## Math 319 Final Exam, Dec 14 2004.

**Name:** **School ID:**

Answer all the following questions, justifying all your statements. Each question is worth 15 points. There are six questions. Good luck!

**1:** Prove ONE of the following results:

EITHER: Let $c$ be a cluster point of the set $\{x_n : n \geq 1\}$. Show that there is a subsequence of $(x_n)$ that converges to $c$.

OR: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $x_n$ are points in $\mathbb{R}$ such that $f(x_n) = n$. Prove that the sequence $(x_n)$ is unbounded.

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| Total | |

**2:** Prove from the definition of limit and the results on the sheet that $\lim \frac{1}{2^n} = 0$.

**3:** Consider the function $f : [0, 2] \to \mathbb{R}$ given by $f(x) = \frac{2}{1+x}$. Prove from the definition that $\lim_{x\to 1} f(x) = 1$.

**4:** Describe examples satisfying the following conditions. Justify your answers.

(i) an infinite subset of $\mathbb{R}$ that has no cluster points.

(ii) a bounded sequence of real numbers that does not converge.

(iii) a function $f : [-1, 1] \to \mathbb{R}$ that is not continuous at $x = 0$.

**5:** Let $(x_n)$ be a monotonic decreasing sequence and set $B = \{x_n : n \geq 1\}$. Show that the point $x_5$ is not a cluster point of $B$.

**6:** Which of the following sequences are monotonic? Which are convergent?

**(i)** $x_n = \dfrac{n+1}{2n-1}$; **(ii)** $x_n = (-1)^n \dfrac{n+1}{2n-1}$; **(iii)** $x_n = \dfrac{n^2+1}{2n-1}$.

**Def 3.1.3** A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms $x_n$ satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called **divergent**.

**Def 3.4.1** Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \dots < n_k < \dots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \dots)$ is called a **subsequence** of $X$.

**Def 4.1.1.** Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a **cluster point** of $A$ if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Let $A$ be a subset of $\mathbb{R}$. A point $c \in \mathbb{R}$ is called a **boundary point of** $A$ if *every* $\epsilon$-neighborhood of $c$ contains a point of $A$ and a point of its complement $\mathbb{R} \setminus A$. ($c$ does NOT have to be in $A$.) A point $c \in A$ is called an **interior point of** $A$ if there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $x$ is entirely contained in $A$.

The subset $A$ of $\mathbb{R}$ is said to be **open** if for every $x \in A$ there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $x$ is entirely contained in $A$. The subset $B$ of $\mathbb{R}$ is said to be **closed** iff its complement $\mathbb{R} \setminus B$ is open.

**Def 4.1.4.** Let $A \subset \mathbb{R}$ and let $c$ be a cluster point of $A$. A function $f : A \to \mathbb{R}$ is said to have **limit** $L$ **at** $c$ if for all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - c| < \delta,\ x \in A \Longrightarrow |f(x) - L| < \epsilon$.

**Def 5.1.1.** Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then $f$ is **continuous at** $c$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta,\ x \in A \Longrightarrow |f(x) - f(c)| < \epsilon$. If $B$ is a subset of $A$ we say that $f$ is **continous on** $B$ if it is continuous at all points $b \in B$.

**Archimedes Principle:** For all $x \in \mathbb{R}$ there is an integer $n > x$.

**Bernoulli inequality:** For all $x \geq 0$ and $n \geq 1$ $(1 + x)^n \geq 1 + nx$.

**Thm 3.1.10 Comparison theorem for limits.** Let $(x_n)$ be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If $(a_n)$ is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq C a_n$ for all $n \geq m$, then $\lim x_n = x$.

**Thm 3.2.2** A convergent sequence of real numbers is bounded.

**Thm 3.3.2 Monotone Convergence Theorem.** A monotone sequence of real numbers is convergent if and only if it is bounded.

**Thm 3.4.2** If $X = (x_n)$ converges to $x \in \mathbb{R}$, every subsequence $X'$ of $X$ converges to $x$.

**3.4.7: Monotone subsequence theorem.** Every sequence has a monotone subsequence.

**3.4.8: Bolzano–Weierstrass theorem.** A bounded sequence of real numbers has a convergent subsequence.

**Thm 4.1.8. Sequential criterion** Let $f : A \to \mathbb{R}$ and $c$ be a cluster point of $A$. Then $\lim_{x \to c} f = L$ iff for every $(x_n)$ in $A \setminus \{c\}$ with limit $c$ the sequence $(f(x_n))$ converges to $L$.

**Thm 5.1.3. Sequential criterion for continuity:** $f : A \to \mathbb{R}$ is continuous at $c \in A$ iff for every $(x_n)$ in $A$ that converges to $c$ the sequence $(f(x_n))$ converges to $f(c)$.

**Thm 5.3.2** Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ is bounded on $I$, i.e. there is $M$ such that $|f(x)| \leq M$ for all $x \in I$.

**Thm 5.3.4** Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ has an absolute maximum and an absolute minimum on $I$, i.e. there are points $c, d$ in $I$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in I$.