Math 319: Definitions and Theorems for Midterm II

Def 3.1.3 A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms x_n satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called **divergent**.

Thm 3.1.10 Comparison theorem for limits. Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ca_n$ for all $n \geq m$, then $\lim x_n = x$.

Thm 3.2.2 A convergent sequence of real numbers is bounded.

Thm 3.2.3 (a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively. Then the sequences $X + Y, X - Y, X \cdot Y$ and cX converge to $x + y, x - y, xy$ and cx respectively.
(b) Moreover if $y \neq 0$ and $y_n \neq 0$ for any n then X/Y converges to x/y.

Thm 3.2.6 If $X = (x_n)$ is a convergent sequence and $a \leq x_n \leq b$ for all n then $a \leq \lim x_n \leq b$.

Thm 3.3.2 Monotone Convergence Theorem. A monotone sequence of real numbers is convergent if and only if it is bounded.

Def 3.4.1 Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \ldots)$ is called a **subsequence** of X.

Thm 3.4.2 If $X = (x_n)$ converges to $x \in \mathbb{R}$ then every subsequence X' of X also converges to x.

3.4.5 Divergence Criterion If a sequence X has two convergent subsequences with different limits, then X is not convergent. If X is unbounded, then it diverges.

3.4.7: Monotone subsequence theorem. Every sequence has a monotone subsequence.

3.4.8: Bolzano–Weierstrass theorem. A bounded sequence of real numbers has a convergent subsequence.

Def 4.1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a **cluster point** of A if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Def 4.1.4. Let $A \subset \mathbb{R}$ and let c be a cluster point of A. A function $f : A \to \mathbb{R}$ is said to have **limit** L **at** c if for all $\epsilon > 0$ there is $\delta > 0$ such that

$$0 < |x - c| < \delta, \ x \in A \Longrightarrow |f(x) - L| < \epsilon.$$

Thm 4.1.8. Sequential criterion Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then $\lim_{x \to c} f = L$ iff for every (x_n) in $A \setminus \{c\}$ that converges to c the sequence $(f(x_n))$ converges to L.