Homework 10 Solutions

§14.6

10. The triple integral is

$$\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y}\sin y\,dzdxdy = \int_0^{\pi/2}\int_0^{y/2}\frac{1}{y}\sin y\,dxdy = \frac{1}{2}\int_0^{\pi/2}\sin y\,dy$$
$$= \frac{1}{2}(-\cos y)\Big|_0^{\pi/2} = \frac{1}{2}$$

12. The value can be found to be approximately

$$\int_0^3\int_0^{2-2y/3}\int_0^{6-2y-3z} ze^{-x^2y^2}dxdzdy \approx 2.118$$

18. We want to set up a triple integral for the volume of the solid bounded by $z = 4 - x^2$ and $z = x^2 + 3y^2$. To find the bound for integral, we look at the intersection of the two surfaces.

$$4 - x^2 = x^2 + 3y^2$$
$$2x^2 + 3y^2 = 4$$

So $x = \pm\sqrt{2 - \frac{3}{2}y^2}$. We can write in integral as

$$\int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}}\int_{-\sqrt{2-\frac{3}{2}y^2}}^{\sqrt{2-\frac{3}{2}y^2}}\int_{x^2+3y^2}^{4-x^2} dzdxdy$$

20. The volume of the solid bounded by the graph of the surface $z = 2xy$ is

$$\int_0^2\int_0^2\int_0^{2xy} dzdxdy = \int_0^2\int_0^2 2xydxdy = 2\int_0^2 xdx\int_0^2 ydy = 4\cdot\frac{x^2}{2}\Big|_0^2 = 8$$

64. Let Q be the cube in the first octant bounded the coordinate planes and $x = 4$, $y = 4$ and $z = 4$. Then the average value of $f(x, y, z = xyz$ over Q is given by

$$\text{Average value} = \frac{1}{V}\iiint_Q f(x,y,z)dV = \frac{1}{64}\int_0^4\int_0^4\int_0^4 xyzdxdydz$$
$$= \frac{1}{64}(\frac{x^2}{2})\Big|_0^4)^3 = 8$$

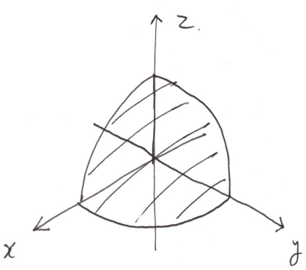
§14.7
The triple integral can be evaluated as follows.

$$A = \int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2 \sin\phi\cos\phi d\rho d\theta d\phi = \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4} \cos^3\theta \sin\phi\cos\phi d\theta d\phi$$
$$= \frac{1}{3}\int_0^{\pi/4} \frac{1}{2}\sin 2\phi d\phi \int_0^{\pi/4} \frac{1}{4}[\cos 3\theta + 3\cos\theta] d\theta = \frac{5\sqrt{2}}{144}$$

22. The solid Q with density ρ has mass

$$m = k\iiint_Q dzdxdy = k\iint_R 12e^{-(x^2+y^2)}dxdy = k\int_0^{\pi/2}\int_0^2 12e^{-r^2} rdrd\theta$$
$$= \frac{k\pi}{3}\frac{e^{-r^2}}{2}\Big|_0^2 = 3\pi k(1-e^{-4})$$

44. The solid we integrate over is as shown.



So we can write the integral in cylindrical coordinates as follows.

$$\int_0^{\pi/2}\int_0^3\int_0^{\sqrt{9-r^2}} \sqrt{r^2+z^2} rdzdrd\theta$$

In spherical coordinates, the integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^3 \sin\phi d\rho d\theta d\phi$$

Evaluating the last integral gives

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^3 \sin\phi d\rho d\theta d\phi = \frac{\pi}{2}\int_0^{\pi/2} \sin\phi d\phi \int_0^3 \rho^3 d\rho = \frac{81\pi}{8}$$

§14.8
6. We have $x = uv - 2u$ and $y = uv$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v-2 & u \\ v & u \end{vmatrix} = u(v-2) - uv = -2u$$

18. First, we calculate the Jacobian for the change of variables.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -\frac{1}{2}$$

Since

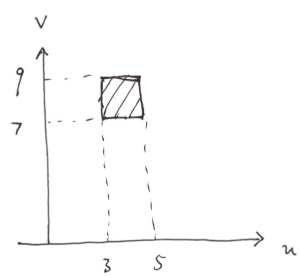
$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$$

we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Then we see that the four vertices are transformed to

$$A^{-1}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix},\ A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$
$$A^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix},\ A^{-1}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$



So, in the new coordiantes, the integral becomes

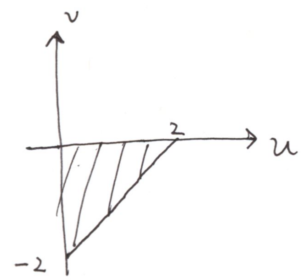
$$\iint_R (2y - x)dA = \int_7^9 \int_3^5 \frac{1}{2}(7u - 3v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{1}{4}(7u^2\Big|_3^5 - 3v^2\Big|_7^9)$$
$$= 4$$

20. The coordinate transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},\ \text{so}\ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The three vertices get mapped to

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



We can evaluate the integral as follows.

$$\iint_R 4(x+y)e^{x-y}dA = \int_{-2}^{0}\int_{0}^{v+2} 4ue^{v}\,du\,dv = \int_{-2}^{0} u^2e^{v}\Big|_0^{v+2} dv$$
$$= \int_{-2}^{0} (v^2+4v+4)e^{v}\,dv$$

Integrating by parts, we find the final answer to be

$$\iint_R 4(x+y)e^{x-y}dA = 2(1-\frac{1}{e^2})$$

36. The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$