Math 319/320 Homework 2

Due Thursday, September 15, 2005

Problem 1. Show that the set S_{odd} of odd (positive and negative) integers is denumberable by

(a) enumerating them and

(b) giving an explicit formula for the corresponding bijection $f : S_{odd} \to \mathbb{N}$.

Hint: Imitate the following example. An **enumeration** of the set $\mathbb{Z}$ of integers is given by $\{0, 1, -1, 2, -2, 3 \ldots\}$. The corresponding explicit bijection $f : \mathbb{Z} \to \mathbb{N}$ is

$$f(0) = 1, \qquad f(k) = 2k, \text{ if } k > 0, \qquad f(k) = -2k + 1, \text{ if } k < 0.$$

Problem 2. Show that for all $n \geq 1$ there is no injection of $\mathbb{N}_n$ onto a *proper* subset of $\mathbb{N}_n$. In other words, any injection $f : \mathbb{N}_n \to \mathbb{N}_n$ is also surjective (and thus bijective.)
(ii) Deduce from (i) that if S is any finite set, there is no injection of S onto a *proper* subset of S.

(iii) Show that (ii) does not hold for the infinite set $S = \mathbb{N}$.

Hint for (i): You can prove this by induction. Or, you can deduce it from Thm B.1 in the textbook (which I did in class) by supposing that there is an injective but nonsurjective map $f : \mathbb{N}_n \to \mathbb{N}_n$ and looking at the composite $g \circ f$ for a suitable map $g : \mathbb{N}_n \to \mathbb{N}_{n-1}$. (Actually, these different approaches boil down to basically the same argument.)

Problem 3. Given a set S we write $\mathcal{P}(S)$ for the set of all its subsets. Note that $\emptyset, S$ are both subsets of S.
(i) List all the subsets of $S = \{1, 2\}$.
(ii) List all the subsets of $S = \{1, 2, 3\}$. (Try to see a relation with (i) – this should give you an idea for the inductive step below.)
(iii) Prove that, for all $n \geq 0$, if a finite set S has n elements then $\mathcal{P}(S)$ has 2^n elements. **Hint:** Use induction. You should have checked this above for $n = 2, 3$, but you need to do the base case $n = 0$ as well as the inductive step.

Problem 4. Prove that $\sqrt{3}$ is irrational.

Problem 5. Let $a, b \in \mathbb{R}$. Show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$. **Hint:** Use order properties in the proof.

Bonus Problem 6. Show that if S is a subset of $\mathbb{N}$ that is not contained in any of the sets $\mathbb{N}_n$ then S is denumerable.