Math 532 - Fall 2019 Solutions to First Examination

1. Let X be a set. Give definitions of the following objects.

(a) An algebra on X is: a nonempty collection of subsets of X that is closed under finite unions and complements, i.e., $\mathscr{A} \subset \mathscr{P}(X)$ is an algebra if for all $A, A_1, ..., A_N \in \mathscr{A}$, $A^c \in \mathscr{A}$ and $\bigcup_{j=1}^N A_j \in \mathscr{A}$.

(b) A σ-algebra on X is: an algebra $\mathscr{M}$ that is closed under countable unions, i.e., if $A_1, A_2, ... \in \mathscr{M}$ then $\bigcup_{j=1}^\infty A_j \in \mathscr{M}$.

(c) A measurable space is: a set X together with a σ-algebra $\mathscr{M} \subset \mathscr{P}(X)$.

(d) A measure is: a function $\mu : \mathscr{M} \to [0, \infty]$, where $\mathscr{M}$ is a σ-algebra, such that $\mu(\varnothing) = 0$ and for any countable collection of pairwise-disjoint sets $\{A_1, A_2, ...\} \subset \mathscr{M}$

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j).$$

(e) An outer measure is: a function $\rho : \mathscr{P}(X) \to [0, \infty]$, where X is a set, such that

(i) $\rho(\varnothing) = 0$,

(ii) ρ is monotonic, i.e., if $A, B \in \mathscr{M}$ and $A \subset B$ then $\rho(A) \leq \rho(B)$, and

(iii) for any countable collection of sets $\{A_1, A_2, ...\} \subset \mathscr{M}$

$$\rho\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \rho(A_j).$$

(f) A Borel set is: an element of the σ-algebra generated by (a.k.a. the smallest σ-algebra that contains) all of the open sets in a topological space.

2. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right continuous function, i.e., $F(x) = F(x+)$ for all $x \in \mathbb{R}$, and let μ_F be the Lebesgue-Stieltjes measure associated to F, i.e., μ_F is the Borel measure defined on the algebra generated by left-half-open intervals by $\mu_F((a, b]) = F(b) - F(a)$.

(a) Compute $\mu_F([a, b])$, with proof.

For any sequence $\varepsilon_j \searrow 0$ we have that $[a, b] = \bigcup_{j \geq 1}(a - \varepsilon_j, b]$. Hence by continuity from above $\mu_F([a, b]) = \lim_{j\to\infty} \mu_F((a - \varepsilon_j, b]) = \lim\limits_{j\to\infty} F(b) - F(a - \varepsilon_j) = F(b) - F(a-)$.

(b) Give necessary and sufficient conditions ensuring that every singleton, i.e., a set with exactly 1 element, is a μ_F-null set.

Applying part (a) with $b = a$ yields, for any $a \in \mathbb{R}$, $\mu_F(\{a\}) = F(a) - F(a-)$. Thus every singleton is a null set if and only if F is left-continuous. Since F is increasing and right-continuous, μ_F annihilates all singletons if and only if F is continuous.

3. Let $g : \mathbb{R}^2 \to [0, \infty]$ satisfy

$$\int_{\mathbb{R}^2} g dm = 4,$$

where m is Lebesgue measure on $\mathbb{R}^2$. Let

$$f_n(x, y) = \pi - e^{-\left(r^2 - n(x+y) + \frac{n^2}{2}\right)}, \quad n \in \mathbb{N},$$

where $r := \sqrt{x^2 + y^2}$.

(a) Determine the function $f(x, y) = \lim_{n\to\infty} f_n(x, y)$.

Note first that $r^2 - n(x + y) + n^2/2 = (x - n/2)^2 + (y - n/2)^2$ is positive. For any point $(x, y) \in \mathbb{R}^2$, if $N > 0$ is given then with $n > 2N + 2\max(|x|, |y|)$ we have $|n/2 - x| \geq \frac{n}{2} - |x| > N$, and hence $|f_n(x, y) - \pi| \leq e^{-(x-n/2)^2-(y-n/2)^2} \leq e^{-2N^2}$, i.e., $\lim_{n\to\infty} |f_n(x, y) - \pi| = 0$, i.e., $f(x, y) = \pi$.

(b) Compute, with justification,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f_n(x, y)g(x, y)dm(x, y).$$

Note that $f_n(x, y) \leq \pi$, and therefore $|f_n g| \leq \pi g$. Since $g \in L^1(\mathbb{R}^2)$ the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f_n g dm = \int_{\mathbb{R}^2} f g dm = \pi \int_{\mathbb{R}^2} g dm = 4\pi.$$

(c) Compute, with justification,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |f_n(x, y)|dm(x, y).$$

We have $\int_{\mathbb{R}^2} |f_n|dm = \int_{\mathbb{R}^2} f_n dm \geq \int_{\mathbb{R}^2} (\pi - 1)dm = +\infty$.

4. (a) Show that for $c \geq 1$ and $0 < A \leq 1$, $\log(1 + A^c) \leq c \cdot A$. (You may use basic calculus.)

Consider the function $f(x) = cx - \log(1+x^c)$ Then $f'(x) = \frac{c(1+x^c-x^{c-1})}{1+x^c} \geq c\frac{1-x^{c-1}}{1+x^c} \geq 0$. Hence f is increasing and $f(0) = 0$, so $cx \geq \log(1 + x^c)$.

Alternatively, consider the function $f(x) = x - \log(1 + x)$. Then $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$,which shows that f is increasing. Thus $f(x) \geq f(0) = 0$ for $x \geq 0$. Hence $cA - \log(1 + A^c) \geq cA - A^c \geq cA - A \geq 0$.

There must be many other possible proofs.

(b) Let $(X, \mathscr{M})$ be a measurable space and let $\mu : \mathscr{M} \to [0, \infty]$ be a positive measure. Suppose $f : X \to [0, \infty]$ satisfies $\int_X f d\mu = \pi$. Show that the function $F : (0, \infty) \to [0, \infty]$ defined by

$$F(c) := \lim_{n\to\infty} \int_X n \log\left(1 + \left(\tfrac{f}{n}\right)^c\right) d\mu$$

is a simple function.

Hint: For $c \geq 1$ Part (a) should help with dominating the integrand. For $c \in (0, 1)$, try Fatou.

If $c < 1$ then by Fatou's Lemma

$$F(c) \geq \int_X \liminf_{n\to\infty} n \log\left(1 + \left(\frac{f}{n}\right)^c\right) d\mu.$$

But by L'Hôpital's Rule if $x > 0$ then

$$\lim_{n\to\infty} n \log(1 + (x/n)^c) = \lim_{\varepsilon\to 0} \frac{\log(1 + (\varepsilon x)^c)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{cx(\varepsilon x)^{c-1}}{1 + (\varepsilon x)^c} = +\infty. \tag{1}$$

Since f is non-negative and has a positive integral, it cannot vanish on a set of full measure, and therefore we see that $F(c) = +\infty$ for $0 < c < 1$.

If $c \geq 1$ then $n \log(1+(x/n)^c) \leq cf$ by part (a), and thus $n \log(1+(f/n)^c) \leq cf \in L^1(\mu)$. By the dominated convergence theorem

$$\lim_{n\to\infty} \int_X n \log\left(1 + \left(\frac{f}{n}\right)^c\right) d\mu = \int_X \lim_{n\to\infty} n \log\left(1 + \left(\frac{f}{n}\right)^c\right) d\mu.$$

When $c > 1$ this limit is 0 (follow (1) with $c \geq 1$), and when $c = 1$ the limit of the integrand is f, and therefore $F(1) = \pi$. Thus

$$F = (+\infty) \cdot \chi_{(0,1)} + \pi\chi_{\{1\}}.$$