Math 203 - Fall 2018
Solutions to Second Examination

1. Consider the function $z = f(x, y)$ is defined implicitly by the equation

$$z^2(1 + \sin(xy)) + e^{2z-y} = 2$$

and the condition $f(0, 2) = 1$. Compute $\frac{\partial f}{\partial x}(0, 2)$.

Solution: Differentiating the equation with respect to x yields

$$2zz_x(1 + \sin(xy)) + z^2 y\cos(xy) + 2z_x e^{2z-y} = 0.$$

Plugging in $(x, y, z) = (0, 2, 1)$ yields $2z_x + 2 + 2z_x = 0$, or

$$z_x(0, 2) = -\frac{1}{2}.$$

Second, almost equivalent solution: One could use the formula

$$z_x = \frac{-F_x}{F_z}$$

where $F(x, y, z) = z^2(1 + \sin(xy)) + e^{2z-y}$. Then

$$F_x = yz^2\cos(xy) \quad \text{and} \quad F_z = 2z(1 + \sin(xy)) + 2e^{2z-y}.$$

Plugging in $(x, y, z) = (0, 2, 1)$ yields $F_x(0, 2, 1) = 2$ and $F_z(0, 2, 1) = 4$, so again

$$z_x(0, 2) = -\frac{1}{2}.$$

2. Consider the function

$$f(x, y) = e^{-(x^2+y^2)}$$

and the unit vector

$$\mathbf{u}_o = \left\langle \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \right\rangle.$$

(a) Calculate the directional derivative $D_{\mathbf{u}_o} f(x, y)$ of f at the point (x, y) along the direction $\mathbf{u}_o$.

(b) Find the maximum value of $D_{\mathbf{u}_o} f(x, y)$ among all points (x, y) in the plane, and the point where this maximum occurs.

Solution: (a)

$$D_{\mathbf{u_o}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u_o} = \left\langle -2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^y)} \right\rangle \cdot \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \sqrt{2}(x-y)e^{-(x^2+y^2)}.$$

(b) We compute that

$$\nabla(D_{\mathbf{u_o}}f(x,y)) = \sqrt{2}e^{-(x^2+y^2)} \langle 1 - 2x(x-y), -1-2y(x-y) \rangle .$$

Therefore the critical points are simultaneous solutions of the two equations

$$2x(x-y) = 1 \quad \text{and} \quad 2y(x-y) = -1.$$

These two equations imply that for any critical point (x, y), $x-y \neq 0$. By taking the quotient of the two equations, we find that if (x, y) is a critical point then $y = -x$. Substituting this relation into the first equation yields $4x^2 = 1$, so $x = \pm\frac{1}{2}$, and together with the relation $y = -x$ we see that the critical points are $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$. Since

$$D_{\mathbf{u_o}}f(\tfrac{1}{2}, -\tfrac{1}{2}) = \sqrt{2}e^{-1/2} \quad \text{and} \quad D_{\mathbf{u_o}}f(-\tfrac{1}{2}, \tfrac{1}{2}) = -\sqrt{2}e^{-1/2},$$

we see that the maximum value of $\sqrt{2/e}$ occurs at the point $(\frac{1}{2}, -\frac{1}{2})$.

3. Find the point (x_o, y_o) whose y-coordinate has the largest possible value among all points on (x, y) lying on the curve

$$(2x-y)^2 + 2(x+3y)^2 = 3.$$

Solution: We are trying to maximize the function $f(x, y) = y$ subject to the constraint $g(x, y) = (2x-y)^2 + 2(x+3y)^2 = 3$. The Lagrange method says that the critical points of this constrained problem occur at the points where the gradient of f is proportional to the gradient of g, i.e., we have the equations

$$\langle 0, 1 \rangle = \lambda \langle 4(2x-y) + 4(x+3y), 2(y-2x) + 12(x+3y) \rangle \quad \text{and} \quad (2x-y)^2 + 2(x+3y)^2 = 3.$$

Since the vector $\langle 0, 1 \rangle$ is non-zero, λ cannot equal 0, and therefore

$$0 = 4(2x-y) + 4(x+3y) = 12x + 8y = (y + \tfrac{3}{2}x).$$

Plugging this equation into the constraint yields

$$3 = (2x-y)^2 + 2(x+3y)^2 = \left(\tfrac{7}{2}x\right)^2 + 2\left(-\tfrac{7}{2}x\right)^2 = 3\left(\tfrac{7}{2}x\right)^2,$$

so that $x = \pm\frac{2}{7}$, and therefore $y = \mp\frac{3}{7}$. Since we are trying to maximize the y value, the point with highest y coordinate is

$$\left(-\tfrac{2}{7}, \tfrac{3}{7}\right).$$

4. Compute the iterated integral

$$\int_0^2 \int_y^{\sqrt{8-y^2}} \int_0^{\sqrt{8-x^2-y^2}} \frac{2\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} dz dx dy.$$

Solution: The smart move here is to change to spherical coordinates. The region of integration

$$\left\{(x,y,z)\ ;\ 0 \le y \le 2, y \le x \le \sqrt{8-y^2}, 0 \le z \le \sqrt{8-x^2-y^2}\right\}$$

is the portion of the ball of radius $2\sqrt{2}$ defined by the spherical coordinates

$$\left\{(\rho,\phi,\theta)\ ;\ 0 \le \rho \le 2\sqrt{2}, 0 \le \phi \le \tfrac{\pi}{2}, 0 \le \theta \le \tfrac{\pi}{4}\right\}.$$

Therefore

$$\begin{aligned}
&\int_0^2 \int_y^{\sqrt{8-y^2}} \int_0^{\sqrt{8-x^2-y^2}} \frac{2\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} dz dx dy \\
&= \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{2\sqrt{2}} \frac{2\sin(\rho^2)}{\rho} \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \left(\int_0^{\pi/4} d\theta\right)\left(\int_0^{\pi/2} \sin\phi d\phi\right)\left(\int_0^{2\sqrt{2}} \sin(\rho^2) 2\rho d\rho\right) \\
&= \frac{\pi}{4}(\cos(0)-\cos(\pi/2))(\cos(0)-\cos(8)) \\
&= \frac{\pi}{4}(1-\cos 8).
\end{aligned}$$