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Math 203 - Fall 2018
Solutions to First Examination

1. Consider the line L given by the vector equation $\mathbf{r}(t) = (4+2t, 1+t, 3-2t)$.

   (a) Find a unit vector $\mathbf{v}$ parallel to this line. (5pts)

   ANSWER: There are two answers, but they are all non-zero scalar multiples of the vector

$$\mathbf{r}'(t) = \langle 2, 1, -2 \rangle .$$

   Since a unit vector was asked for, we must divide $\mathbf{r}'(t)$ or its negative by the norm $||\mathbf{r}'(t)|| = 3$. Therefore the answer is

$$\mathbf{v} = \frac{1}{3}\langle 2, 1, -2 \rangle \quad \text{or} \quad -\frac{1}{3}\langle 2, 1, -2 \rangle .$$

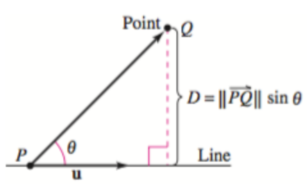
   (b) Find the equation for the plane perpendicular to $\mathbf{v}$ and passing through the point $(-1, 2, 0)$. (10pts)

   ANSWER: A point on this plane with coordinates (x, y, z) must satisfy

$$\langle x+1, y-2, z \rangle \cdot \mathbf{v} = 0, \quad \text{which is the same as} \quad 2(x+1) + y - 2 - 2z = 0.$$

   (c) Find the distance from the origin to the line L. (10pts)

   ANSWER: In the picture below, take Q to be the origin $(0, 0, 0)$. The distance is obtained by choosing any point P on the line, finding the vector $\vec{PQ}$, and projecting it onto a unit vector parallel to the line:



   The quantity D is the picture is $||(PQ) \times \mathbf{u}||$, the magnitude of the cross product.

   Here the unit vector is $\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \pm\frac{1}{3}\langle 2, -1, -2 \rangle$. We will choose $P = \mathbf{r}(-1) = (2, 0, 5)$, so that $PQ = \langle -2, 0, -5 \rangle$. (This choice makes one of the coordinates 0, and keeps the other components as integers, which makes the cross product easier to compute. But you can make any other choice; the outcome will be the same.) Then we have

$$PQ \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -5 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \left\langle \frac{-5}{3}, -\frac{14}{3}, \frac{2}{3} \right\rangle ,$$

   and therefore $||PQ \times \mathbf{u}|| = \frac{1}{3}\sqrt{25 + 196 + 4} = \frac{\sqrt{225}}{3} = 5.$

2. Consider the curve

$$\mathbf{r}(t) = \left\langle 4\sin(t^2 - \pi t), 2e^{4(t-\pi)} \right\rangle .$$

(a) Find the unit tangent vector $\mathbf{T}(2\pi)$ at the point $\mathbf{r}(\pi)$. (10pts)

(A typo was corrected in the room at the exam. We seek $\mathbf{T}(\pi)$, not $\mathbf{T}(2\pi)$)

ANSWER: $\mathbf{r}'(t) = \left\langle 4(2t-\pi)\cos(t^2 - \pi t), 8e^{4(t-\pi)} \right\rangle$, so $\mathbf{r}'(\pi) = \langle 4\pi, 8\rangle$, and therefore

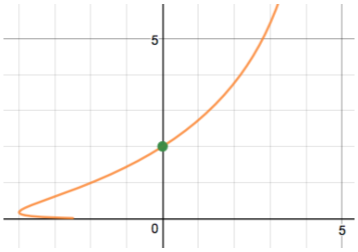
$$\mathbf{T}(\pi) = \frac{\mathbf{r}'(\pi)}{||\mathbf{r}'(\pi)||} = \frac{1}{\sqrt{\pi^2+4}} \langle \pi, 2\rangle .$$

(b) Find the unit normal vector $\mathbf{N}(\pi)$ at the point $\mathbf{r}(\pi)$. (15pts)

ANSWER: Since $\mathbf{N}(t) \perp \mathbf{T}(t)$ and we are in two dimensions, there are only two possibilities for $\mathbf{N}(\pi)$:

$$\mathbf{N}(\pi) = \frac{1}{\sqrt{\pi^2+4}} \langle -2, \pi\rangle \quad \text{or} \quad \mathbf{N}(\pi) = \frac{1}{\sqrt{\pi^2+4}} \langle 2, -\pi\rangle .$$

To decide which of these is the correct answer, we have to see which way the curve bends. First, $\mathbf{r}(\pi) = \langle 0, 2\rangle$. If we take t very close to π but slightly larger than π, then both components of $\mathbf{r}(t)$ increase, but the first component increases much slower than the second component. Therefore the curve is concave up,



so the unit normal points northwest: $\mathbf{N}(\pi) = \frac{1}{\sqrt{\pi^2+4}} \langle -2, \pi\rangle$.

SECOND SOLUTION: We know that the acceleration vector $\mathbf{a}(t) = \mathbf{r}''(t)$ at any time t only has components in the directions of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. (Since the curve lies in the plane, this fact is obvious, but this solution would work even if the curve were a space curve.) Since $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are perpendicular, if we write $\mathbf{a}(t) = f(t)\mathbf{T}(t) + g(t)\mathbf{N}(t)$ then

$$f(t) = \mathbf{a}(t) \cdot \mathbf{T}(t),$$

and therefore $\mathbf{N}(\mathbf{t})$ is parallel to the vector

$$\mathbf{a}(t) - f(t)\mathbf{T}(t).$$

It is easy to calculate the acceleration:

$$\begin{aligned}\mathbf{a}(t) &= \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}\left\langle 4(2t-\pi)\cos(t^2-\pi t), 8e^{4(t-\pi)}\right\rangle \\ &= \left\langle -4(2t-\pi)^2\sin(t^2-\pi t) + 8\cos(t^2-\pi t), 32e^{4(t-\pi)}\right\rangle .\end{aligned}$$

Thus $\mathbf{a}(\pi) = \langle 8, 32\rangle$. Since $\mathbf{T}(\pi) = \frac{\langle \pi, 2\rangle}{\sqrt{\pi^2+4}}$, $f(\pi) = \frac{8\pi+64}{\sqrt{\pi^2+4}}$. Therefore

$$\begin{aligned}\mathbf{a}(\pi) - f(\pi)\mathbf{T}(\pi) &= \langle 8, 32\rangle - \frac{8\pi+64}{\sqrt{\pi^2+4}}\frac{\langle \pi, 2\rangle}{\sqrt{\pi^2+4}} \\ &= \frac{1}{\pi^2+4}\left\langle 8(\pi^2+4) - (8\pi+64)\pi, 32(\pi^2+4) - 2(8\pi+64)\right\rangle \\ &= \frac{1}{\pi^2+4}\left\langle 32-64\pi, 32\pi^2-16\pi\right\rangle = \frac{32\pi-16}{\pi^2+4}\left\langle -2, \pi\right\rangle .\end{aligned}$$

Thus $\mathbf{N}(\pi)$ is the unit vector in the direction of $\langle -2, \pi\rangle$, so

$$\mathbf{N}(\pi) = \frac{1}{\sqrt{\pi^2+4}}\left\langle -2, \pi\right\rangle .$$

(You get 10/15 if you choose the other normal vector.)

Alternatively, you could try to compute $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$ and $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$, but this is a much longer computation, and is very likely to result in errors. Nevertheless, if you computed it this way, and correctly, then you will get full points. Partial credit will depend a lot on your work.

3. Find the length of the curve

$$\mathbf{R}(t) = \left(2e^t, 2e^{-t}, 2\sqrt{2}t\right), \qquad 0 \leq t \leq 3.$$

`Hint:` $(a + \frac{1}{a})^2 = ?$

ANSWER: First, the speed is

$$||\mathbf{R}'(t)|| = ||\left\langle 2e^t, -2e^{-t}, 2\sqrt{2}\right\rangle|| = 2\sqrt{e^{2t} + e^{-2t} + 2} \overset{\text{by the hint}}{=} 2\sqrt{(e^t + e^{-t})^2} = 2(e^t + e^{-t})$$

The length is the integral of speed with respect to time:

$$\ell = \int_0^3 ||\mathbf{R}'(t)||dt = \int_0^3 2(e^t + e^{-t})dt = 2\left(e^t - e^{-t}\right)_{t=0}^{t=3} = 2(e^3 - e^{-3}).$$

4. For the function

$$f(x, y, z) = z^2 \sin\left(\frac{xy}{z^2}\right),$$

calculate $\frac{\partial f}{\partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)$ and $\frac{\partial^2 f}{\partial x^2}$. (5+10+10pts)

ANSWER: First,

$$\frac{\partial f}{\partial y} = z^2 \cos\left(\frac{xy}{z^2}\right) \cdot \frac{x}{z^2} = x \cos\left(\frac{xy}{z^2}\right).$$

Next, Clairaut's Theorem says that mixed partial derivatives can be taken in any order. So to compute $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)$ we can just compute the partial derivative of $\frac{\partial f}{\partial y}$ with respect to z. We get

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(x \cos\left(\frac{xy}{z^2}\right)\right) = \frac{2x^2y}{z^3} \sin\left(\frac{xy}{z^2}\right).$$

The last one, $\frac{\partial^2 f}{\partial x^2}$, is just computed directly:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(y \cos\left(\frac{xy}{z^2}\right)\right) = -\frac{y^2}{z^2} \sin\left(\frac{xy}{z^2}\right).$$