

Math 319 Final Exam, Dec 14 2004.

Name:

School ID:

Answer all the following questions, justifying all your statements. Each question is worth 15 points. There are six questions. Good luck!

1: Prove ONE of the following results:

EITHER: Let c be a cluster point of the set $\{x_n : n \geq 1\}$. Show that there is a subsequence of (x_n) that converges to c .

OR: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and suppose that x_n are points in \mathbb{R} such that $f(x_n) = n$. Prove that the sequence (x_n) is unbounded.

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Total	

2: Prove from the definition of limit and the results on the sheet that $\lim \frac{1}{2^n} = 0$.

3: Consider the function $f : [0, 2] \rightarrow \mathbb{R}$ given by $f(x) = \frac{2}{1+x}$. Prove from the definition that $\lim_{x \rightarrow 1} f(x) = 1$.

4: Describe examples satisfying the following conditions. Justify your answers.

(i) an infinite subset of \mathbb{R} that has no cluster points.

(ii) a bounded sequence of real numbers that does not converge.

(iii) a function $f : [-1, 1] \rightarrow \mathbb{R}$ that is not continuous at $x = 0$.

5: Let (x_n) be a monotonic decreasing sequence and set $B = \{x_n : n \geq 1\}$. Show that the point x_5 is not a cluster point of B .

6: Which of the following sequences are monotonic? Which are convergent?

$$(i) \quad x_n = \frac{n+1}{2n-1}; \quad (ii) \quad x_n = (-1)^n \frac{n+1}{2n-1}; \quad (iii) \quad x_n = \frac{n^2+1}{2n-1}.$$

Def 3.1.3 A sequence $X = (x_n)$ in \mathbb{R} is said to **converge** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms x_n satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called **divergent**.

Def 3.4.1 Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \dots < n_k < \dots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \dots)$ is called a **subsequence** of X .

Def 4.1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a **cluster point** of A if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Def 4.1.4. Let $A \subset \mathbb{R}$ and let c be a cluster point of A . A function $f : A \rightarrow \mathbb{R}$ is said to have **limit L at c** if for all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - c| < \delta$, $x \in A \implies |f(x) - L| < \epsilon$.

Def 5.1.1. Let $A \subset \mathbb{R}$, let $f : A \rightarrow \mathbb{R}$ and let $c \in A$. Then f is **continuous at c** if for every $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta$, $x \in A \implies |f(x) - f(c)| < \epsilon$. If B is a subset of A we say that f is **continuous on B** if it is continuous at all points $b \in B$.

Archimedes Principle: For all $x \in \mathbb{R}$ there is an integer $n > x$.

Bernoulli inequality: For all $x \geq 0$ and $n \geq 1$ $(1 + x)^n \geq 1 + nx$.

Thm 3.1.10 Comparison theorem for limits. Let (x_n) be a sequence in \mathbb{R} and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ca_n$ for all $n \geq m$, then $\lim x_n = x$.

Thm 3.2.2 A convergent sequence of real numbers is bounded.

Thm 3.3.2 Monotone Convergence Theorem. A monotone sequence of real numbers is convergent if and only if it is bounded.

Thm 3.4.2 If $X = (x_n)$ converges to $x \in \mathbb{R}$, every subsequence X' of X converges to x .

3.4.7: Monotone subsequence theorem. Every sequence has a monotone subsequence.

3.4.8: Bolzano–Weierstrass theorem. A bounded sequence of real numbers has a convergent subsequence.

Thm 4.1.8. Sequential criterion Let $f : A \rightarrow \mathbb{R}$ and c be a cluster point of A . Then $\lim_{x \rightarrow c} f = L$ iff for every (x_n) in $A \setminus \{c\}$ with limit c the sequence $(f(x_n))$ converges to L .

Thm 5.1.3. Sequential criterion for continuity: $f : A \rightarrow \mathbb{R}$ is continuous at $c \in A$ iff for every (x_n) in A that converges to c the sequence $(f(x_n))$ converges to $f(c)$.

Thm 5.3.2 Let $I = [a, b]$ be a closed bounded interval and $f : I \rightarrow \mathbb{R}$ be continuous on I . Then f is bounded on I , i.e. there is M such that $|f(x)| \leq M$ for all $x \in I$.

Thm 5.3.4 Let $I = [a, b]$ be a closed bounded interval and $f : I \rightarrow \mathbb{R}$ be continuous on I . Then f has an absolute maximum and an absolute minimum on I , i.e. there are points c, d in I such that $f(c) \leq f(x) \leq f(d)$ for all $x \in I$.