Math 532 - Fall 2019
Solutions to Second Examination

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1. Let $p > 0$ and let $f \in L^p(\mathbb{R}^n)$.

   (a) Prove that if $||f||_p \le 1$, $\varepsilon > 0$ and $E_\varepsilon(f) = \{x \in \mathbb{R}^n \ ; \ |f(x)| > \varepsilon\}$ then $m(E_\varepsilon(f)) \le \varepsilon^{-p}$.

   (b) Suppose now that and $\{f_n\} \subset L^p(\mathbb{R}^n)$ and $f_n \to f$ in $L^p(\mathbb{R}^n)$. Show that $f_n \to f$ in measure.

   Solution:

   (a) For any $g \in L^p(\mathbb{R}^n)$ we have

   $$m(E_\varepsilon(g)) = \int_{E_\varepsilon(g)} dm \le \frac{1}{\varepsilon^p} \int_{E_\varepsilon(g)} |g|^p dm \le \frac{||g||_p^p}{\varepsilon^p}.$$

   Since $||f||_p = 1$, taking $g = f$ yields the desired estimate.

   (b) According to the proof of part (a), for any $N > 0$ one has $m(\{|f_n - f| > \varepsilon\}) \le \varepsilon^{-p} ||f_n - f||_p^p$. Thus

   $$\lim_{n \to \infty} m(\{|f_n - f| > \varepsilon\}) = 0$$

   for every $\varepsilon > 0$, as desired.

2. Let μ and ν be positive measures on $(X, \mathscr{M})$ such that $\nu \ll \mu$, and write $\lambda = \mu + \nu$.

   (a) Show that $\quad 0 \le \dfrac{d\nu}{d\lambda} < 1 \quad \mu$-a.e.

   (b) Show that $\quad \dfrac{d\nu}{d\mu} = \dfrac{\frac{d\nu}{d\lambda}}{1 - \frac{d\nu}{d\lambda}}$.

   Solution:

   (a) If $\lambda(E) = 0$ then $0 \le \mu(E) + \nu(E) = \lambda(E) = 0$, so $\nu \ll \lambda$ and $\mu \ll \lambda$. By the Lebesgue-Radon-Nikodym Theorem there are non-negative functions $\frac{d\mu}{d\lambda}$ and $\frac{d\nu}{d\lambda}$ such that

   $$d\nu = \frac{d\nu}{d\lambda} d\lambda \quad \text{and} \quad d\mu = \frac{d\mu}{d\lambda} d\lambda.$$

   Moreover, any two such functions agree on the complement of a λ-null set. Hence

   $$d\lambda = d\mu + d\nu = \left(\frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda} \right) d\lambda,$$

and $\frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda} = 1$ λ-almost everywhere. This shows $0 \leq \frac{d\nu}{d\lambda} \leq 1$ λ-almost everywhere, and since $\mu \ll \lambda$, μ-almost everywhere. It remains to show that the set $F = \{\frac{d\nu}{d\lambda} = 1\}$ is μ-null. But we know that $\frac{d\mu}{d\lambda} = 0$ λ-almost everywhere on F, so

$$0 = \int_F \frac{d\mu}{d\lambda} d\lambda = \int_F d\mu = \mu(F).$$

(b) By part (a) and Lebesque-Radon-Nikodym we know that $d\mu = \frac{d\mu}{d\lambda} d\lambda$, $d\nu = \frac{d\nu}{d\mu} d\mu$ and $\frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda} = 1$. By uniqueness of Lebesque-Radon-Nikodym we must have $\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} = \frac{d\nu}{d\lambda}$, and therefore

$$\frac{d\nu}{d\mu} = \frac{\frac{d\nu}{d\lambda}}{\frac{d\mu}{d\lambda}} = \frac{\frac{d\nu}{d\lambda}}{1 - \frac{d\nu}{d\lambda}},$$

as required.

3. Show that if $f \in L^1_{\ell oc}(\mathbb{R}^n)$ and f is continuous at $x \in \mathbb{R}^n$ then x is in the Lebesgue set of f, i.e.,

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dm(y) = 0.$$

Solution: Let $\varepsilon > 0$. We are asked to show that there exists $\delta > 0$ such that

$$0 < r < \delta \Rightarrow \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dm(y) < \varepsilon.$$

To establish the latter statement, note that since f is continuous at x, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $y \in B_\delta(x) \Rightarrow |f(x) - f(y)| < \varepsilon$. We choose this δ. If $0 < r < \delta$ then $B_r(x) \subset B_\delta(x)$, and hence

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dm(y) < \frac{\varepsilon}{m(B_r(x))} \int_{B_r(x)} dm(y) = \varepsilon,$$

which is what we wanted to show.

4. Let $K_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $K_\alpha(x, y) = \frac{1}{(1+|x-y|)^\alpha}$ for all $x, y \in \mathbb{R}^n$, where $\alpha > 0$ is a real number. For which values of α is the operator $T_\alpha : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ defined by

$$T_\alpha f(x) := \int_{\mathbb{R}^n} K_\alpha(x, y) f(y) dm(y)$$

well-defined and bounded? Justify your answer.

Solution: First observe that

$$\begin{aligned} ||T_\alpha f||_1 &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{f(y)}{(1 + |x-y|)^\alpha} dm(y) \right| dm(x) \right) \\ &\leq \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(y)|}{(1 + |x-y|)^\alpha} dm(y) \right) dm(x) \right) \\ &= \left(\int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \frac{1}{(1 + |x-y|)^\alpha} dm(x) \right) dm(y) \right), \end{aligned}$$

where the last equality is by Fubini-Tonelli. Moreover, if f is a non-negative function then the inequality is an equality.

By translation-invariance of Lebesgue measure

$$C_\alpha := \int_{\mathbb{R}^n} \frac{1}{(1+|x-y|)^\alpha} dm(y)$$

is independent of x, and by using polar coordinates centered at $x \in \mathbb{R}^n$ one has

$$C_\alpha = \sigma_n \int_0^\infty \frac{r^{n-1}dr}{(1+r)^\alpha} \in (0, +\infty].$$

Since

$$\int_0^\infty \frac{r^{n-1}dr}{(1+r)^\alpha} = \int_0^1 \frac{r^{n-1}dr}{(1+r)^\alpha} + \int_1^\infty \frac{r^{n-1}dr}{(1+r)^\alpha} \leq 1 + \int_1^\infty r^{n-\alpha-1}dr$$

and

$$\int_0^\infty \frac{r^{n-1}dr}{(1+r)^\alpha} \geq \int_1^\infty \frac{r^{n-1}dr}{(1+r)^\alpha} = \int_1^\infty \frac{r^{n-1-\alpha}dr}{(1+r^{-1})^\alpha} \geq \frac{1}{2^\alpha} \int_1^\infty r^{n-1-\alpha}dr,$$

we see that C_α is finite if and only if $\alpha > n$. Therefore if $\alpha > n$ then

$$||T_\alpha f||_1 \leq C_\alpha ||f||_1.$$

On the other hand, if $0 < \alpha \leq n$ then $||T_\alpha f||_1 = +\infty$ for any nonegative $f \in L^1(\mathbb{R}^n)$. (For example, if f is the characteristic function of a set of finite Lebesgue measure.)