration: $S_{odd} = \{1, -1, 3, -3, 5, -5, \dots\}$

A bijection $f : S_{odd} \to \mathbb{N}$:

$$f(2k+1) = 2k+1, \;\; k \geq 0, \qquad f(-(2k+1)) = 2k+2, \;\; k \geq 0.$$

Problem 2. (i) *Show that for all $n \geq 1$ there is no injection of $\mathbb{N}_n$ onto a proper subset of $\mathbb{N}_n$.*

Let's prove this by induction. When $n = 1$ $\mathbb{N}_n = \{1\}$ has one element and there is only one map $f : \mathbb{N}_n \to \mathbb{N}_n$, namely the identity (which is bijective.)

Suppose this is true when $n = k$ and consider the case $n = k+1$. Let $f : \mathbb{N}_{k+1} \to \mathbb{N}_{k+1}$ be injective but not surjective. We show below that we can always use f to construct a new map $g : \mathbb{N}_k \to \mathbb{N}_k$ which is injective but not surjective. This contradicts the inductive hypothesis. Hence f cannot exist.

To define g;

case (i): $k+1 \notin f(\mathbb{N}_{k+1})$. Then $f(j) < k+1$ for all $j \leq k+1$. Define $g : \mathbb{N}_k \to \mathbb{N}_k$ by setting $g(j) = f(j), j \in \mathbb{N}_k$. This is injective because f is. I claim that the element $p := f(k+1)$ is not in $g(\mathbb{N}_k)$. To see this, note that because f is injective, there is only one element of $\mathbb{N}_{k+1}$, namely $k+1$ itself, that is mapped to p by f. Therefore $g^{-1}(p)$ is the empty set.

case (ii): suppose that $k+1 = f(j)$ for some $j \leq k+1$. Then because f is not surjective there is $p \leq k$ that is not in the image of f. Define a new map $F : \mathbb{N}_{k+1} \to \mathbb{N}_{k+1}$ by setting:

$$F(i) = f(i), i \in \mathbb{N}_{k+1}, i \neq j, \qquad F(j) = p.$$

Then F is injective but not surjective because $k+1 \notin F(\mathbb{N}_{k+1})$. Now construct g from F as in case (i).

(ii)*Deduce from (i) that if S is any finite set, there is no injection of S onto a proper subset of S.*

Suppose that $f : S \to S$ is an injection. Since S is finite there is a positive integer n and a bijection $g : \mathbb{N}_n \to S$. Then the composite $g^{-1} \circ f \circ g : \mathbb{N}_n \to \mathbb{N}_n$ is injective, since g, f and g^{-1} are. Therefore $g^{-1} \circ f \circ g$ is surjective by part (i). Therefore f is surjective.

(iii)*Show that (ii) does not hold for the infinite set $S = \mathbb{N}$.*

Define $f : \mathbb{N} \to \mathbb{N}$ by $f(k) = k+1)$ for all k.

Problem 3. (i) The subsets of $\{1,2\}$ are

$$\emptyset, \;\; \{1\}, \;\; \{2\}, \;\; \{1,2\}.$$

The subsets of $\{1,2,3\}$ are

$$\begin{array}{cccc} \emptyset, & \{1\}, & \{2\}, & \{1,2\}, \\ \{3\}, & \{1,3\}, & \{2,3\}, & \{1,2,3\}. \end{array}$$

(iii) *Prove that if a set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.*

If S has one element then it has two subsets, the empty set and the set S. Thus $\mathcal{P}(S)$ has 2 elements.

Now suppose the statement is true for $n = k$ and consider a set S with $k + 1$ elements. Suppse $x \in S$ and let $S_0 = S \setminus \{x\}$. Then S_0 has k elements. Every subset of S either lies in S_0 or contains x. By the inductive hypothesis there are 2^k subsets of S_0. Every subset T of S that contains x corresponds to a unique subset T_0 of S_0, namely $T_0 = T \setminus \{x\}$. Moreover, given a subset T_0 of S_0 the subset $T = S_0 \cup \{x\}$is a subset of S containing $\{x\}$. Therefore there is a bijective correspondence between the subsets of S_0 and the subsets of S that contain x. Hence there are 2^k subsets of this kind. So altogether $\mathcal{P}(s)$ has $2^k + 2^k = 2^{k+1}$ elements. Hence the statement holds for $n = k + 1$, and so holds for all n by induction.

Problem 4._Prove that $\sqrt{3}$ is irrational._

Suppose that $\sqrt{3}$ is rational are write it as p/q where p, q are positive integers with no common divisor. Then $3 = p^2/q^2$, that is $3q^2 = p^2$. Therefore 3 divides p^2. We claim that 3 must divide p. If not, we may write $p = 3k + r$ for some integer k and where $r = 1$ or 2. Then $p^2 = (3k+r)^2 = 9k^2+6k+r^2$. Since 3 divides p^2, 3 must divide $r^2 = p^2 - 3(3k^2+2k)$. But r^2 is either 1 or 4. So this is impossible. Hence 3 divides p. Therefore $p = 3k$ and the equation $3q^2 = p^2 = 9k^2$ gives $q^2 = 3k^2$. Therefore 3 divides q^2. Arguing as above, it follows that 3 divides q. But this contradicts our asumption that p and q have no common divisor. Hence the original assumption was wrong: $\sqrt{3}$ cannot be rational.

Problem 5. _Show that $a^2 + b^2 = 0$ iff $a = 0$ and $b = 0$._

If $a = 0 = b$ then $a^2 = 0 = b^2$, so $a^2 + b^2 = 0$.

Therefore we need to show the converse: if $a^2 + b^2 = 0$ then $a = 0 = b$. We will do this by proving the contrapositive. That is, if at least one of a, b is nonzero then $a^2 + b^2 \neq 0$. (in fact in this case we will see that $a^2 + b^2 > 0$.)

To do this, suppose first that $a \neq 0$. Then either $a > 0$ or $-a > 0$ by the trichotomy rule (Def 2.1.5(iii)). If $a > 0$ then $a^2 > 0$ by Def 2.1.5(ii). Similarly, if $-a > 0$ then $(-a)^2 > 0$. Since $(-a)^2 = a^2$, we find that for any $a \neq 0$ $a^2 > 0$. Also, $a^2 \geq 0$ for all $a \in \mathbb{R}$.

Now suppose that at least one of a, b is nonzero. By renaming them if necessary, we may suppose that $a \neq 0$. Then $a^2 > 0$ and $b^2 \geq 0$. Hence $a^2 + b^2 > 0$. (This follows from Def 2.1.5 (i) if $b^2 > 0$ and is obvious if $b^2 = 0$ since in this case $a^2 + b^2 = a^2 > 0$ by hypothesis.)

This completes the proof.

Bonus Problem 6. _Show that if S is a subset of $\mathbb{N}$ that is not contained in any of the sets $\mathbb{N}_n$ then S is denumerable._

We will define a bijection $\mathbb{N} \to S$ using the principle of induction. The statement $P(n)$ is: there is an injection $f : \mathbb{N}_n \to S$ so that

i) $f(n) \geq n$, and

ii) if $s \in S$ is not in the image $f(\mathbb{N}_n)$ then $s > f(j)$ for all $j \in \mathbb{N}_n$.

By the well ordering principle S has a smallest element, say s_1. Define $f(1) = s_1$. This map satisfies $P(1)$. Suppose by induction that f is defined on $\mathbb{N}_k$ and satisfies (i), (ii) above. Set $f(k+1)$ equal to the smallest element in the set $S \setminus f(\mathbb{N}_k)$. Then $f(k+1)$ is larger that all the elements $f(j), j \leq k$ by (ii). Hence $f(k+1) > f(k) \geq k$ by (i). Hence $f(k+1) \geq k+1$. So (i) holds for f on $\mathbb{N}_{k+1}$. Also because $f(k+1)$ is the smallest element in $S \setminus f(\mathbb{N}_k)$, every element in $S \setminus f(\mathbb{N}_{k+1})$ is larger than every element in $f(\mathbb{N}_{k+1})$. So (ii) holds for f on $\mathbb{N}_{k+1}$.

Therefore we may define $f : \mathbb{N}_n \to S$ satisfying (i) and (ii) for all n.

It follows from (ii) and the inductive construction that

$$f(1) < f(2) < f(3) < \dots$$

ie if $i < j$ then $f(i) < f(j)$. Hence f is injective. Suppose that f is not surjective and let $s \in S \subset \mathbb{N}$ be not in its image. Consider properties (i) and (ii) in the case $k = s$. By hypothesis s is not in the image $f(\mathbb{N}_s)$. Therefore by (ii) s is strictly larger than all elements in the image $f(\mathbb{N}_s)$. Hence $s > f(s)$. But this contradicts (i).