Homework 8 solutions

§13.9

14. The problem can be translated into the following form: find the maximum of xyz under the constraint $x^2+y^2+z^2=r^2$, $x,y,z>0$. After converting $z=\sqrt{r^2-x^2-y^2}$, our problem is to find the maximum of $f(x,y)=xy\sqrt{r^2-x^2-y^2}$ for $x,y>0$.

Now we find critical points of f. One can compute the gradient of f, and it is the following.

$$\nabla f=\left\langle \frac{y}{\sqrt{r^2-x^2-y^2}}(r^2-2x^2-y^2), \frac{x}{\sqrt{r^2-x^2-y^2}}(r^2-x^2-2y^2)\right\rangle.$$

Thus, the critical points satisfy the equation $r^2-2x^2-y^2=r^2-x^2-2y^2=0$. Solving it, we get a single critical point $(x,y)=(\frac{1}{\sqrt{3}}r,\frac{1}{\sqrt{3}}r)$. To use second partials test, we need to compute all the second partial derivatives:

$$f_{xx}=\frac{xy(-3r^2+2x^2+3y^2)}{(r^2-x^2-y^2)^{\frac{3}{2}}},\quad f_{yy}=\frac{xy(-3r^2+3x^2+2y^2)}{(r^2-x^2-y^2)^{\frac{3}{2}}},$$
$$f_{xy}=\frac{r^4-3r^2(x^2+y^2)+(2x^4+3x^2y^2+2y^4)}{(r^2-x^2-y^2)^{\frac{3}{2}}}.$$

At the critical point $(x,y)=(\frac{1}{\sqrt{3}}r,\frac{1}{\sqrt{3}}r)$, we have $f_{xx}=f_{yy}=-\frac{4}{\sqrt{3}}r$ and $f_{xy}=-\frac{2}{\sqrt{3}}r$. Hence $d=f_{xx}f_{yy}-(f_{xy})^2=4r^2>0$ and $f_{xx}=-\frac{4}{\sqrt{3}}r<0$. Using the second partials test, we conclude the critical point point $x=y=z=\frac{1}{\sqrt{3}}r$ is a relative maximum point.

18. The question is to find the maximum of $H=-x\ln x-y\ln y-z\ln z$ under the constraint $x+y+z=1$. From the constraint, we can eliminate the variable z as $z=1-x-y$, and hence we have a two-variable function $H(x,y)=-x\ln x-y\ln y-(1-x-y)\ln(1-x-y)$.

The gradient of H is

$$\nabla H=\langle -\ln x+\ln(1-x-y), -\ln y+\ln(1-x-y)\rangle.$$

From it, the critical points of H are the solutions of the equation $-\ln x+\ln(1-x-y)=-\ln y+\ln(1-x-y)=0$. Solving the equation yields a single critical point $(x,y)=\left(\frac{1}{3},\frac{1}{3}\right)$. To use the second partials test, we need all the second partial derivatives

$$H_{xx}=-\frac{1}{x}-\frac{1}{1-x-y},\quad H_{xy}=-\frac{1}{1-x-y},\quad H_{yy}=-\frac{1}{y}-\frac{1}{1-x-y}.$$

Thus, at the critical point $x=y=\frac{1}{3}$, we have $d=H_{xx}H_{yy}-(H_{xy})^2=(-6)\times(-6)-(-3)^2=27>0$ and $H_{xy}=-6<0$. This tells us $x=y=z=\frac{1}{3}$ is a relative maximum point. At this point, it achieves the maximum value $H=-\frac{1}{3}\ln\left(\frac{1}{3}\right)\times 3=\ln 3$.

§13.10

10. Define a function $g(x,y) = 2x + 4y$. We want to compute the minimum of $f(x,y) = \sqrt{x^2+y^2}$ under the constraint $g(x,y) = 15$. Apply the Lagrange multiplier method. To do so, we need to solve the equations $\nabla f = \lambda \nabla g$ and $g = 15$. Computing ∇f and ∇g, this is

$$\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \lambda \langle 2, 4 \rangle, \quad 2x + 4y = 15.$$

Manipulating the equations, we have $y = 2x = 4\lambda\sqrt{x^2+y^2}$, whence $x = \frac{3}{2}, y = 3$. Thus, the minimum value is $f(\frac{3}{2}, 3) = \frac{3}{2}\sqrt{5}$.

28. The distance between two points $(4,0,0)$ and (x,y,z) is measured by $\sqrt{(x-4)^2+y^2+z^2}$. Thus, we need to minimize the function $f(x,y,z) = (x-4)^2+y^2+z^2$ under the constraint $g(x,y,z) = \sqrt{x^2+y^2} - z = 0$. Apply the Lagrange multiplier method. The gradients are $\nabla f = \langle 2(x-4), 2y, 2z \rangle$ and $\nabla g = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$. For simplicity, we exclude the possibility $x = y = z = 0$ to use the fraction $\frac{1}{\sqrt{x^2+y^2}}$ freely. Lagrange multiplier method requires us to solve the equations

$$\langle 2(x-4), 2y, 2z \rangle = \lambda \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle, \quad z = \sqrt{x^2+y^2}.$$

Manipulating the equalities, we get

$$2(x-4) = \frac{\lambda}{\sqrt{x^2+y^2}} x, \quad 2y = \frac{\lambda}{\sqrt{x^2+y^2}} y, \quad \frac{\lambda}{\sqrt{x^2+y^2}} = -2.$$

From the first and third equality, we get $x = 2$. From the second and third equality, we get $y = 0$. Hence $z = \sqrt{2^2+0^2} = 2$. That is, the minimum value of the function f is $f(2,0,2) = 8$. Therefore, the minimum distance is $2\sqrt{2}$.

46. Let $g(x,y,z) = x^2+y^2+z^2$ and $h(x,y,z) = x - z$. Use the Lagrange multiplier method. We need to solve the equation $\nabla T = \lambda \nabla g + \mu \nabla h$, $g = 50$ and $h = 0$. Computing the gradients, these are

$$\langle 2x, 2y, 0 \rangle = \langle 2\lambda x + \mu, 2\lambda y, 2\lambda z - \mu \rangle, \quad x^2+y^2+z^2 = 50, \quad x = z.$$

One can solve these and get the solutions $(x,y,z) = (0, \pm 5\sqrt{2}, 0), (\pm 5, 0, \pm 5)$. Now $T(0, \pm 5\sqrt{2}, 0) = 150$ and $T(\pm 5, 0, \pm 5) = 125$. This means the maximum temperature is 150.

48. We want to compute the minimum perimeter $f(l,h) = \left(\frac{\pi}{2}+1\right) l + 2h$ under the fixed area constraint $g(l,h) = \frac{\pi}{8} l^2 + lh = A$. Using Lagrange multiplier method, we need to solve the equation

$$\lambda \left\langle \frac{\pi}{2} + 1, 2 \right\rangle = \left\langle \frac{\pi}{4} l + h, l \right\rangle, \quad \frac{\pi}{8} l^2 + lh = A.$$

Solving the equations leads $l = 2\sqrt{\frac{A}{\frac{\pi}{2}+2}}$ and $h = \sqrt{\frac{A}{\frac{\pi}{2}+2}}$. These values give us the minimum perimeter $2\sqrt{\left(\frac{\pi}{2}+2\right) A}$. Note that we had $l = 2h$.

§14.1

12. We have the sequence of identities

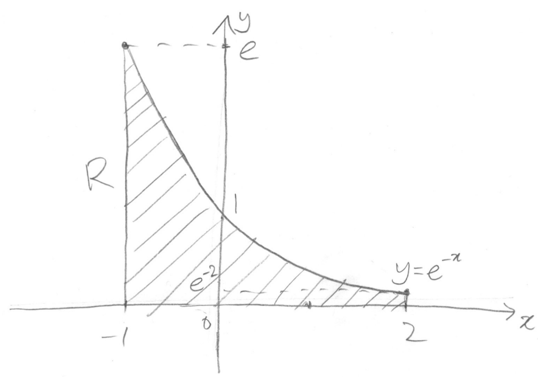
$$\int_{-1}^{1}\int_{-2}^{2}(x^2-y^2)dydx = \int_{-1}^{1}\left(x^2y-\frac{y^3}{3}\Big|_{-2}^{2}\right)dx$$
$$= \int_{-1}^{1}\left(4x^2-\frac{16}{3}\right)dx = \frac{4}{3}x^3-\frac{16}{3}x\Big|_{-1}^{1} = -8.$$

28. We have the sequence of identities

$$\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} 3r^2\sin\theta drd\theta = \int_0^{\frac{\pi}{4}}\left(r^3\sin\theta\Big|_0^{\cos\theta}\right)d\theta$$
$$= \int_0^{\frac{\pi}{4}}\sin\theta\cos^3\theta d\theta = -\int_1^{\frac{1}{\sqrt{2}}}u^3du = \frac{3}{16}.$$

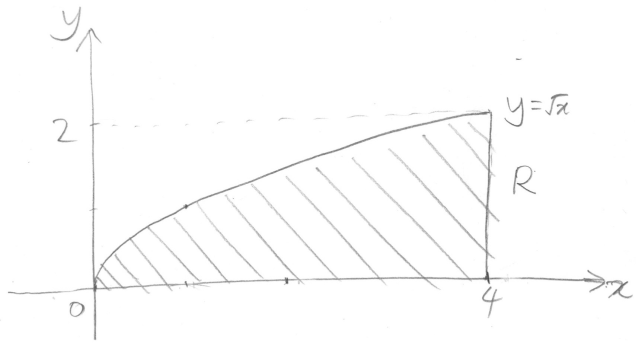
In the last part, we have used the substitution $u=\cos\theta$.

48. From the picture, we can change the order of the integration in the following way.



$$\int_{-1}^{2}\int_0^{e^{-x}} f(x,y)dydx = \iint_R f(x,y)dA = \int_0^{e^{-2}}\int_{-1}^{2} f(x,y)dxdy + \int_{e^{-2}}^{e}\int_{-1}^{-\ln y} f(x,y)dxdy.$$

66. We have the sequence of identities



$$\int_0^2 \int_{y^2}^4 \sqrt{x}\sin x dx dy = \iint_R \sqrt{x}\sin x dA = \int_0^4 \int_0^{\sqrt{x}} \sqrt{x}\sin x dy dx$$
$$= \int_0^4 x\sin x dx = -x\cos x + \sin x\Big|_0^4 = -4\cos 4 + \sin 4.$$

The first line is the change of order of integrations, and the second line is further computations of the order-changed integration.

§14.2

6. Dividing the rectangle $[0,4]\times[0,2]$ by eight 1×1 squares, we have eight centers of squares

$$p_1 = \left(\frac{1}{2}, \frac{1}{2}\right),\ p_2 = \left(\frac{1}{2}, \frac{3}{2}\right),\ \cdots,\ p_7 = \left(\frac{7}{2}, \frac{1}{2}\right),\ p_8 = \left(\frac{7}{2}, \frac{3}{2}\right).$$

Hence the approximation is

$$\sum_{i=1}^{8} f(p_i)\Delta A_i = \frac{1}{\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+1\right)} \times 1 + \frac{1}{\left(\frac{1}{2}+1\right)\left(\frac{3}{2}+1\right)} \times 1 + \cdots + \frac{1}{\left(\frac{7}{2}+1\right)\left(\frac{3}{2}+1\right)} \times 1$$
$$= \frac{7936}{4725} \approx 1.68.$$

18. Draw the picture and convince yourself that we have two different ways to compute the given double integration:

$$\iint_R \frac{y}{1+x^2} dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dy dx$$
$$= \int_0^2 \int_{y^2}^4 \frac{y}{1+x^2} dx dy.$$

We will use the first one. Computing it further, we have

$$\int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dydx = \int_0^4 \left(\frac{y^2}{2(1+x^2)} \Bigg|_0^{\sqrt{x}} \right) dx = \int_0^4 \frac{x}{2(1+x^2)} dx$$
$$= \frac{1}{4} \int_0^{16} \frac{1}{1+u} du = \frac{1}{4} \ln(1+u) \Bigg|_0^{16} = \frac{1}{4} \ln 17.$$

During the computation, we have used the substitution $u = x^2$.

36. The intersection locus of the given two graphs $z = x^2 + y^2$ and $z = 2x$ can be obtained by solving the equation $z = x^2 + y^2 = 2x$. Solving it, the (x, y)-coordinate of the intersection locus is given by $(x-1)^2 + y^2 = 1$, a circle of center $(1, 0)$ and radius 1. The volume between the two graphs can be computed by integrating the difference of z-values of the two graphs, along the region $(x-1)^2 + y^2 \le 1$. Writing R as the region

$$R = \{(x, y) : (x-1)^2 + y^2 \le 1\},$$

the desired volume can be measured by the double integration $\iint_R (2x - (x^2 + y^2)) dA$.

58. Let $R = [0, 2] \times [0, 4]$ be the region on which we are measuring the temperature. The average temperature on the region is

$$\frac{1}{\text{area}(R)} \iint_R T dA = \frac{1}{8} \int_0^2 \int_0^4 (20 - 4x^2 - y^2) dydx$$
$$= \frac{1}{8} \int_0^2 \left((20 - 4x^2)y - \frac{y^3}{3} \Bigg|_0^4 \right) dx = \int_0^2 \left(-2x^2 + \frac{22}{3} \right) dx = -\frac{2}{3}x^3 + \frac{22}{3}x \Bigg|_0^2 = \frac{28}{3}.$$