# SAMPLING SEQUENCES FOR BERGMAN SPACES FOR $p < 1$

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Abstract. We provide a proof of the sufficiency direction of Seip's characterization of sampling sequences for Bergman spaces for $p < 1$ based on the methods of Berndtsson and Ortega-Cerdà.

## 1. Introduction

For $0 < p < \infty$ and $\phi$ a function subharmonic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, define $F^p_\phi$ to be the set of functions analytic in $\mathbb{D}$ satisfying

$$\|f\|_{\phi,p} = \left\{ \int_{\mathbb{D}} |f(z)|^p \frac{e^{-\phi(z)}}{1-|z|^2} dA(z) \right\}^{1/p} < \infty,$$

where $dA$ denotes Lebesgue area measure.

We say that a sequence $\Gamma = \{\gamma_n\}$ of distinct points in the disk is a *sampling sequence for* $F^p_\phi$ if there exist positive constants $K_1$ and $K_2$ such that

$$K_1 \|f\|^p_{\phi,p} \le \sum_n |f(\gamma_n)|^p e^{-\phi(\gamma_n)}(1-|\gamma_n|^2) \le K_2 \|f\|^p_{\phi,p}$$

for all $f \in F^p_\phi$.

Letting $\phi(z) = \log \frac{1}{1-|z|^2}$, we obtain the standard Bergman space $A^p$ and the corresponding sampling sequences, which were characterized by Seip [5] for $p = 2$ using methods that were extended to the case $1 \le p < \infty$ by the first named author [4]. Berndtsson and Ortega-Cerdà [1] showed, using an altogether different proof, that a variation of Seip's density condition from [5] is actually sufficient to give sampling sequences in $F^2_\phi$. While it does not appear that the arguments of Seip can be modified to work for $A^p$ when $0 < p < 1$, the techniques of [1], as was conjectured in [2], can be adapted to $F^p_\phi$ (and hence $A^p$) for $0 < p < 1$. The purpose of this note is to show how this can be done.

1991 *Mathematics Subject Classification.* 30H05, 46E15.

*Key words and phrases.* Bergman space, sampling.

We introduce the definitions necessary for the statement of the theorem we will prove.

The sequence $\Gamma = \{\gamma_n\}$ is said to be *uniformly discrete* if

$$\delta(\Gamma) = \inf_{n \neq m} |\phi_{\gamma_n}(\gamma_m)| > 0,$$

where

$$\phi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}$$

is the standard involutive Möbius transformation. The disk of centre $\zeta$ and radius $r$ in this metric will be denoted by $\Delta(\zeta, r)$.

In the disk it is useful to consider the invariant Laplacian $\tilde{\Delta} = (1-|z|^2)^2 \partial^2/\partial z \partial \overline{z}$, and for a measure $\mu$ and a function $g$, the invariant convolution $\mu * g$, defined by

$$(\mu * g)(z) = \frac{1}{\pi} \int_{\mathbb{D}} g(\phi_z(\zeta)) \frac{d\mu(\zeta)}{(1 - |\zeta|^2)^2}.$$

Consider now the measure $\nu = \pi \sum_n (1 - |\gamma_n|^2)^2 \delta_{\gamma_n}$, where $\delta_z$ is the Dirac-delta measure at the point $z$, and for $1/2 < r < 1$ the function

$$\xi_r(\zeta) = \begin{cases} \frac{1}{c_r} \log \frac{1}{|\zeta|^2} & \text{if } 1/2 < |\zeta| < r, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_r$ is such that $\int_{\mathbb{D}} \xi_r(\zeta) \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} = 1$.

We are now in a position to state the main theorem of this note.

**Main Theorem.** *Suppose a sequence $\Gamma$ is uniformly discrete, and $\phi$ is a $C^2$ subharmonic function with uniformly bounded invariant Laplacian $\tilde{\Delta}\phi$. If there exists $r < 1$ and $\delta > 0$ such that*

$$(\nu * \xi_r)(z) > \frac{2}{p} \tilde{\Delta} \phi(z) + \delta$$

*for all $z \in \mathbb{D}$, then $\Gamma$ is a sampling sequence for $F^p_\phi$.*

Seip [5] introduces the following definitions. For $\Gamma$ uniformly discrete, $z \in \mathbb{D}$ and $1/2 < r < 1$, let

$$D(\Gamma, r) = \frac{\sum_{1/2 < |\gamma_n| < r} \log \frac{1}{|\gamma_n|}}{\log \frac{1}{1-r}}$$

and

$$D^-(\Gamma) = \liminf_{r \to 1} \inf_{z \in \mathbb{D}} D(\phi_z(\Gamma), r).$$

We then have the following theorem, as stated in [4].

**Theorem A.** *Let $1 \leq p < \infty$. A uniformly discrete sequence $\Gamma$ is a sampling sequence for $A^p$ if and only if $D^-(\Gamma) > 1/p$.*

A calculation shows that

$$\frac{D(\phi_z(\Gamma), r)}{(\nu * \xi_r)(z)} = \frac{1}{2}\frac{c_r}{\log\frac{1}{1-r}} \to \frac{1}{2} \qquad \text{as } r \to 1.$$

Moreover, since $\tilde{\Delta}\phi(z) = 1$ if $\phi(z) = \log\frac{1}{1-|z|^2}$, the sufficiency direction of Theorem A will follow from the Main Theorem, for $0 < p < \infty$.

As mentioned above, the proof is based on the techniques used in [1]. Our main interest lies in proving the Main Theorem when $0 < p < 1$, thus completing Theorem A, but the proof works, without modification, when $1 \leq p < \infty$. With the reader of the paper [1] in mind, we will employ the same notation as in that article.

The paper is organized as follows. In the next section we recall some of the notation from [1] that is necessary to prove the Main Theorem, and we then prove the main theorem given a collection of technical lemmas. Finally, we complete these technicalities, some of which were claimed without proof in [1], so we have included details here for the convenience of the reader.

## 2. Proof of the Main Theorem

For $0 < t, \epsilon < 1$, consider the functions

$$\chi_\epsilon = \frac{t}{\epsilon^2}\chi_{\Delta(0,\epsilon)} \quad \text{and} \quad \nu_\epsilon = \nu * \chi_\epsilon.$$

Note that

$$\nu_\epsilon dA * \xi_r - \nu * \xi_r = (\nu * \xi_r)dA * \chi_\epsilon - \nu * \xi_r,$$

which approaches 0 as $\epsilon \to 0$ and $t \to 1$. Here we have used the fact that $f dA * g = g dA * f$ and

$$\mu * (h dA * g) = (\mu * h) dA * g \tag{1}$$

whenever $h$ is radial. We can therefore choose $r$ and $t$ close to 1 and $\epsilon$ close to 0 so that

$$\nu_\epsilon dA * \xi_r(z) > \frac{2}{p}\tilde{\Delta}\phi(z) + \frac{\delta}{2}$$

for all $z \in \mathbb{D}$. Consider now the function

$$v = \frac{p}{2}(\nu_\epsilon - \nu_\epsilon dA * \xi_r) dA * E,$$

where $E(z) = \log|z|^2$. Since $E$ is the fundamental solution of the invariant Laplacian (with respect to the invariant convolution), we see that

$$\tilde{\Delta} v = \frac{p}{2}(\nu_\epsilon - \nu_\epsilon dA * \xi_r),$$

so that the function $\psi = \phi + v$ satisfies

$$\tilde{\Delta}\psi \leq \frac{p}{2}\nu_\epsilon - \frac{\delta}{2}.$$

We require the following four lemmas to complete the proof of the theorem.

**Lemma 1.** *There are positive constants $C_r$ and $C_\epsilon$ such that*

$$-C_\epsilon \le v(z) \le 0 \qquad \text{for all } z \in \mathbb{D}. \tag{2}$$

*Moreover,*

$$|v(z) - t\log \epsilon^p| \le C_r \tag{3}$$

*for all $z \in \mathbb{D}$ with $\rho(z,\gamma_n) < \epsilon$ for some $n$.*

**Lemma 2.**

$$\frac{\delta}{2}\int_{\mathbb{D}} |h(z)|^p \frac{e^{-\psi(z)}}{(1-|z|^2)} dA(z) \le \frac{t}{\epsilon^2}\sum_n \int_{\Delta(\gamma_n,\epsilon)} |h(z)|^p \frac{e^{-\psi(z)}}{(1-|z|^2)} dA(z) \tag{4}$$

*for all $h \in F^p_\phi$.*

**Lemma 3.** *There is a constant $C > 0$ such that for each $h \in F^p_\phi$ and $a \in \mathbb{D}$, there exists $\tilde{h}_a \in F^p_\phi$ such that $\tilde{h}_a(a) = h(a)$ and*

$$\frac{1}{C} e^{-\phi(a)} |\tilde{h}_a(z)|^p \le |h(z)|^p e^{-\phi(z)} \le C e^{-\phi(a)} |\tilde{h}_a(z)|^p$$

*for all $z \in \Delta(a, 1/2)$.*

**Lemma 4.** *There is a constant $C > 0$ such that*

$$\frac{1}{\epsilon^2}\int_{\Delta(a,\epsilon)} |g(z)|^p dA(z) \le C|g(a)|^p(1-|a|^2)^2 + C\epsilon^p \int_{\Delta(a,1/2)} |g(z)|^p dA(z)$$

*for all $g \in F^p_\phi$.*

We take these lemmas as given and proceed with the proof. Suppose that $h \in F^p_\phi$.

Then

$$
\begin{aligned}
&\frac{\delta}{2}\int_{\mathbb{D}}|h(z)|^p\frac{e^{-\psi(z)}}{1-|z|^2}dA(z) \le \frac{t}{\epsilon^2}\sum_n\int_{\Delta(\gamma_n,\epsilon)}|h(z)|^p\frac{e^{-\psi(z)}}{1-|z|^2}dA(z)\\
&\quad= \frac{t}{\epsilon^2}\sum_n\int_{\Delta(\gamma_n,\epsilon)}|h(z)|^p\frac{e^{-\phi(z)}e^{-v(z)}}{1-|z|^2}dA(z)\\
&\quad\le Ct\epsilon^{-pt-2}\sum_n\int_{\Delta(\gamma_n,\epsilon)}|h(z)|^p\frac{e^{-\phi(z)}}{1-|z|^2}dA(z)\\
&\quad\le Ct\epsilon^{-pt-2}\sum_n\frac{1}{1-|\gamma_n|^2}\int_{\Delta(\gamma_n,\epsilon)}|h(z)|^pe^{-\phi(z)}dA(z)\\
&\quad\le Ct\epsilon^{-pt-2}\sum_n\frac{e^{-\phi(\gamma_n)}}{1-|\gamma_n|^2}\int_{\Delta(\gamma_n,\epsilon)}|\tilde{h}_n(z)|^pdA(z)\\
&\quad= Ct\epsilon^{-pt}\sum_n\frac{e^{-\phi(\gamma_n)}}{1-|\gamma_n|^2}\int_{\Delta(\gamma_n,\epsilon)}|\tilde{h}_n(z)|^pdA(z)\\
&\quad\le Ct\epsilon^{-pt}\sum_n\frac{e^{-\phi(\gamma_n)}}{1-|\gamma_n|^2}\left\{C|\tilde{h}_n(\gamma_n)|^p(1-|\gamma_n|^2)^2+Ct\epsilon^p\int_{\Delta(\gamma_n,1/2)}|\tilde{h}_n(z)|^pdA(z)\right\}\\
&\quad\le Ct\epsilon^{-pt}\sum_n e^{-\phi(\gamma_n)}(1-|\gamma_n|^2)|h(\gamma_n)|^p+Ct\epsilon^{p-pt}\sum_n\int_{\Delta(\gamma_n,1/2)}|h(z)|^p\frac{e^{-\phi(z)}}{1-|z|^2}dA(z)\\
&\quad\le Ct\epsilon^{-pt}\sum_n e^{-\phi(\gamma_n)}(1-|\gamma_n|^2)|h(\gamma_n)|^p+Ct\epsilon^{p-pt}\int_{\mathbb{D}}|h(z)|^p\frac{e^{-\phi(z)}}{1-|z|^2}dA(z).
\end{aligned}
$$

The first line is Lemma 2, the third follows from Lemma 1, while the fourth follows from the fact that

$$\frac{1-|\gamma_n|^2}{1-|z|^2}\le\frac{4}{1-\epsilon^2}$$

for all $z\in\Delta(\gamma_n,\epsilon)$. The fifth line is a consequence of Lemma 3, where $\tilde{h}_n=\tilde{h}_{\gamma_n}$, while the seventh follows from Lemma 4 and the eighth from Lemma 3.

If we take $\epsilon$ small enough, we arrive at the lower sampling inequality. The upper inequality follows from the fact that $\Gamma$ is uniformly discrete. ■

## 3. Technicalities

We consider now the proofs of the four lemmas.

*Proof of Lemma 1.* We enumerate $\Gamma$ once and for all, and let $\Gamma_N$ be the first $N$ terms of $\Gamma$. We then set

$$\nu_N=\pi\sum_{\gamma\in\Gamma_N}(1-|\gamma|^2)^2\delta_\gamma \quad\text{and}\quad w_N=\nu_N*E-(\nu_N*E)dA*\xi_r.$$

Several applications of (1) yield

$$w_N(z) = \sum_{\gamma\in\Gamma_N} \left( \log|\varphi_z(\gamma)|^2 - \int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \right).$$

Our first claim is that for each compact set $K \subset \mathbb{D}$ there exists an integer $N = N(K)$ such that for all $z \in K$ and $M > N$,

$$w_M(z) = w_N(z).$$

Indeed, fix $\gamma \in \Gamma$ and $z \in \mathbb{D}$ such that $|\varphi_\gamma(z)| > r$. The function $\zeta \mapsto \log|\varphi_\gamma(\zeta)|^2$ is harmonic in the domain $V_r(\gamma) = \{\zeta \ : \ \frac{1}{2} < |\varphi_\gamma(\zeta)| < r\}$, and thus

$$\begin{aligned}
&\int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \\
&= \int_{V_r(\gamma)} \xi_r(\varphi_\zeta(\gamma)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \\
&= \int_{1/2<|u|<r} \xi_r(u) \log|\varphi_z \circ \varphi_\gamma(u)|^2 \frac{dA(u)}{\pi(1-|u|^2)^2} = \log|\varphi_z(\gamma)|^2.
\end{aligned}$$

The second equality follows from the invariance of the measure $dA(\zeta)/(\pi(1-|\zeta|^2)^2)$, while the third equality comes from the mean value property for harmonic functions and the fact that $\xi_r(\zeta)dA(\zeta)/(\pi(1-|\zeta|^2)^2)$ is a radial probability measure. We therefore have that

$$w_N(z) = \sum_{\gamma\in\Gamma_N\cap\Delta(z,r)} \left( \log|\varphi_z(\gamma)|^2 - \int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \right).$$

The claim thus follows from the fact that for a given compact set $K$, there is only a finite number $N$ of points $\gamma \in \Gamma$ such that for some $z \in K$, $|\varphi_\gamma(z)| \le r$.

We set

$$w = \lim_{N\to\infty} w_N,$$

where the limit is taken in the locally uniform topology. In other words, $w$ is the ordered sum

$$w(z) = \sum_{\gamma\in\Gamma} \left( \log|\varphi_z(\gamma)|^2 - \int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \right).$$

Since $\tilde{\Delta}(\nu_N * E) = \nu_N$ is a positive measure, $\nu_N * E$ is subharmonic. Therefore, again since $\xi_r(\zeta)dA(\zeta)/(\pi(1-|\zeta|^2)^2)$ is a radial probability measure, $\nu_N * E \le \nu_N * E * \xi_r$, i.e., $w_N \le 0$. It follows that $w \le 0$. Since $v = \frac{p}{2}(wdA * \chi_\epsilon)$, this implies the right hand side of (2).

Turning our attention now to (3), we wish to show first that there exists a constant $E_r > 0$ such that for every $\gamma \in \Gamma$,

$$\left| w(z) - \log|\varphi_z(\gamma)|^2 \right| \leq E_r$$

whenever $z \in \Delta(\gamma, \sigma)$, where $\sigma = \delta(\Gamma)/2$. By the above remarks, and since $\Gamma$ is uniformly discrete, there exists an integer $N$, depending only on $r$, such that

$$w(z) = \sum_{j=0}^{N} \left( \log|\varphi_z(\gamma_j)|^2 - \int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma_j)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \right),$$

where $\gamma_0 = \gamma$ and $\gamma_1, ..., \gamma_N$ are the members of $\Gamma$ in $\Delta(\gamma, \frac{r+\sigma}{1+r\sigma})$. It follows that

$$\begin{aligned} w(z) - \log|\varphi_z(\gamma)|^2 = & - \int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \\ & + \sum_{j=1}^{N} \log|\varphi_z(\gamma_j)|^2 \\ & \quad - \sum_{j=1}^{N} \int_{\mathbb{D}} \xi_r(\varphi_\zeta(\gamma_j)) \log|\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2}. \end{aligned}$$

Now, for any $t \in \Gamma$ the integral

$$I_t = \int_{\mathbb{D}} \xi_r(\varphi_\zeta(t)) \log|\varphi_z(\zeta)|^{-2} \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2}$$

may be estimated as follows:

$$\begin{aligned} I_t & \leq \frac{1}{c_r} \int_{\frac{1}{2}<|\varphi_\zeta(t)|<r} \log|\varphi_\zeta(t)|^{-2} \log|\varphi_z(\zeta)|^{-2} \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \\ & \leq \frac{1}{c_r} \log 4 \int_{\Delta(z,1/2)} \log|\varphi_z(\zeta)|^{-2} \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} + \log 4 \int_{\mathbb{D}} \xi_r(\varphi_\zeta(t)) \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} \\ & = \frac{2}{c_r} \log 4 \int_0^{1/2} \frac{s \log s^{-2}}{(1-s^2)^2} ds + \log 4 =: D_r. \end{aligned}$$

We thus obtain that

$$\begin{aligned} \left| w(z) - \log|\varphi_z(\gamma)|^2 \right| & \leq |I_\gamma| + \sum_{j=1}^{N} \log|\varphi_z(\gamma_j)|^{-2} + \sum_{j=1}^{N} |I_{\gamma_j}| \\ & \leq D_r(N+1) + N \log \sigma^{-2} =: E_r. \end{aligned}$$

Next, we need to estimate the convolution product

$$F_\epsilon(z) = \left(\log|\varphi_\gamma(\cdot)|^2 dA * \chi_\epsilon\right)(z) = \frac{t}{\epsilon^2}\int_{\Delta(0,\epsilon)} \log|\varphi_\gamma\circ\varphi_z(\zeta)|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2}.$$

It is easy to verify that, with $u = \varphi_z(\gamma)$,

$$\varphi_\gamma\circ\varphi_z(\zeta) = \lambda\varphi_u(\zeta),$$

where $|\lambda| = 1$. Thus, changing variables, we have

$$F_\epsilon(z) = \frac{t}{\epsilon^2}\int_{\Delta(u,\epsilon)} \log|\zeta|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2}.$$

Then

$$\begin{aligned}
F_\epsilon(z) - t\log\epsilon^2 &= \frac{t}{\epsilon^2}\int_{\Delta(u,\epsilon)} \log\left|\frac{\zeta}{2\epsilon}\right|^2 \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} + \frac{t}{\epsilon^2}\log 4\epsilon^2 \int_{\Delta(u,\epsilon)} \frac{dA(\zeta)}{\pi(1-|\zeta|^2)^2} - t\log\epsilon^2 \\
&= 4t\int_{\frac{1}{2\epsilon}\Delta(u,\epsilon)} \log|\zeta|^2 \frac{dA(\zeta)}{\pi(1-4\epsilon^2|\zeta|^2)^2} + \frac{t}{\epsilon^2}\log 4\epsilon^2 \frac{\epsilon^2}{1-\epsilon^2} - t\log\epsilon^2 \\
&= 4t\int_{\frac{1}{2\epsilon}\Delta(u,\epsilon)} \log|\zeta|^2 \frac{dA(\zeta)}{\pi(1-4\epsilon^2|\zeta|^2)^2} + t\frac{\log 4}{1-\epsilon^2} + t\frac{\epsilon^2\log\epsilon^2}{1-\epsilon^2}.
\end{aligned}$$

The second line follows from a change of variables and the fact that the hyperbolic area of a disk of radius $\epsilon$ is $\frac{\epsilon^2}{1-\epsilon^2}$. Since $|u| < \epsilon$, $\frac{1}{2\epsilon}\Delta(u,\epsilon) \subseteq \mathbb{D}$, and so the absolute value of the integral in the last line is bounded by

$$\frac{1}{1-4\epsilon^2}\int_{\mathbb{D}} \log\frac{1}{|\zeta|^2} dA(\zeta),$$

which is seen to converge. We thus have a constant $C$ such that for sufficiently small $\epsilon$,

$$|\log|\phi_\gamma(\cdot)|^2 dA * \chi_\epsilon(z) - t\log\epsilon^2| \le C, \qquad \text{whenever } |\phi_\gamma(z)| < \epsilon.$$

Therefore, we have that for $|\varphi_\gamma(z)| < \epsilon$,

$$\begin{aligned}
\left|wdA * \chi_\epsilon(z) - t\log\epsilon^2\right| &\le \left|wdA * \chi_\epsilon(z) - \log|\varphi_\gamma(\cdot)|^2 dA * \chi_\epsilon(z)\right| \\
&\qquad + \left|\log|\varphi_\gamma(\cdot)|^2 dA * \chi_\epsilon(z) - t\log\epsilon^2\right| \\
&\le D_r + M =: C_r.
\end{aligned}$$

Since $v = \frac{p}{2}(wdA * \chi_\epsilon)$, the proof of (3) is complete.

Finally, we wish to prove the left inequality of (2). First, if $|\varphi_z(\gamma)| < \delta(\Gamma)/2$ for some $\gamma \in \Gamma$, then (3) gives a universal lower bound for $v(z)$. On the other hand, if $z$ is isolated away from $\Gamma$, then a look at the way $w$ was estimated above shows that $v(z)$ is bounded from below by a negative number of even smaller modulus. This completes the proof. ■

*Proof of Lemma 2.* Let us note that the right hand side of (4) is

$$\int_{\mathbb{D}} \frac{|h(z)|^p e^{-\psi(z)} \nu_\epsilon(z)}{1-|z|^2} dA(z).$$

We set $U = |h|^p e^{-\psi}$. Then $\log U + \psi = p \log |h|$ is a subharmonic function. Thus

$$0 \leq \Delta \log U + \Delta \psi = \frac{1}{U} \Delta U - \frac{1}{U^2} |U_z|^2 + \Delta \psi \leq \frac{1}{U} \Delta U + \Delta \psi.$$

It follows from the nonnegativity of $U$ and the estimate (1) on $\tilde{\Delta}\psi$ above, that

$$\tilde{\Delta} U \geq -U \tilde{\Delta} \psi \geq -\frac{p}{2} U \nu_\epsilon + U \frac{\delta}{2}.$$

Dividing by $1-|z|^2$ and integrating yields

$$\frac{\delta}{2} \int_{\mathbb{D}} \frac{U(z)}{1-|z|^2} dA(z) \leq \frac{p}{2} \int_{\mathbb{D}} \frac{U(z)}{1-|z|^2} \nu_\epsilon(z) dA(z) + \int_{\mathbb{D}} (1-|z|^2) \Delta U(z) dA(z).$$

We would like to show that the second integral on the right is nonpositive. If $U$ were compactly supported, we could use integration by parts to shift the Laplacian to $1-|z|^2$. Since $\Delta(1-|z|^2) = -1$ and $U \geq 0$, we would be done. So instead, one "cuts things off" as follows. Let $\chi_t \geq 0$, $0 << t < 1$ be a function which is identically one on $[0, t]$, supported compactly in $[0, 1)$, with additional properties to be described shortly. Then

$$\int_{\mathbb{D}} (1-|z|^2) \chi_t(|z|) \Delta U(z) dA(z) = \int_{\mathbb{D}} \Delta \left( (1-|z|^2) \chi_t(|z|) \right) U(z) dA(z).$$

Recalling that on radial functions,

$$\Delta = \frac{1}{4} (\partial_r^2 + \frac{1}{r} \partial_r),$$

one computes that

$$\Delta \left( (1-|z|^2) \chi_t(|z|) \right) = -\chi_t(|z|) - |z| \chi_t'(|z|) + (1-|z|^2) \Delta \chi_t(|z|).$$

One then has

$$\int_{\mathbb{D}} (1-|z|^2) \chi_t(|z|) \Delta U(z) dA(z) = \int_{\mathbb{D}} -\chi_t(|z|) U(z) dA + I_t,$$

where

$$J_t = -\pi \int_{\mathbb{D}} \left( (1-|z|^2)^2 |z| \chi_t'(|z|) U(z) + (1-|z|^2) \tilde{\Delta} \chi_t(|z|) \right) \frac{dA(z)}{\pi(1-|z|^2)^2}$$

Now, by Lemma 1 and the hypotheses on $h$, $U$ is integrable with respect to the Poincaré area, and thus with respect to Euclidean area. Hence as $t \to 1$,

$$\int_{\mathbb{D}} \chi_t(|z|) U(z) dA(z) \to \int_{\mathbb{D}} U(z) dA(z) \geq 0.$$

We claim that with a good choice of $\chi_t$, the integral $J_t \to 0$ as $t \to 1$. To see this, simply choose $\chi_t$ so that it has bounded invariant Laplacian, uniformly in $t$. (Examples of this are easy enough to construct. For instance, let $f$ be a smooth function on the nonnegative real line, which is supported on $[0, 1/2]$ and is identically 1 on $[0, 1/4]$. Then just take

$$\chi_t(|z|) := f\left( \frac{|z|^2 - t}{1 - \frac{1+t}{2}} \right).$$

The boundedness of the invariant Laplacian is easy to check.) Because $\chi_t$ is radial, this will also give a bound on the gradient. One can then apply the dominated convergence theorem. This completes the proof. ■

*Proof of Lemma 3.* Since $\phi$ is subharmonic with bounded invariant Laplacian, we may apply the Riesz decomposition theorem. Thus, if $G$ is a fixed Green operator, one has

$$\phi = G(\tilde{\Delta}\phi) + f_a$$

for some harmonic function $f_a$ in a neighborhood of the closed disc $\overline{\Delta(a, 1/2)}$. Let $g_a$ be a holomorphic function whose real part is $f_a$, and set $q_a = g_a - g_a(a)$. Then

$$|\phi - \phi(a) - 2\Re(q_a)| \leq K.$$

Now, let $\tilde{h}_a(z) = h(z) e^{-2p^{-1} q_a(z)}$. Then

$$|h(z)|^p e^{-\phi(z)} = |\tilde{h}_a(z)|^p e^{-\phi(z) + 2\Re(q_a(z))} \leq C e^{-\phi(a)} |\tilde{h}_a(z)|^p,$$

where $C = e^K$. The other inequality in the lemma follows similarly. ■

*Proof of Lemma 4.* Let $z \in \Delta(a, \epsilon)$ and write $g(z) - g(a) = \int_a^z g'(u) du$, so that

$$\begin{aligned} |g(z)| &\leq |g(a)| + \left| \int_a^z g'(u) du \right| \\ &\leq |g(a)| + |z - a| \sup_{u \in \Delta(a,\epsilon)} |g'(u)|, \end{aligned}$$

which implies that

$$
\begin{aligned}
|g(z)|^p &\leq 2^p \left\{ |g(a)|^p + |z-a|^p \sup_{u \in \Delta(a,\epsilon)} |g'(u)|^p \right\} \\
&\leq 2^p \left\{ |g(a)|^p + C|z-a|^p \sup_{u \in \Delta(a,\epsilon)} (1-|u|^2)^{-2-p} \int_{\Delta(u,\epsilon)} |g(\zeta)|^p dA(\zeta) \right\} \\
&\leq 2^p \left\{ |g(a)|^p + C|z-a|^p (1-|a|^2)^{-2-p} \int_{\Delta(a,1/2)} |g(\zeta)|^p dA(\zeta) \right\}.
\end{aligned}
$$

The second line follows from the standard estimate (see [3], for example)

$$
|g'(u)|^p \leq C(1-|u|^2)^{-2-p} \int_{\Delta(u,\epsilon)} |g(\zeta)|^p dA(\zeta).
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{\epsilon^2} \int_{\Delta(a,\epsilon)} |g(z)|^p dA(z) \\
&\quad \leq C \frac{1}{\epsilon^2} |g(a)|^p \int_{\Delta(a,\epsilon)} dA(z) \\
&\qquad + C \frac{1}{\epsilon^2} (1-|a|^2)^{-2-p} \int_{\Delta(a,1/2)} |g(\zeta)|^p dA(\zeta) \int_{\Delta(a,\epsilon)} |a-z|^p A(z).
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{\Delta(a,\epsilon)} |a-z|^p A(z) &\leq \epsilon^p \int_{\Delta(a,\epsilon)} |1-\overline{a}z|^p dA(z) \\
&\leq \epsilon^{p+2} (1-|a|^2)^{p+2} \int_{\Delta(0,\epsilon)} |1-\overline{a}z|^{-4-p} dA(z) \\
&\leq C\epsilon^{p+2} (1-|a|^2)^{p+2},
\end{aligned}
$$

the result follows. ■

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