A compactification of $(\mathbf{C}^*)^4$ with no non-constant meromorphic functions

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Abstract

For each 2-dimensional complex torus T, we construct a compact complex manifold $X(T)$ with a $\mathbf{C}^2$-action, which compactifies $(\mathbf{C}^*)^4$ such that the quotient of $(\mathbf{C}^*)^4$ by the $\mathbf{C}^2$-action is biholomorphic to T. For a general T, we show that $X(T)$ has no non-constant meromorphic functions.

1 Introduction

There is a well-known example, due to J.-P. Serre, of a Zariski open subset of a ruled surface over an elliptic curve, which is Stein, but not affine ([Ha] 6.3). This example plays an interesting role in complex analysis, for example in the theory of local cohomology of analytic sheaves (e.g. [KP]) and the theory of nef vector bundles ([DPS] 1.7). The purpose of this note is to extend this construction to the dimension 4 by interpreting it from the view-point of additive group action. Of course, it may be possible to have more direct generalization of Serre's construction to higher dimensions. But our approach via additive group action reveals a number of interesting features of the resulting 4-dimensional compact complex manifold. This complex manifold is interesting in the following aspects.

A well-known conjecture in the study of compactifications of $\mathbf{C}^n$ is the following:

Conjecture Every compactification of $\mathbf{C}^n$ is Moishezon. Namely, a compact complex manifold containing $\mathbf{C}^n$ as a Zariski open subset has n algebraically independent meromorphic functions.

Although this is true in dimension 2, it is completely open in higher-dimensions, except for some partial results in dimension 3 (cf. [PS]). Even under the additional assumption that the compactifying divisor is smooth, in which case it is conjectured that the compactification is $\mathbf{P}_n$, or under the assumption that the compactification is Kähler, the problem remains unsolved. One may ask the same question for compactifications of $(\mathbf{C}^*)^n$. But our construction will give a negative answer:

Theorem 1 *Let T be any complex torus of dimension 2. Then there exists a compact Kähler 4-fold $X = X(T)$ and a smooth divisor $D \subset X$ such that D is biholomorphic to $\mathbf{P}_1 \times T$ and $X - D$ is biholomorphic to $(\mathbf{C}^*)^4$.*

Since the image of a Moishezon manifold is Moishezon, if T is not an abelian variety, $X(T)$ is a compactification of $(\mathbf{C}^*)^4$ which is not Moishezon.

One partial answer to the above conjecture is the result of Gellhaus [Ge] that any equivariant compactification of $\mathbf{C}^n$ is Moishezon. In other words, if $\mathbf{C}^n$ acts on an n-dimensional compact complex manifold with a faithful orbit, the manifold has n algebraically independent meromorphic functions. One may ask the following question as a generalization of this result:

If $\mathbf{C}^n$ acts on an m-dimensional compact complex manifold, $m \geq n$, with a faithful orbit, does the manifold have at least n algebraically independent meromorphic functions?

In fact, answering question of this type is believed to be one of the possible approaches to the above conjecture. However, our manifold $X(T)$ gives a negative answer again. There is a $\mathbf{C}^2$-action on $X(T)$ with faithful orbits by construction, but

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Theorem 2 *For a general torus T, the manifold $X(T)$ has no non-constant meromorphic functions.*

Our example suggests that construction of meromorphic functions on the compactification of $\mathbf{C}^n$ may be a very delicate problem.

One of the advantage of the view-point of additive group action in our construction is that it can be easily generalized to other cases. In principle, when there is a quotient Y' of a complex manifold Y by a lattice in $\mathbf{C}^k$ we can get a $\mathbf{C}^k$-action on $(\mathbf{C}^*)^k \times Y$ and a compactification of $(\mathbf{C}^*)^k \times Y$ which is a $\mathbf{P}_k$-bundle over Y'. For example, it is straight-forward to generalize our construction to a compactification of $(\mathbf{C}^*)^{2n}$, which is a $\mathbf{P}_n$-bundle over an n-dimensional complex torus.

2 An approach to Serre's example via C-action

It is instructive first to give a construction of Serre's example from the view-point of $\mathbf{C}$-action on $\mathbf{C}^* \times \mathbf{C}^*$, to clarify the construction in Theorem 1.

Let $\alpha \in \mathbf{C} - \mathbf{R}$. Consider the $\mathbf{C}$-action on $\mathbf{C}^* \times \mathbf{C}^*$ given by

$$s \cdot (x, z) = (e^s x, e^{\alpha s} z) \tag{1}$$

Since α and 1 are independent over $\mathbf{Z}$, the map $s \mapsto s \cdot p$ is injective for any $p \in \mathbf{C}^* \times \mathbf{C}^*$, i.e., the action is faithful. Moreover, since α and 1 are independent over $\mathbf{R}$ the same map is actually proper. Indeed, if $\{s_j\}_{j\in\mathbf{N}}$ is a divergent sequence in $\mathbf{C}$ such that $\{\log |e^{s_j} x|\}_{j\in\mathbf{N}}$ is bounded, then the real part of s_j is confined to a strip of finite width in the s-plane for all j. Thus the imaginary part of s_j diverges. But since α has non-zero imaginary part, $\log |e^{\alpha s} z|$ is unbounded.

It follows from general theory of Lie group actions that the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the action (1) is a Riemann surface B. We claim in fact that it is an elliptic curve. Indeed, this action realizes $\mathbf{C}^* \times \mathbf{C}^*$ as a locally trivial $\mathbf{C}$-bundle over B, and thus in particular, B is homotopy equivalent to $\mathbf{C}^* \times \mathbf{C}^*$. Since every noncompact Riemann surface has no second homology, we see that B must be compact. The homology of B then forces it to be an elliptic curve. In fact, one can check that B is the torus $\mathbf{C}/(\mathbf{Z} + \alpha\mathbf{Z})$.

Now, since $\mathrm{Aut}(\mathbf{C})$ is an affine group, the bundle $\mathbf{C}^* \times \mathbf{C}^* \to B$ is an affine bundle with fibers $\mathbf{C}$. Thus we can attach ∞ to each fiber and obtain a $\mathbf{P}_1$-bundle over the elliptic curve B. Equivalently, the affine transition functions of the bundle $\mathbf{C}^* \times \mathbf{C}^* \to B$ can be homogenized so as to define a rank 2 vector bundle $E \to B$ whose projectivization $\mathbf{P}(E) \to B$ has a distinguished section, and the complement of this section is $\mathbf{C}^* \times \mathbf{C}^*$.

The construction outlined above can be carried out quite explicitly, and the reader is invited to do so and obtain in particular the following additional facts.

- The bundle $\mathbf{P}(E) \to B$ is real analytically isomorphic to $\mathbf{P}_1 \times B$. Thus the section of this bundle has self intersection 0. However, since the complement of this section is Stein, the section is holomorphically rigid.
- The vector bundle $E \to B$ is flat, i.e., it can be given transition functions which are locally constant.
- In fact, $E \to B$ is a non-split extension of $\mathcal{O}$ by $\mathcal{O}$, and thus the algebraic structure inherited by $\mathbf{C}^* \times \mathbf{C}^*$ from $\mathbf{P}(E)$ is not affine.

Remark. A famous problem in complex analysis is to determine whether or not $\mathbf{C}^* \times \mathbf{C}^*$ contains an open subset biholomophic to $\mathbf{C}^2$ ([RR] Appendix). Perhaps one can show that no open subset of $\mathbf{P}(E)$ is biholomorphic to $\mathbf{C}^2$.

3 Proof of theorem 1

Every complex torus T is biholomorphic to one of the form $\mathbf{C}^2/(\mathbf{Z}e_1+\mathbf{Z}e_2+\mathbf{Z}\lambda+\mathbf{Z}\mu)$, where e_1, e_2 are the standard unit vectors in $\mathbf{C}^2$ and $\{e_1, e_2, \lambda, \mu\}$ are independent over $\mathbf{R}$. We call $\{\lambda, \mu\}$ *normalized lattice vectors* for T. In terms of normalized lattice vectors $\lambda = (\lambda^1, \lambda^2)$ and $\mu = (\mu^1, \mu^2)$, T can be obtained as a quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the $\mathbf{Z}^2$ action

$$(m, n) \cdot (z, w) = (ze^{2\pi\sqrt{-1}(m\lambda^1+n\mu^1)}, we^{2\pi\sqrt{-1}(m\lambda^2+n\mu^2)}).$$

We denote the quotient map by $(z, w) \mapsto [z, w]$.

Fix a torus $T = \mathbf{C}^2/(\mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}\lambda + \mathbf{Z}\mu)$ and consider the following $\mathbf{C}^2$ action on $(\mathbf{C}^*)^4$.

$$(s, t) * (x, y, z, w) := (e^s x, e^t y, e^{\lambda^1 s+\mu^1 t} z, e^{\lambda^2 s+\mu^2 t} w), \tag{2}$$

where $\lambda = \lambda^1 e_1 + \lambda^2 e_2$, and similarly for μ.

First, notice that this is a faithful action. Indeed, for fixed $p \in (\mathbf{C}^*)^4$, if $(s, t) * p = (s', t') * p$ then

$$\begin{aligned} s - s' &= 2\pi\sqrt{-1}m_1 \\ t - t' &= 2\pi\sqrt{-1}m_2 \\ (\lambda^1 s + \mu^1 t) - (\lambda^1 s' + \mu^1 t') &= 2\pi\sqrt{-1}k_1 \\ (\lambda^2 s + \mu^2 t) - (\lambda^2 s' + \mu^2 t') &= 2\pi\sqrt{-1}k_2 \end{aligned}$$

for some integers m_1, m_2, k_1, k_2. Thus $m_1\lambda + m_2\mu - k_1 e_1 - k_2 e_2 = 0$ and so $m_1 = m_2 = k_1 = k_2 = 0$.

It is possible to show directly, as in section 2, that the map $(s, t) \mapsto (s, t) * p$ is an embedding of $\mathbf{C}^2$ into $(\mathbf{C}^*)^4$. This also follows from the next proposition. Let $\pi : (\mathbf{C}^*)^4 \to T$ be the holomorphic map defined by

$$\pi(x, y, z, w) = \left[ze^{-(\lambda^1 \log x+\mu^1 \log y)}, we^{-(\lambda^2 \log x+\mu^2 \log y)}\right].$$

Proposition 3.1 *The map π is the quotient map for the action $*$. That is to say,*

$$\pi(x, y, z, w) = \pi(x', y', z', w') \iff (s, t) * (x, y, z, w) = (x', y', z', w')$$

for some $(s, t) \in \mathbf{C}^2$.

*Proof.*Suppose $\pi(x, y, z, w) = \pi(x', y', z', w')$. Then

$$\frac{z'}{z} = e^{\left[\lambda^1(\log(x'/x)+2\pi\sqrt{-1}c)+\mu^1(\log(y'/y)+2\pi\sqrt{-1}d)\right]}$$

and

$$\frac{w'}{w} = e^{\left[\lambda^2(\log(x'/x)+2\pi\sqrt{-1}c)+\mu^2(\log(y'/y)+2\pi\sqrt{-1}d)\right]}$$

for some integers c and d. The reader can observe that it is possible to choose a single branch of the logarithm so that all the numbers appearing in these equations make sense. Now let $s = \log(x'/x) + 2\pi\sqrt{-1}c$ and $t = \log(y'/y) + 2\pi\sqrt{-1}d$. Then

$$\frac{x'}{x} = e^s \qquad \text{and} \qquad \frac{y'}{y} = e^t,$$

and so $(s, t) * (x, y, z, w) = (x', y', z', w')$. $\square$

From general theory of Lie group actions, it follows that the bundle $\pi : (\mathbf{C}^*)^4 \to T$ is a locally trivial $\mathbf{C}^2$ bundle. However, we will show this more directly.

To this end, let $D = D_1$ be a fundamental domain of T in $\mathbf{C}^2$, e.g., D is the convex hull of the 16 vertices $\{b_1e_1 + b_2e_2 + b_3\lambda + b_4\mu \mid b_1, b_2, b_3, b_4 \in \{0,1\}\ \}$, and let $D_2, ..., D_N$ be translates of D in $\mathbf{C}^2$ such that

$$\overline{D} \subset \bigcup_{j=1}^{N} D_j.$$

Let $\mathcal{D}_j$ be the image of D_j in $\mathbf{C}^* \times \mathbf{C}^*$ under the map $(z,w) \mapsto (e^{2\pi\sqrt{-1}z}, e^{2\pi\sqrt{-1}w})$. The restriction to $\mathcal{D}_j$ of the projection $p : \mathbf{C}^* \times \mathbf{C}^* \to T$ is biholomorphic onto its image Δ_j. The bundle structure of $(\mathbf{C}^*)^4 \to T$ is now defined as follows. Let $Y_j = \pi^{-1}(\Delta_j) \subset (\mathbf{C}^*)^4$ and let $\varphi_j : \Delta_j \times \mathbf{C}^2 \to Y_j$ be given as follows. Suppose $(\zeta, \eta) \in \mathcal{D}_j$. Then

$$\varphi_j([\zeta,\eta],(s,t)) = (e^s, e^t, e^{\lambda^1 s+\mu^1 t}\zeta, e^{\lambda^2 s+\mu^2 t}\eta). \tag{3}$$

This map is well defined because D_j is a fundamental domain, and thus $\mathcal{D}_j$ contains a unique (ζ,η) projecting onto $[\zeta,\eta]$.

It can be verified that the map φ_j is biholomorphic, but we will actually write down the inverse. To this end, fix $\xi = (x,y,z,w) \in \pi^{-1}(\Delta_j)$, and choose a branch of log such that $\log x$ and $\log y$ are well defined. Define the integers $m = m_j(\xi)$ and $n = n_j(\xi)$ to be those integers such that

$$\left(e^{-(\lambda^1 \log x+\mu^1 \log y+2\pi\sqrt{-1}(\lambda^1 m+\mu^1 n))}z, e^{-(\lambda^2 \log x+\mu^2 \log y+2\pi\sqrt{-1}(\lambda^2 m+\mu^2 n))}w\right) \in \mathcal{D}_j.$$

Then

$$\begin{aligned}\varphi_j^{-1}(\xi) \quad = \quad & \Big(\Big[e^{-(\lambda^1 \log x+\mu^1 \log y+2\pi\sqrt{-1}(\lambda^1 m+\mu^1 n))}z, e^{-(\lambda^2 \log x+\mu^2 \log y+2\pi\sqrt{-1}(\lambda^2 m+\mu^2 n))}w\Big], \\ & \log x + 2\pi\sqrt{-1}m_j(\xi), \log y + 2\pi\sqrt{-1}n_j(\xi)\Big).\end{aligned}$$

We leave it to the reader to verify that φ_j^{-1} is well defined. The main thing is that φ_j^{-1} is continuous, even though the chosen branch of log, as well as m and n, are not.

It follows from this discussion that the transition functions $g_{ij} = \varphi_j^{-1} \circ \varphi_i$ for π are of the form

$$g_{ij}([z,w])(s,t) = (s + 2\pi\sqrt{-1}m_{ij}, t + 2\pi\sqrt{-1}n_{ij}) \tag{4}$$

for some integers m_{ij} and n_{ij}. In particular, they are locally constant. We summarize this as follows.

Proposition 3.2 *The fiber bundle $\pi : (\mathbf{C}^*)^4 \to T$ is affine and flat.*

The transition functions g_{ij} for the affine bundle above can be used to construct a vector bundle $E \to T$ whose transition functions are given by

$$G_{ij}([z,w]) \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2\pi\sqrt{-1}m_{ij} & 1 & 0 \\ 2\pi\sqrt{-1}n_{ij} & 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ s \\ t \end{pmatrix}$$

Evidently the projectivization $X := \mathbf{P}(E)$ of E is a $\mathbf{P}_2$ bundle over T. Moreover, even though the coordinate functions r, s, t are not globally defined, the divisor $D = (r = 0) \subset X$ is well defined, i.e.,

$$G_{ij}([z,w]) \begin{pmatrix} 0 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ t \end{pmatrix}$$

From this, we also see that D is a trivial $\mathbf{P}_1$-bundle over T. Since the projectivization of a vector bundle over a compact Kähler manifold is itself Kähler, X is Kähler. The proof of Theorem 1 is complete.

4 Proof of theorem 2

Let us return to the affine bundle $\pi : (\mathbf{C}^*)^4 \to T$ in Proposition 3.2. Though somewhat abusive, we will also denote the $\mathbf{P}_2$-bundle $X \to T$ by π.

Lemma 4.1 *The affine bundle π does not have a section or an affine subbundle of rank 1.*

*Proof.*A section of π gives a compact complex torus in $(\mathbf{C}^*)^4$, which contradicts the maximum principle. Under the local trivialization $\varphi_j : \Delta_j \times \mathbf{C}^2 \to Y_j \subset (\mathbf{C}^*)^4$ in (3), an affine subbundle is given by a linear equation

$$a_j + b_j s + c_j t = 0$$

where a_j, b_j, c_j are holomorphic functions on Δ_j. The transition functions 4 give the relations

$$a_i + b_i s + c_i t = a_j + b_j(s + 2\pi\sqrt{-1}m_i j) + c_j(t + 2\pi\sqrt{-1}n_i j).$$

We then have that b_j and c_j define global holomorphic functions on T. Thus they are constant and the functions a_j on Δ_j satisfy

$$a_i - a_j = bm_i j + cn_{ij}.$$

Thus the $\mathbf{Z}$-valued cocyles $\{m_{ij}\}$ and $\{n_{ij}\}$ become linearly dependent in $H^1(T, \mathcal{O})$.

But the $\mathbf{C}^2$-bundle π is precisely the quotient of the trivial $\mathbf{C}^2$-bundle on the universal cover $\mathbf{C}^2$ of T where $\gamma \in \Gamma$ acts by $(p, q) \mapsto (p + \gamma, q + \gamma)$. Thus the two cocycles are linearly independent. □

For the rest of this section, we assume that T has no nonconstant meromorphic functions or curves, and that every line bundle on T is flat. This is true for a general choice of T.

Lemma 4.2 *There cannot be two algebraically independent meromorphic functions on X.*

*Proof.*To obtain a contradiction, suppose that f and g are two independent meromorphic functions on X. Since T has no nonconstant meromorphic function, possibly after perturbing f and g, we can assume that there is an irreducible component Z of the variety $(f = g = 0)$ whose intersection with the generic fiber of π is a finite set disjoint from D, the compactifying divisor. Let $\mathcal{A} \subset T$ be the set of points t such that either $\pi^{-1}(t) \cap Z$ is not finite, or else $\pi^{-1}(t) \cap Z \cap D \neq \emptyset$. Since $\mathcal{A}$ is a proper analytic subvariety of T, it must be finite.

For $t \in T - \mathcal{A}$, let ζ_t be the center of mass of the set-with-multiplicity $\pi^{-1}(t) \cap Z$, and let Z' be the set $\{\zeta_t \mid t \in T - \mathcal{A}\}$. Then Z' is a holomorphic section of the $\mathbf{C}^2$-bundle $\mathbf{P}(E) - D = (\mathbf{C}^*)^4$ over $T - \mathcal{A}$, and thus extends to a section of $\mathbf{P}(E) - D$ over T by Hartogs extension, a contradiction to Lemma 4.1. □

Let us say that a meromorphic function f on M has fiberwise linear levels if for each $c \in \mathbf{P}_1$, the level sets $(f = c)$ intersect the fibers of π in hyperplanes.

Lemma 4.3 *If f is a nonconstant meromorphic function on X, then f has fiberwise linear levels.*

*Proof.*Note first that the $\mathbf{C}^2$-action (2) on $(\mathbf{C}^*)^4$ extends holomorphically to an action on X, which fixes D pointwise. Moreover, the action preserves fibers and is linear on them. If f is a meromorphic function on X with non-linear fiber levels, then by pulling back f with the $\mathbf{C}^2$ action, we could produce a second meromorphic function g with different level foliation on the fibers. Thus g and f would be algebraically independent, contradicting lemma 4.2. □

An easy consequence of the assumption that every line bundle on T is flat is

Lemma 4.4 *Let L be a line bundle on T. If there exists a non-zero map of line bundles $L \to \mathcal{O}$, then $L = \mathcal{O}$.*

Now we can complete the proof of Theorem 2 by

Lemma 4.5 *The manifold X has no meromorphic functions with fiberwise linear levels.*

Proof. A level set of such a meromorphic function defines a rank-2 subbundle $F \subset E$ such that $\mathbf{P}F \neq D$. From the transition functions, we have the exact sequence

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0.$$

By Lemma 4.4, F must surject to $\mathcal{O}$ and $\mathbf{P}F \cap (\mathbf{P}E - D)$ defines a rank-1 affine subbundle of $\pi : (\mathbf{C}^*)^4 \to T$, a contradiction to Lemma 4.1. $\square$

References

[DPS] Demailly, J.-P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. J. Alg. Geom. **3** (1994) 295-345

[G] Gellhaus, C.: Äquivariante Kompaktifizierungen des $\mathbf{C}^n$. Math. Zeit. **206** (1991) 211-217

[H] Hartshorne, R.: *Ample subvarieties of algebraic varieties.* Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin-Heidelberg-New York 1970

[KP] Kosarew, S., Peternell, T.: Formal cohomology, analytic cohomology and non-algebraic manifolds. Compositio Math. **74** (1990) 299-325

[PS] Peternell, T., Schneider, M.: Compactifications of $\mathbf{C}^n$: A survey. Proc. Symp. Pure Math. **52** Part 2 (1991) 455-466

[RR] Rosay, J.-P., Rudin, W.: Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. Trans. Amer. Math. Soc. **310** (1988) 47-86

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