Homework 4. Solutions.

12.4 Problems 14, 18, 28, 36, 70

14. First, we have

$$r'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$$

and

$$||r'(t)|| = \sqrt{4 + 64\sin^2 t\cos^2 t}$$

At the point $(1, \sqrt{3}, 1)$, $t = \frac{\pi}{6}$, so

$$r'(\frac{\pi}{6}) = \langle \sqrt{3}, -1, 2\sqrt{3}\rangle$$

$$||r'(\frac{\pi}{6})|| = 4$$

and thus

$$T(\frac{\pi}{6}) = \langle \frac{\sqrt{3}}{4}, \frac{-1}{4}, \frac{\sqrt{3}}{2}\rangle$$

The parametric equations for the tangent line at this point are

$$x(t) = \frac{\sqrt{3}}{4}t + 1$$
$$y(t) = \frac{-1}{4}t + \sqrt{3}$$
$$z(t) = \frac{\sqrt{3}}{2}t + 1$$

18. Recall that the normal vector is given by $N(t) = \dfrac{T'(t)}{||T'(t)||}$ where $T(t) = \dfrac{r'(t)}{||r'(t)||}$. We have

$$r'(t) = \langle \sqrt{2}, e^t, -e^{-t}\rangle$$
$$||r'(t)|| = e^t + e^{-t}$$

where we have used that $(2 + e^{2t} + e^{-2t}) = (e^t + e^{-t})^2$. Then,

$$T(t) = \langle \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^t}{e^t + e^{-t}}, \frac{-e^{-t}}{e^t + e^{-t}}\rangle$$
$$T'(t) = \langle \frac{-\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}, \frac{-e^t(e^t - e^{-t})}{(e^t + e^{-t})^2} + \frac{e^t}{(e^t + e^{-t})}, \frac{e^{-t}(e^t - e^{-t})}{(e^t + e^{-t})^2} + \frac{e^{-t}}{(e^t + e^{-t})}\rangle$$
$$= \langle \frac{-\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2}\rangle$$
$$||T'(t)|| = \frac{\sqrt{2}}{e^t + e^{-t}}$$

Hence

$$N(t) = \langle \frac{-(e^t - e^{-t})}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}\rangle$$

28. We use the fact that $a_T = \dfrac{v \cdot a}{||v||}$ and $a_N = \sqrt{||a||^2 - a_T^2}$ to compute the two components of acceleration. Since $v(t) = \langle e^t, -e^{-t}\rangle$, $a(t) = \langle e^t, e^{-t}\rangle$, and $||v(t)|| = \sqrt{e^{2t} + e^{-2t}} = ||a(t)||$ we have

$$a_T = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}},\ a_T(0) = 0$$

$$a_N = \frac{2}{\sqrt{e^{2t} + e^{-2t}}},\ a_N(0) = \sqrt{2}$$

36. Since $||r'(t)|| = \sqrt{5}$, we have $a_T = ||r'(t)||' = 0$ and thus $a_N = \sqrt{||a||^2 - a_T^2} = ||a|| = 1$

70. True.

12.5 Problems 16, 18, 22, 36, 62

16. The length of $r(t)$ between $t = 0$ to $t = \frac{\pi}{2}$ is given by

$$L = \int_0^{\frac{\pi}{2}} ||r'(t)||dt = \int_0^{\frac{\pi}{2}} \sqrt{((\cos t + t\sin t)')^2 + ((\sin t - t\cos t)')^2 + ((t^2)')^2}dt = \int_0^{\frac{\pi}{2}} \sqrt{5t^2}dt = \frac{5\pi^2}{8}$$

18. a) The arclength parameter s is computed

$$s = \int_0^t ||r'(u)||du = \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2 + 1}du = \int_0^t \sqrt{5}du = \sqrt{5}t$$

b) From a) we get that $t = \dfrac{s}{\sqrt{5}}$, thus

$$r(s) = \langle 2\cos(\frac{s}{\sqrt{5}}), 2\sin(\frac{s}{\sqrt{5}}), \frac{s}{\sqrt{5}}\rangle$$

c) When $t = 1$, notice that $s = \sqrt{5} \cdot 1$, hence $r(\sqrt{5}) = \langle 2\cos(1), 2\sin(1), 1\rangle$. Similarly, when $s = 4$ we have $r(4) = \langle 2\cos(\frac{4}{\sqrt{5}}), 2\sin(\frac{4}{\sqrt{5}}), \frac{4}{\sqrt{5}}\rangle$

d) To verify $||r'(s)|| = 1$ we directly compute the norm with respect to the arclength parameter s. This gives

$$r'(s) = \langle -2\sin(\frac{s}{\sqrt{5}})\frac{1}{\sqrt{5}}, 2\cos(\frac{s}{\sqrt{5}})\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$$

hence $||r'(s)|| = \sqrt{\dfrac{4}{4} + \dfrac{1}{5}} = 1$

22. Since s is an arclength parameter, the curvature $K = ||r''(s)||$. We have $r''(s) = \langle -\cos s, -\sin s, 0\rangle$, so $K = 1$.

36. Using the equations derived in the textbook, we see that one way to obtain the curvature K is by computing

$$K = \frac{a_N}{\left(\frac{ds}{dt}\right)^2} = \frac{\sqrt{||a||^2 - a_T^2}}{\left(\frac{ds}{dt}\right)^2} = \frac{\sqrt{||a||^2 - a_T^2}}{\left(\frac{ds}{dt}\right)^2} = \frac{\sqrt{||a||^2 - \left(\frac{d^2s}{dt^2}\right)}}{\left(\frac{ds}{dt}\right)^2}$$

where s is the arclength parameter. Observe that $\frac{ds}{dt} = ||r'(t)||$, so since

$$\begin{aligned} r'(t) &= \langle 2e^{2t}, -e^{2t}\sin t + 2e^{2t}\cos t, e^{2t}\cos t + 2e^{2t}\sin t\rangle \\ ||r'(t)||^2 &= 4e^{4t} + e^{4t}\left((2\cos t - \sin t)^2 + (2\sin t + \cos t)^2\right) \\ &= 9e^{4t} \end{aligned}$$

From here we also get that

$$\left(\frac{d^2s}{dt^2}\right)^2 = 36e^{4t}$$

since $\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}\left(3e^{2t}\right)$
Finally,

$$\begin{aligned} a(t) &= \langle 4e^{2t}, -4e^{2t}\sin t + 3e^{2t}\cos t, 4e^{2t}\cos t + 3e^{2t}\sin t\rangle \\ ||a(t)||^2 &= 41e^{4t} \end{aligned}$$

hence

$$\begin{aligned} K &= \frac{\sqrt{41e^{4t} - 36e^{4t}}}{9e^{4t}} \\ &= \frac{\sqrt{5}e^{2t}}{9e^{4t}} \\ &= \frac{\sqrt{5}}{9e^{2t}} \end{aligned}$$

62. a) We compute the length as follows

$$L = \int_0^2 ||r'(t)||dt = \int_0^2 \sqrt{1+4t^2}dt$$

after the substitution $u = 2t$, *we have*

$$= \frac{1}{2}\int_0^4 \sqrt{1+u^2}du$$

Now we use trig. substitution with $u = \tan\theta$ *to obtain*

$$= \frac{1}{2}\int_0^{\arctan(4)} \sec^3(\theta)d\theta$$

This integral can be solved using integration by parts with $u = \sec\theta, v = \int sec^2\theta d\theta$, *so we get*

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big|_0^{\arctan(4)}\right)$$

Using right triangles we can compute that $\sec(\arctan(4)) = \sqrt{17}$, *thus*

$$= \frac{1}{4}\left(4\sqrt{17} + \ln|\sqrt{17}+4|\right)$$

b) The curvature is given by $K(t) = \dfrac{||T'(t)||}{||r'(t)||}$. We have

$$\begin{aligned} r'(t) &= \langle 1, 2t\rangle \\ ||r'(t)|| &= \sqrt{1+4t^2} \\ T(t) &= \langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\rangle \\ T'(t) &= \langle \frac{2}{(1+4t^2)^{\frac{3}{2}}}, \frac{-4t}{(1+4t^2)^{\frac{3}{2}}}\rangle \\ ||T'(t)|| &= \frac{2}{1+4t^2} \end{aligned}$$

so $K(t) = \dfrac{2}{\sqrt{1+4t^2}}$

c) The curvature decreases from $t = 0$ to $t = 2$.

13.1 Problems 12, 18, 28, 47-50, 78

12.

a) $g(1, 0) = \ln(1) = 0$

b) $g(0, -t^2) = \ln|-t^2| = 2\ln t$

c) $g(e, 0) = \ln(e) = 1$

d) $g(e, e) = \ln(2e) = \ln(2) + 1$

18. We have

$$g(x,y) = \int_x^y \frac{1}{t}dt = \ln|t|\Big|_x^y$$
$$= \ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|$$

Thus

a) $g(4,1) = \ln\left(\frac{1}{4}\right)$

b) $g(6,3) = \ln\left(\frac{1}{2}\right)$

c) $g(2,5) = \ln\left(\frac{5}{2}\right)$

d) $g(\frac{1}{2},7) = \ln 14$

28. The domain is all numbers x, y satisfying $6x^2 - y^2 \leq 9$ and the range is all nonnegative real numbers $z \geq 0$.

47.C
48.D
49.B
50.A

78.

a) $W(15,9) = \frac{1}{6}$

b) $W(15,3) = \frac{1}{2}$

a) $W(12,7) = \frac{1}{5}$

a) $W(5,2) = \frac{1}{3}$