Math 319 Homework 10

due Thursday December 8

Problem 1. Given $x \in \mathbb{R}$ let $[[x]]$ be the greatest integer $n \in \mathbb{Z}$ that is $\leq x$. Thus $[[2]] = 2$ and $[[-1/2]] = -1$. Graph the following functions and determine their points of continuity:

(a) $f(x) := [[x^2]]$ for $x \in \mathbb{R}$,

(b) $g(x) := \dfrac{[[x]]}{x}$ for $x > 0$,

(c) $h(x) := [[2\sin(x)]]$, for $x \in \mathbb{R}$.

Hint: Exercise 4 in Sec 5.1 is similar, and has an answer at the back of the book.

Problem 2. Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$ and that $f(b) = g(b)$. Now define $h : a, c \to \mathbb{R}$ by setting

$$h(x) = \{ \begin{array}{ll} f(x), & \text{if } x \in [a, b] \\ g(x), & \text{if } x \in (b, c]. \end{array}$$

(i) Draw the graphs of functions f, g that satisfy these conditions,

(ii) Prove from the definition that h is continuous at b. (It follows that h is continuous on $[a, c]$, but I am not asking you to prove this.)

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous everywhere.

(i) Suppose that $f(x) = 1$ for all rational numbers x. Show that $f(x) = 1$ for all x.

(ii) What can you say about the values of f if all you know is that $f(1/n) = 0$ for all $n \in \mathbb{N}$?

Problem 4. The function $h(x) := (x - 3)(x - 4)(x - 5)(x - 6)(x - 8)$ has five roots in the interval $[0, 9]$. Which root is found by the bisection method? Which root is found if you start with the interval $[2, 9]$. Explain your answers.

In the following two questions your answers may quote any theorems from Sections up to 5.2 but not those from Sec 5.3. Problem 5 asks you to prove part of Thm 5.3.4. (Don't do this by repeating the argument given in the book substituting $-f$ for f, but do the argument for f directly.) Problem 6 uses similar ideas.

Problem 5. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and you know that f is bounded below. Prove that there is $c \in [0, 1]$ such that $f(c) = \inf\{f(x); x \in [0, 1]\}$.

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $c \notin f([0, 1])$. Show that there is $\epsilon > 0$ such that $|f(x) - c| > \epsilon$ for all $x \in [0, 1]$.

Hint: Argue by contradiction.