NEW ESTIMATES FOR THE MINIMAL $\mathbf{L^2}$ SOLUTION OF $\bar{\partial}$ AND APPLICATIONS TO GEOMETRIC FUNCTION THEORY IN WEIGHTED BERGMAN SPACES

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1. INTRODUCTION

The goal of this paper is to study two problems of geometric function theory in weighted Bergman spaces in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. We treat both problems by making use of a theorem, which we prove, that gives improved estimates for the solution of minimal L^2 norm for the $\bar{\partial}$ equation. The technique we use to establish these improved estimates is a new method which we call *double twisting*. The double twisting technique is broad in scope, and should have many more applications, which we hope to demonstrate in future work.

The particular kind of geometry we are interested in primarily concerns $\mathscr{C}^2$-smooth weights φ whose curvature forms $dd^c\varphi$ satisfy

$$0 < m\Theta \leq dd^c\varphi \leq M\Theta \tag{1}$$

for some positive constants $m < M$, where $\Theta := -dd^c \log(1 - |z|^2)$ is the Bergman Kähler metric on the unit ball. The weighted Bergman spaces in question are the Banach spaces $\mathscr{F}^p(dV_\Theta, \varphi)$ consisting of functions $f \in \mathcal{O}(\mathbb{B})$ such that

$$\int_{\mathbb{B}} |f|^p e^{-p\varphi} dV_\Theta < +\infty, \quad 1 \leq p < \infty, \quad \text{or} \quad \sup_{\mathbb{B}} |f| e^{-\varphi} < +\infty$$

in the case $p = \infty$. Here dV_Θ is the volume form associated to Θ. These Banach spaces lie inside the larger spaces $L^p(dV_\Theta, \varphi)$, defined in the same way except that the functions are just measurable instead of holomorphic. For $p = 2$ one has the Bergman projection $P : L^2(dV_\Theta, \varphi) \to \mathscr{F}^2(dV_\Theta, \varphi)$, which is an integral operator whose integral kernel K is called the *Bergman kernel* (c.f. Section 3).

Our first main result is the following theorem on the off-diagonal decay of the Bergman kernel.

THEOREM 1. *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy*

$$(n + 2\sigma)\,\Theta \leq 2dd^c\varphi \leq M\Theta \tag{2}$$

for some constant $\sigma > 1/4$. Let K denote the Bergman kernel for $P : L^2(dV_\Theta, \varphi) \to \mathscr{F}^2(dV_\Theta, \varphi)$. Define

$$\alpha_\sigma := \frac{1 + \sqrt{4\sigma - 1}}{2}.$$

Then there is a constant $C > 0$ depending only on M and σ such that for all $z, w \in \mathbb{B}$,

$$|K(z, \bar{w})| e^{-\varphi(z) - \varphi(w)} \leq C e^{-\alpha_\sigma d_\Theta(z,w)},$$

where d_Θ is the so-called hyperbolic distance, i.e., the Riemannian distance in the Kähler metric Θ.

Theorem 1 is an analog for the unit ball of a theorem of Christ ($n = 1$) [C-1991] and Delin ($n \geq 2$) [D-1998] in $\mathbb{C}^n$. It also bears resemblance to related results in the unweighted case, starting with work going back to Kerzman [Ke-1972], and to more recent and considerably harder results of McNeal [M-1994]. Finally we note that, while in the $\mathbb{C}^n$ case the function $z \mapsto K(z, 0)e^{-\varphi(z)-\varphi(0)}$ is in L^p for any $p \in (0, \infty]$,

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this is not the case for the Bergman kernel of the orthogonal projection $P : L^2(dV_\Theta, \varphi) \to \mathscr{F}^2(dV_\Theta, \varphi)$, no matter how large $\varepsilon > 0$ is, as the reader can see by looking at the case $\varphi(z) = c\log(1-|z|^2)$ for $c > 0$ (c.f. Example 8.3 and Remark 8.4). Therefore applications to questions of boundedness and compactness of Toeplitz operators would have a very different form from the results established in [SV-2011]. The reader interested in the questions of boundedness and compactness of Toeplitz operators should have a look at the preprint [ARS-2011] of Abate, Raissy and Saracco, who work mostly in unweighted spaces but over more general domains.

Our second main theorem concerns L^∞ estimates for the $\bar\partial$-equation.

THEOREM 2. *For any pair of constants $M, \varepsilon > 0$ there is a constant $C > 0$ such that if $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfies*

$$\left(\tfrac{n}{2} + \tfrac{1}{4} + \varepsilon\right)\Theta \le dd^c\varphi \le M\Theta \tag{3}$$

and θ is a $(0,1)$-form such that

$$\bar\partial\theta = 0 \quad \textit{and} \quad \sup_B |\theta|_\Theta e^{-\varphi} < +\infty,$$

then there is a function u such that

$$\bar\partial u = \theta \quad \textit{and} \quad \sup_B |u|e^{-\varphi} \le C \sup_B |\theta|_\Theta e^{-\varphi}.$$

Here $|\cdot|_\Theta$ is the pointwise metric for $(0,1)$-forms induced from the Kähler form Θ.

Let us make a few remarks about these results. Theorem 2 is an improvement of a result of Berndtsson [B-1997, Corollary 2'], in the sense that the latter has the somewhat stronger assumption

$$\left(\tfrac{(n+1)^2}{2} + \varepsilon\right)\Theta \le dd^c\varphi$$

in place of (3).

In fact, both of the above theorems make some use Kohn's solution of the $\bar\partial$-Neumann problem, as we will explain later. More precisely, we prove that the solution of minimal norm for the $\bar\partial$-Neumann problem satisfies some rather strong L^2-estimates. In fact, we have the following theorem.

THEOREM 3. *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ be a weight in the unit ball satisfying*

$$(n + 2\sigma)\Theta \le 2dd^c\varphi \le M\Theta$$

for some numbers $M > 0$ and $\sigma > 1/4$. Let θ be a smooth $(0,1)$-form on $\mathbb{B}$ such that

$$\bar\partial\theta = 0 \quad \textit{and} \quad \int_{\mathbb{B}} |\theta|^2_\Theta e^{-2\varphi} dV_\Theta < +\infty.$$

Let $w \in \mathbb{B}$ be any point. Then the solution U of the equation $\bar\partial U = \theta$ whose norm

$$\int_{\mathbb{B}} |U|^2 e^{-2\varphi} dV_\Theta$$

is minimal, also satisfies the estimate

$$\int_{\mathbb{B}} e^{-2\alpha_\sigma d_\Theta(\cdot,w)} |U|^2 e^{-2\varphi} dV_\Theta \le C \int_{\mathbb{B}} e^{-2\alpha_\sigma d_\Theta(\cdot,w)} |\theta|^2_\Theta e^{-2\varphi} dV_\Theta,$$

where the constant C is independent of w, of θ and of φ.

Theorem 3 is a consequence of the main new technique developed in the present article: the double twisting technique. The technique is a merger of two different twisting techniques that are known in the literature; the first, introduced by Ohsawa and further developed by a number of people, is rather well-known and has been used to great effect in many areas of complex analysis, complex geometry and algebraic geometry. The second technique, introduced independently by Berndtsson and McNeal, is less well-known, and has not been internalized in the same way, in part because some of the things one can establish with the second method can also be established with the first twisting method. An explanation, and some examples, of these two twisting techniques and what one does with them, is given in Section 5. A clear understanding of the two methods is important for the derivation of the double-twisted techinque developed in the following section, namely Section 6.

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2. REVIEW OF GEOMETRY IN THE UNIT BALL

2.1. **Metrics and measures.** Let $\mathbb{B}$ (or $\mathbb{B}_n$ if needed) denote the unit ball in $\mathbb{C}^n$. We may sometimes denote the unit disk $\mathbb{B}_1$ by $\mathbb{D}$. A ball of radius r and center z in $\mathbb{C}^n$ is denoted $\mathbb{B}(z,r)$. Define the functions

$$\rho_w(z) =: \rho(z,w) := \log \frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}, \quad w \in \mathbb{B}.$$

In particular, $\rho_o(z) := \rho(z,0)$. We write

$$\omega_e = dd^c|z|^2 \qquad \text{and} \qquad \Theta := dd^c\rho_o$$

for the Euclidean Kähler form in $\mathbb{C}^n$ and the *Bergman* Kähler form in $\mathbb{B}$, respectively. Here and below,

$$d^c = \frac{\sqrt{-1}}{4}(\partial - \bar\partial), \quad \text{and thus} \quad dd^c = \frac{\sqrt{-1}}{2}\partial\bar\partial.$$

Let λ denote Lebesgue measure in $\mathbb{C}^n$;

$$d\lambda = \frac{\omega_e^n}{n!}$$

is the measure associated to the Euclidean volume form. Similarly

$$\begin{aligned} dV_\Theta := \frac{\Theta^n}{n!} &= \frac{1}{n!2^n}\left(\frac{\sum dz^j \wedge d\bar z^j}{1-|z|^2} + \frac{\bar z \cdot dz \wedge z \cdot d\bar z}{(1-|z|^2)^2}\right)^n \\ &= \frac{1}{n!2^n}\left(\left(\frac{\sum dz^j \wedge d\bar z^j}{1-|z|^2}\right)^n + n\left(\frac{\sum dz^j \wedge d\bar z^j}{1-|z|^2}\right)^{n-1} \wedge \frac{\bar z \cdot dz \wedge z \cdot d\bar z}{(1-|z|^2)^2}\right) \\ &= \frac{1}{n!}\left(n!\frac{d\lambda(z)}{(1-|z|^2)^n} + n\cdot(n-1)!\frac{|z|^2 d\lambda(z)}{(1-|z|^2)^{n+1}}\right) \\ &= e^{(n+1)\rho_o(z)}d\lambda(z) \end{aligned}$$

is the volume form associated to the Bergman metric.

As usual, the length $|\alpha|_\Theta$ of a $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_{\bar j} d\bar z^j$ induced by the Bergman metric is defined by

$$|\alpha|^2_\Theta \frac{\Theta^n}{n!} = \frac{1}{2(n-1)!}\overline{\alpha} \wedge \alpha \wedge \Theta^{n-1}$$

A calculation shows that

$$|\alpha|^2_\Theta = (1-|z|^2)^2\left(\sum_{j=1}^n |\alpha_{\bar j}|^2\right) + (1-|z|^2)\left(\sum_{1\le m,p\le n} |z^m\alpha_{\bar p} - z^p\alpha_{\bar m}|^2\right).$$

In particular, for any function $h : \mathbb{B} \to \mathbb{C}$ the $(0,1)$-form $h(z)\bar{\partial}|z|^2$ has Bergman square-length

$$\left|h(z)\bar{\partial}|z|^2\right|_\Theta^2 = (1-|z|^2)^2 \sum_{j=1}^{n} |h(z)z^j|^2. \tag{4}$$

2.2. **Automorphisms of $\mathbb{B}$.** Recall that the group $\mathrm{Aut}(\mathbb{B})$ of holomorphic diffeomorphisms, or *automorphisms*, of the unit ball in $\mathbb{C}^n$ contains the involutions

$$F_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \quad a \in \mathbb{B} - \{0\}, \quad F_0(z) = -z,$$

where $P_a z = \frac{\langle z,a\rangle}{|a|^2} a$, $Q_a = I - P_a$ and $s_a = \sqrt{1-|a|^2}$. Moreover, the Schwarz Lemma shows that any automorphism of $\mathbb{B}$ is of the form UF_a for some unitary U. Note that, since $F_a(0) = a$, $\mathrm{Aut}(\mathbb{B})$ acts transitively on the unit ball. Next, the formula

$$1 - |F_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\langle z,a\rangle|^2}, \tag{5}$$

which is easily checked, implies that

$$F_a^*\Theta = dd^c F_a^*\rho_o = dd^c\rho_a = \Theta.$$

Thus Θ is $\mathrm{Aut}(\mathbb{B})$-invariant. In particular, if θ is a $(0,1)$-form then $\overline{(F_z^*\theta)} \wedge F_z^*\theta \wedge \Theta^{n-1} = F_z^*\left(\overline{\theta} \wedge \theta \wedge \Theta^{n-1}\right)$, which shows that

$$|F_z^*\theta|_\Theta^2 = F_z^*|\theta|_\Theta^2. \tag{6}$$

2.3. **The hyperbolic distance.** The Riemannian distance, with respect to the Bergman metric Θ, between two points is invariant under automorphisms of the ball, i.e., the distance between two points z and w is the same as the distance between 0 and $UF_z(w)$, where U is any unitary transformation. Since there exists a unitary transformation such that $UF_z(w) = |F_z(w)|e_1$, it suffices to compute the distance between 0 and λe_1, where $\lambda > 0$. The latter is the length of the curve $t \mapsto te_1$, $0 \le t \le \lambda$, which computes as

$$d_\Theta(0, \lambda e_1) = \int_0^\lambda \frac{dt}{1-t^2} = \frac{1}{2}\log\frac{1+\lambda}{1-\lambda}.$$

Thus

$$d_\Theta(z,w) = \frac{1}{2}\log\frac{1+|F_z(w)|}{1-|F_z(w)|}.$$

In particular, note that

$$0 \le d_\Theta(z,w) - \frac{1}{2}\rho(z,w) \le \log 2. \tag{7}$$

2.4. **The pseudo-hyperbolic distance.** The function

$$\delta(z,w) := |F_z(w)|$$

has the following properties.

(P1) (Symmetry) $\delta(z,w) = \delta(w,z)$.
(P2) (Non-degeneracy) $\delta(z,w) = 0$ if and only if $z = w$.
(P3) (Group invariance) $\delta(z,w) = \delta(0, F_z(w))$.
(P4) (Triangle inequality) $\delta(x,z) \le \delta(x,y) + \delta(y,z)$.

The first three properties are rather straightforward to verify, and the fourth is a little more technical but still elementary. The proof of (P4) can be found in [St-94, Lemma 7.3].

DEFINITION 2.1. The function $\delta : \mathbb{B} \times \mathbb{B} \to [0, \infty)$ is called the *pseudo-hyperbolic distance*. For $z \in \mathbb{B}$ and $r \in (0, 1)$, the set

$$\mathbb{E}(z, r) := \{\zeta \in \mathbb{B} \; ; \; \delta(z, \zeta) < r\}$$

is called the pseudo-hyperbolic ball of radius r and center z. $\diamond$

REMARK 2.2. Note that

$$d_\Theta = \frac{1}{2} \log \frac{1+\delta}{1-\delta} \qquad \text{and} \qquad \delta = \tanh(e^{d_\Theta}).$$

In particular a pseudo-hyperbolic ball of radius r is a hyperbolic ball of radius $\frac{1}{2} \log \frac{1+r}{1-r}$. $\diamond$

3. GENERALIZED BERGMAN SPACES: BACKGROUND AND NOTATION

3.1. **Generalized Bergman spaces.** Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ be a function, μ a measure on $\mathbb{B}$, and $p \in [1, \infty)$. We define the spaces

$$L^p(\mu, \varphi) := \left\{ f \; ; \; ||f||_p := \left(\int_{\mathbb{B}} |f|^p e^{-p\varphi} d\mu \right)^{1/p} < +\infty \right\}$$

and set

$$\mathscr{F}^p(\mu, \varphi) := L^p(\mu, \varphi) \cap \mathcal{O}(\mathbb{B}).$$

Similarly one can define

$$L^\infty(\mu, \varphi) = \{ f \; ; \; \mu\text{-Ess. Sup.} |f e^{-\varphi}| < +\infty \} \qquad \text{and} \qquad \mathscr{F}^\infty(\mu, \varphi) := L^\infty(\mu, \varphi) \cap \mathcal{O}(\mathbb{B}).$$

Under reasonable assumptions for the measure μ, the subspaces $\mathscr{F}^p(\mu, \varphi) \subset L^p(\mu, \varphi)$ are closed for all $1 \le p \le \infty$. These spaces are called *generalized Bergman spaces*; the space $\mathscr{F}^p(d\lambda, 0)$ is the classical Bergman space.

If $p = 2$ we have a bounded orthogonal projection $L^2(\mu, \varphi) \to \mathscr{F}^2(\mu, \varphi)$, called the *Bergman projection*. This projection is an integral operator, whose integral kernel, called the *Bergman kernel*, is denoted $K(z, \bar{w})$ in the present paper, or $K_{\mu,\varphi}(z, \bar{w})$ when we wish to emphasize the weight and measure that define the underlying Hilbert space.

3.2. **The Bergman kernel.** Let $\varphi : \mathbb{B} \to \mathbb{R}$ be a $\mathscr{C}^2$-smooth weight function such that

$$2\delta\Theta \le 2dd^c\varphi \le M\Theta$$

for some positive constants δ and M. The subspace $\mathscr{F}^2(dV_\Theta, \varphi)$ of $L^2(dV_\Theta, \varphi)$ is closed (as one can see from (9) of Proposition 4.4 and Montel's Theorem), and the Bergman projection

$$P = P_{dV_\Theta, \varphi} : L^2(dV_\Theta, \varphi) \to \mathscr{F}^2(dV_\Theta, \varphi)$$

is the integral operator whose kernel $K(z, \bar{w}) = K_{dV_\Theta, \varphi}(z, \bar{w})$ is given by

$$K(z, \bar{w}) = \sum_{j=1}^{\infty} f_j(z) \overline{f_j(w)},$$

where $\{f_j\}$ is any orthonormal basis for $\mathscr{F}^2(dV_\Theta, \varphi)$. For a fixed $z \in \mathbb{B}$ we can select a basis $\{g_j\}_{j \ge 2}$ for the subspace $S_z \subset \mathscr{F}^2(dV_\Theta, \varphi)$ of all weighted-squared-integrable holomorphic functions vanishing at z. The inequality (9) of Proposition 4.4 (for $p = 2$) shows that evaluation at z is a bounded linear functional (where on the set $\mathbb{C}$ of possible values $f(z)$ of all holomorphic functions $f \in \mathscr{F}^2(dV_\Theta, \varphi)$ we put the norm $|\cdot| e^{-\varphi(z)}$) and therefore S_z has codimension 1 or 0. In view of Proposition 5.6, there are non-vanishing weighted square-integrable holomorphic functions at any point z, and hence $\mathscr{F}^2(dV_\Theta, \varphi) = S_z \oplus \mathbb{C} f_1$ for some $f_1 \in \mathscr{F}^2(dV_\Theta, \varphi)$ with $||f_1||_{2,\varphi} = 1$, unique up to a unimodular constant factor. We therefore have

$$K(z, \bar{w}) = f_1(z)\overline{f_1(w)} + \sum_{j \ge 2} g_j(z)\overline{g_j(w)} = f_1(z)\overline{f_1(w)}.$$

We note in particular that

$$K(z,\bar z)=\sup_{||f||=1}|f(z)|^2, \tag{8}$$

and that the supremum is a maximum.

3.3. **Invariance property.** We start with the following trivial proposition.

PROPOSITION 3.1. *Let $\varphi\in\mathscr{C}^2(\mathbb{B})$ and set $\varphi_z(\zeta):=\varphi\circ F_z(\zeta)$. Then the Bergman kernels $K_{dV_\Theta,\varphi}$ satisfy the relation*

$$K_{dV_\Theta,\varphi}(z,w)e^{-\varphi(z)-\varphi(w)}=K_{dV_\Theta,\varphi_z}(0,F_z(w))e^{-\varphi_z(0)-\varphi_z(F_z(w))},$$

and in particular $K_{dV_\Theta,\varphi}(z,\bar z)e^{-2\varphi(z)}=K_{dV_\Theta,\varphi_z}(0,0)e^{-2\varphi_z(0)}$ for all $z,w\in\mathbb{B}$.

Proof. Observe that if $F_z\in\mathrm{Aut}(\mathbb{B})$ and $f\in\mathscr{F}^2(dV_\Theta,\varphi)$ then

$$\int_{\mathbb{B}}|f(\zeta)|^2e^{-2\varphi(\zeta)}dV_\Theta(\zeta)=\int_{\mathbb{B}}|f(F_z(w))|^2e^{-2\varphi\circ F_z(w)}dV_\Theta(w).$$

Thus the basis $\{f_j\circ F_z\}$ is orthonormal for $\mathscr{F}^2(dV_\Theta,\varphi_z)$ whenever $\{f_j\}$ is an orthonormal basis for $\mathscr{F}^2(dV_\Theta,\varphi)$. We therefore have that

$$\begin{aligned}K_{dV_\Theta,\varphi_z}(0,F_z(w))e^{-\varphi_z(0)-\varphi_z(F_z(w))} &=\sum_j f_j(F_z(0))\overline{f_j(F_z(F_z(w)))}e^{-\varphi\circ F_z(0)-\varphi\circ F_z(F_z(w))}\\ &=\sum_j f_j(z)\overline{f_j(w)}e^{-\varphi(z)-\varphi(w)}\\ &=K(z,\bar w)e^{-\varphi(z)-\varphi(w)}.\end{aligned}$$

This completes the proof. □

REMARK 3.2. Because $dd^c\varphi_z=F_z^*dd^c\varphi$ and Θ is $\mathrm{Aut}(\mathbb{B})$-invariant, Proposition 3.1 is particularly useful for curvature constraints like those of (1). ◇

4. WEIGHTS WHOSE CURVATURE IS COMPARABLE TO THE BERGMAN METRIC FORM

4.1. **Uniform estimates for the dd^c equation.** The following lemma is fundamental. It is similar to but slightly stronger than Lemma 6 in [L-2001]. Our proof is slightly different as well.

LEMMA 4.1. *There exists a constant $C>0$ with the following property. Let ω be a $\mathscr{C}^2$-smooth, closed $(1,1)$-form on a neighborhood of the closed half-ball $\overline{\mathbb{B}(0,1/2)}$ in $\mathbb{C}^n$, such that*

$$-M\Theta\le\omega\le M\Theta$$

for some positive constant M. Then there exist a function $\psi\in\mathscr{C}^2(\mathbb{B}(0,1/2))$ such that

$$dd^c\psi=\omega\quad\textit{and}\quad\sup_{\mathbb{B}(0,1/2)}(|\psi|+|d\psi|)\le CM.$$

Proof. We assume that ω has compact support in $\mathbb{B}(0,3/4)$. Suppose first that $n=1$. Then one takes

$$\mathbb{D}(0,1/2)\ni z\mapsto\psi(z):=\int_{\mathbb{D}}\log|z-\zeta|^2\omega(\zeta).$$

Note that $\omega=h\Theta$ for some real-valued function h. A standard argument using integration-by-parts shows that

$$\frac{(1-|z|^2)^2}{\pi}\frac{\partial^2\psi}{\partial z\partial\bar z}=h.$$

The function ψ is clearly bounded by the constant

$$M \sup_{z\in\mathbb{D}(0,1/2)} \int_{\mathbb{D}(0,3/4)} \left|\log|\zeta - z|^2\right| \frac{dA(\zeta)}{(1-|\zeta|^2)^2}$$

while the derivative is controlled by

$$M \sup_{z\in\mathbb{D}(0,1/2)} \int_{\mathbb{D}(0,3/4)} \frac{|z-\zeta|^{-1}dA(\zeta)}{(1-|\zeta|^2)^2}.$$

Thus we have the stated result.

In higher dimensions, write $\omega = \omega_{i\bar{j}}\frac{\sqrt{-1}}{2}dz^i \wedge d\bar{z}^j$. Then as in the 1-dimensional case, the function

$$\psi(z) := \int_{\mathbb{D}} \omega_{1\bar{1}}(\zeta, z^2, ..., z^n)\log|z^1 - \zeta|^2 dA(\zeta)$$

then satisfies

$$\frac{1}{\pi}\frac{\partial^2\psi}{\partial z^1\partial\bar{z}^1} = \omega_{1\bar{1}}.$$

From the condition $d\omega = 0$, we see that, when i and j are both different from 1,

$$\begin{aligned} \frac{1}{\pi}\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^j} &= \int_{\mathbb{D}} \frac{\partial^2\omega_{1\bar{1}}}{\partial z^i\partial\bar{z}^j}(\zeta, z^2, ..., z^n)\log|z^1-\zeta|^2 dA(\zeta) \\ &= \int_{\mathbb{D}} \frac{\partial^2\omega_{i\bar{j}}}{\partial\zeta\partial\bar{\zeta}}(\zeta, z^2, ..., z^n)\log|z^1-\zeta|^2 dA(\zeta) \\ &= \omega_{i\bar{j}}(z). \end{aligned}$$

Note that, with $z' = (z^2, ..., z^n)$,

$$\Theta_{1\bar{1}} = \frac{1-|z'|^2}{(1-|z|^2)^2}.$$

Therefore, in the higher-dimensional setting, ψ is bounded in $\mathscr{C}^1$-norm by

$$M \sup_{z\in\mathbb{B}(0,1/2)} \left(\int_{\mathbb{D}(0,3/4)} \left|\log|\zeta - z^1|^2\right| \frac{(1-|z'|^2)dA(\zeta)}{(1-|\zeta|^2-|z'|^2)^2} + \int_{\mathbb{D}(0,3/4)} |z-\zeta|^{-1}\frac{(1-|z'|^2)dA(\zeta)}{(1-|\zeta|^2-|z'|^2)^2} \right).$$

The proof is complete. □

4.2. **Uniform local pluriharmonic recentering of weights.**

COROLLARY 4.2. *Let $z \in \mathbb{B}$ and let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy $-M\Theta \le dd^c\varphi \le M\Theta$ for some positive constant M. Then for each $r \in (0, 1/4]$ there exists $F \in \mathcal{O}(\mathbb{E}(z, r))$ and a constant $C_r > 0$ that depends on M and r but not on z, such that $F(z) = 0$ and the function $\psi := \varphi - \varphi(z) - \operatorname{Re} F$ satisfies*

$$\sup_{\mathbb{E}(z,r)} |\psi| + |d\psi| \le C_r.$$

Proof. By group invariance we may assume $z = 0$. Applying Lemma 4.1 to the form $\omega = dd^c\varphi$, we obtain a function ψ such that $dd^c\psi = dd^c\varphi$ with the appropriate $\mathscr{C}^1$-estimates. The function $\eta := \varphi - \varphi(0) + \psi(0) - \psi$ is then pluriharmonic, and therefore is the real part of a holomorphic function F. The imaginary part of F can be taken to be the function $\int_0^z d^c\eta$, and so F vanishes at 0. The proof is complete. □

REMARK 4.3. The assumption that $r \le 1/4$ (as well as the estimate in the ball of Radius 1/2 obtained in Lemma 4.1) is made solely for convenience of the proof we used above. It is nor hard to prove the result for any $r \in (0, 1)$, as long as we allow the constants to depend on r. We will use the version that considers all $r \in (0, 1)$ only for certain statements that are not decisive to the main vein of the article. ⋄

4.3. **Weighted Bergman inequalities.**

PROPOSITION 4.4 (Weighted Bergman inequalities). *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy $-M\Theta \le \sqrt{-1}\partial\bar\partial\varphi \le M\Theta$ for some positive constant M. If $f \in \mathscr{F}^p(\varphi)$ then for each $r \in (0,1)$ there exists a positive constant C_r such that*

$$(|f|^p e^{-p\varphi})(z) \le C_r^p \int_{\mathbb{E}(z,r)} |f|^p e^{-p\varphi} dV_\Theta. \tag{9}$$

and

$$|\nabla(|f|^p e^{-p\varphi})|_\Theta(z) \le C_r^p \int_{\mathbb{E}(z,r)} |f|^p e^{-p\varphi} dV_\Theta \tag{10}$$

Proof. First note that if φ satisfies the curvature bounds, then so does $\varphi \circ F_a$. Therefore it suffices to prove the result for $z = 0$. To achieve the latter, use Corollary 4.2 to replace $e^{-\varphi}$ by $|e^H|e^{-\varphi(0)}$ for some bounded holomorphic H, and thereby reduce to the unweighted setting, in which case the result is classical. □

4.4. **Slow growth of Bergman functions.**

COROLLARY 4.5. *Let φ be as in Propositions 4.4 and let $a > 0$ be given. Then there is a number $\delta > 0$ with the following property. If $z \in \mathbb{B}$, $f \in \mathscr{F}^p(dV_\Theta, \varphi)$ and $|f(z)|e^{-\varphi(z)} \ge a||f||_p$ then $|f(w)|e^{-\varphi(w)} \ge (a/2)||f||_p$ for all $w \in \mathbb{E}(z,\delta)$.*

Proof. Otherwise (10) in Proposition 4.4 is violated. □

5. TWISTED TECHNIQUES

Twisted-$\bar\partial$ techniques have become important in a number of branches of complex analysis and analytic geometry. As the authors see it, there are two types of ways to apply the twisted methods. The first and most common is an *a priori* estimate for the twisted $\bar\partial$ operator. This technique is used, among other places, in the proof of the Ohsawa-Takegoshi extension theorem and all of its variants. In the present paper we use it to state and prove Theorem 5.4, which is most likely due to Ohsawa, and is rather similar to work of Donnelly-Fefferman.

The second technique, though it has been around for almost as long as the first, is seen somewhat less often. So far as we can tell, it was introduced independently by Berndtsson [B-1997] and McNeal, though McNeal did not publish it. This technique is to use the twisted basic identity to obtain better L^2 estimates on the minimal L^2 solution of the $\bar\partial$ equation.

Now, if one is working in the ball, Theorem 5.4 is stronger than Hörmander's Theorem for solving $\bar\partial$ with L^2-estimates. It stands to reason that there might be a strengthened version of the Berndtsson-McNeal technique for improved estimates for the minimal solution of $\bar\partial$. As stated, this is not really true, but to understand in what sense it could be true, it is worth explaining how one obtains the improved Hörmander Theorem. In fact, the improvement of Hörmander's Theorem is achieved by solving a twisted $\bar\partial$ equation, i.e., an equation of the form

$$Tu = \theta$$

where $Tu = \bar\partial(\sqrt{\tau}u)$. After the fact, the solution u of the T-equation is transformed into a solution $U := \sqrt{\tau}u$ of the $\bar\partial$-equation, and it is the denominator of τ obtained in the L^2-norm of U that provides the improvement.

To obtain a similar improvement for the Berndtsson-McNeal technique, one should try to find improved estimates for the minimal solution of the T-equation, rather than the $\bar\partial$-equation.

In this section, we begin with the twisted identity and obtain from it the twisted basic estimate used in the Donnelly-Fefferman-Ohsawa technique. We then prove Ohsawa's Theorem using this *a priori* estimate, and use it to prove a couple of interesting results, one of which we will need in the sequel. Finally, we will recall the Berndtsson-McNeal technique for obtaining improved L^2 estimates for the $\bar\partial$-equation.

In the next section, we apply the Berndtsson-McNeal technique to the *a priori* estimate to obtain improved estimates for the minimal solution of the twisted $\bar{\partial}$-equation.

5.1. **The twisted basic identity.** Let ψ be a smooth weight function on a domain Ω with smooth boundary of codimension 1 in $\mathbb{C}^n$, cut out by a smooth defining function ρ such that $|d\rho| \equiv 1$ on $\partial\Omega$. In [SV-2011] we recalled how one obtains from the *Basic Identity*

$$||\bar{\partial}^*_\psi \alpha||^2 + ||\bar{\partial}\alpha||^2 = \int_\Omega \sum_{i,j} \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} d\lambda + ||\overline{\nabla}\alpha||^2 + \int_{\partial\Omega} \sum_{i,j} \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} dS \tag{11}$$

the so-called *Twisted Basic Identity*:

$$\begin{aligned} &||\sqrt{\tau}\bar{\partial}^*_\psi \alpha||^2 + ||\sqrt{\tau}\bar{\partial}\alpha||^2 \\ &= \int_\Omega \sum_{i,j} \left(\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} - \frac{\partial^2 \tau}{\partial z^i \partial \bar{z}^j} \right) \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} d\lambda + 2\mathrm{Re} \int_\Omega \bar{\partial}^*_\psi \alpha \left(\sum_i \overline{\frac{\partial \tau}{\partial z^i} \alpha_{\bar{i}}} \right) e^{-\psi} d\lambda \\ &\quad + ||\sqrt{\tau}\overline{\nabla}\alpha||^2 + \int_{\partial\Omega} \sum_{i,j} \tau \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} dS. \end{aligned} \tag{12}$$

Here $\tau : \Omega \to (0, \infty)$ is a smooth function, and both identities hold for all smooth $(0,1)$-forms in the domain of $\bar{\partial}^*_\psi$, as the reader can verify.

As a first consequence, we have the twisted basic estimate which seems to have been discovered independently by McNeal and Siu.

THEOREM 5.1 (Twisted Basic Estimate). *Let $\psi : \Omega \to \mathbb{R}$ be a smooth weight function. Let τ and A be positive functions with τ smooth. Then for any smooth $(0,1)$-form α in the domain of $\bar{\partial}^*_\psi$,*

$$\begin{aligned} &\int_\Omega (\tau + A) |\bar{\partial}^*_\psi \alpha|^2 e^{-\psi} d\lambda + \int_\Omega \tau |\bar{\partial}\alpha|^2 e^{-\psi} d\lambda \\ &\geq \int_\Omega \sum_{i,j} \left(\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} - \frac{\partial^2 \tau}{\partial z^i \partial \bar{z}^j} - \frac{1}{A} \frac{\partial \tau}{\partial z^i} \frac{\partial \tau}{\partial \bar{z}^j} \right) \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} d\lambda \\ &\quad + \int_\Omega \tau |\overline{\nabla}\alpha|^2 e^{-\psi} d\lambda + \int_{\partial\Omega} \sum_{i,j} \tau \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} dS. \end{aligned} \tag{13}$$

Proof. Since

$$2\mathrm{Re} \int_\Omega \bar{\partial}^*_\psi \alpha \overline{\sum_i \frac{\partial \tau}{\partial z^i} \alpha_{\bar{i}}} e^{-\psi} \geq - \int_\Omega \frac{1}{A} \sum_{i,j} \frac{\partial \tau}{\partial z^i} \frac{\partial \tau}{\partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} - \int_\Omega A |\bar{\partial}^*_\psi \alpha|^2 e^{-\psi},$$

the result follows immediately from the twisted basic identity (12). □

Integration-by-parts shows that

$$\int_\Omega \bar{\partial}^*_\psi \alpha \overline{\sum_i \frac{\partial \tau}{\partial z^i} \alpha_{\bar{i}}} e^{-\psi} d\lambda = - \int_\Omega \tau \sum_i (\bar{\partial}\bar{\partial}^*_\psi \alpha)_{\bar{i}} \overline{\alpha_{\bar{i}}} e^{-\psi} + ||\sqrt{\tau}\bar{\partial}^*_\psi \alpha||^2,$$

for smooth α in the domain of $\bar{\partial}^*_\psi$, and thus (12) implies the following result.

THEOREM 5.2. *For any smooth $(0,1)$-form α in the domain of $\bar{\partial}^*$, one has the identity*

$$\begin{aligned} &2\mathrm{Re} \int_\Omega \tau \left\langle \bar{\partial}\bar{\partial}^*_\psi \alpha, \alpha \right\rangle e^{-\psi} d\lambda + ||\sqrt{\tau}\bar{\partial}\alpha||^2 = ||\sqrt{\tau}\bar{\partial}^*_\psi \alpha||^2 \\ &+ \int_\Omega \sum_{i,j} \left(\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} - \frac{\partial^2 \tau}{\partial z^i \partial \bar{z}^j} \right) \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} d\lambda + ||\sqrt{\tau}\overline{\nabla}\alpha||^2 + \int_{\partial\Omega} \sum_{i,j} \tau \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\psi} d\lambda. \end{aligned} \tag{14}$$

REMARK 5.3. The identity (14) is a twisted version of what has been called the *$\partial\bar{\partial}$-Bochner-Kodaira Identity* by Siu in [S-1982]. $\diamond$

5.2. **A sharpened version of Hörmander's Theorem.** In this section, we prove the following theorem.

THEOREM 5.4 (Ohsawa). *Let φ be a locally integrable function such that $2dd^c\varphi \geq (n+2\delta)\Theta$ for some positive constant δ. Then there is a positive constant C such that for any $(0,1)$-form θ satisfying*

$$\bar{\partial}\theta = 0 \quad \textit{and} \quad \int_{\mathbb{B}} |\theta|^2_{\Theta} e^{-2\varphi} dV_{\Theta} < +\infty$$

there exists a measurable function U such that

$$\bar{\partial}U = \theta \quad \textit{and} \quad \int_{\mathbb{B}} |U|^2 e^{-2\varphi} dV_{\Theta} \leq \frac{C}{\delta}\int_{\mathbb{B}} |\theta|^2_{\Theta} e^{-2\varphi} dV_{\Theta}.$$

Moreover, U is smooth whenever θ is smooth.

REMARK 5.5. Note that, since $e^{-2\varphi}dV_{\Theta} = e^{-2\varphi+(n+1)\rho_o}d\lambda$, our hypothesis does not imply that, with respect to Lebesgue measure, the weight in question has positive curvature. In particular, although it looks a lot like Hörmander's Theorem, Theorem 5.4 cannot be deduced from Hörmander's Theorem. $\diamond$

Proof of Theorem 5.4. Set $\psi := 2\varphi + n\rho_o$, $\tau = e^{-\rho_o}$ and $A = \frac{\tau}{\delta}$. Then as $\sqrt{-1}\partial\rho_o \wedge \bar{\partial}\rho_o \leq \Theta$, we find that

$$\tau\sqrt{-1}\partial\bar{\partial}\psi - \partial\bar{\partial}\tau - \frac{\sqrt{-1}}{A}\partial\tau\wedge\bar{\partial}\tau = \tau\left(2dd^c\varphi + (n+1)dd^c\rho_o - (1+\delta)\partial\rho_o\wedge\bar{\partial}\rho_o\right) \geq \tau\delta\Theta.$$

For functions f and $(0,1)$-forms β let

$$Tf := \bar{\partial}(\sqrt{\tau}f) \quad \text{and} \quad S(\beta) := \sqrt{\tau}\bar{\partial}\beta.$$

From Theorem 5.1 we therefore deduce that for any smooth $(0,1)$-forms α in the domain of $\bar{\partial}^*_{\psi}$, one has the a priori estimate

$$\int_{\mathbb{B}} \tau \sum_{i,j=1}^{n} \Theta_{\bar{i}j}\alpha_{\bar{i}}\overline{\alpha_{\bar{j}}}e^{-\psi}d\lambda \leq \frac{1}{\delta}\int_{\mathbb{B}} |S\alpha|^2 e^{-\psi}d\lambda + \frac{2}{\delta^2}\int_{\mathbb{B}} |T^*\alpha|^2 e^{-\psi}d\lambda.$$

Now let θ be as in the statement of the theorem. Then for any smooth $(0,1)$-form α in the domain of T^* and in the domain of S (which agree with the domains of $\bar{\partial}^*$ and $\bar{\partial}$ respectively) we have

$$\left|\int_{\mathbb{B}} \sum \alpha_{\bar{i}}\overline{\theta_{\bar{i}}}e^{-\psi}d\lambda\right|^2 \leq \left(\int_{\mathbb{B}} \tau^{-1}|\theta|^2_{\Theta}e^{-\psi}d\lambda\right) \times \left(\int_{\mathbb{B}} \tau \sum_{i,j=1}^{n} \Theta_{\bar{i}j}\alpha_{\bar{i}}\overline{\alpha_{\bar{j}}}e^{-\psi}d\lambda\right). \tag{15}$$

Let

$$\mathscr{L}_{\theta}(T^*\alpha) := \int_{\mathbb{B}} \sum \alpha_{\bar{i}}\overline{\theta_{\bar{i}}}e^{-\psi}d\lambda.$$

Then the estimate (15) implies that $\mathscr{L}_{\theta}$ is a continuous linear functional on the subspace of $L^2(\lambda, \frac{1}{2}\psi)$ containing all the smooth functions $T^*\alpha$ where α is a smooth $\bar{\partial}$-closed form in the domain of $\bar{\partial}^*_{\psi}$. The norm on $\mathscr{L}_{\theta}$ on this subspace is at most

$$\left(\int_{\mathbb{B}} \tau^{-1}|\theta|^2_{\Theta}e^{-\psi}d\lambda\right)^{1/2} = \left(\int_{\mathbb{B}} |\theta|^2_{\Theta}e^{-2\varphi}dV_{\Theta}\right)^{1/2}.$$

By the Hahn-Banach Theorem, we may extend $\mathscr{L}_{\theta}$ to the entire Hilbert space with the same norm, and therefore by the Riesz Representation Theorem there exists $u \in L^2(\lambda, \varphi)$ representing $\mathscr{L}_{\theta}$. Restricting to the original subspace, we have

$$\int_{\mathbb{B}} u\overline{T^*\alpha}e^{-\psi}d\lambda = \int_{\mathbb{B}} \sum \theta_{\bar{i}}\overline{\alpha_{\bar{i}}}e^{-\psi}d\lambda,$$

which means that

$$Tu = \theta \quad \text{and} \quad \int_{\mathbb{B}} |u|^2 e^{-\psi} d\lambda \le \frac{C}{\delta} \int_{\mathbb{B}} |\theta|^2_{\Theta} e^{-2\varphi} dV_{\Theta}.$$

If we set $U := \sqrt{\tau} u$ then we find that

$$\bar\partial U = \theta \quad \text{and} \quad \int_{\mathbb{B}} |U|^2 e^{-2\varphi} dV_{\Theta} \le \frac{C}{\delta} \int_{\mathbb{B}} |\theta|^2_{\Theta} e^{-2\varphi} dV_{\Theta}.$$

Finally, the smoothness assertion follows from the interior elliptic regularity of $\bar\partial$. □

5.3. **An application: Uniform interpolation at a point and Carleson's condition.** When we study the Bergman kernel in the next section, we will make use of the following result.

PROPOSITION 5.6 (Uniform 1-point interpolation). *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy $(n+2\delta)\Theta \le 2dd^c\varphi \le M\Theta$ in $\mathbb{B}$, for some positive constants δ and M. Then there exists a positive constant C, depending only on δ and M, such that for each $p \in [1,\infty]$ and each $z \in \mathbb{B}$ there exists $f \in \mathscr{F}^p(dV_{\Theta}, \varphi)$ satisfying*

$$f(z) = e^{\varphi(z)} \quad \textit{and} \quad \int_{\mathbb{B}} |f|^p e^{-p\varphi} dV_{\Theta} \le C^p.$$

Proof. Once again, we shall reduce to the case $z = 0$. To this end, assume that the result holds for $z = 0$, in particular for any weight that satisfies the curvature hypotheses. Let $z \in \mathbb{B}$. Then as before, the weight $\varphi_z := \varphi \circ F_z$ satisfies the same curvature hypotheses as φ. Let $g \in \mathcal{O}(\mathbb{B}^n)$ satisfy $g(0)e^{-\varphi_z(0)} = 1$ and $\int_{\mathbb{B}} |g|^p e^{-p\varphi_z} dV_{\Theta} \le C$. Define $f := g \circ F_z$. Then $f(z)e^{-\varphi(z)} = g(0)e^{-\varphi \circ F_z(0)} = 1$, while

$$\int_{\mathbb{B}} |f|^p e^{-p\varphi} dV_{\Theta} = \int_{\mathbb{B}} |g|^p e^{-p\varphi_z} dV_{\Theta},$$

so that f solves the interpolation problem at z.

We assume from now on that $z = 0$. Fix a smooth function $\chi \in \mathscr{C}^\infty_o(\mathbb{B}(0,1/4))$ such that $\chi|_{\mathbb{B}(0,1/8)} \equiv 1$. Define the $(0,1)$-form $\theta(\zeta) := e^{F(\zeta)+\varphi(0)}\bar\partial\chi(\zeta)$, where $F \in \mathcal{O}(\mathbb{B}(0,1/4))$ is the holomorphic function from Corollary 4.2 applied to the weight $\varphi - \frac{n}{2}\rho_o$. In particular,

$$e^{-n\rho_o} \lesssim e^{-2(\varphi - \varphi(0) - \mathrm{Re}\, F)} \lesssim e^{-n\rho_o}.$$

Now, since θ is supported on $\mathbb{B}(0,1/4) - \mathbb{B}(0,1/8)$, one has

$$\int_{\mathbb{B}} |\theta|^2 \frac{e^{-(2\varphi - \delta\rho_o)}}{|\zeta|^{2n}} dV_{\Theta} \lesssim \int_{\mathbb{B}(0,1/4)} e^{-2(\varphi-\varphi(0)-\mathrm{Re}\, F)} e^{(1+\delta)\rho_o} d\lambda < +\infty.$$

We may therefore apply Theorem 5.4 with the weight $n\log|\zeta|^2 + \varphi(\zeta) - \frac{\delta}{2}\rho_o(\zeta)$ to obtain a function u such that $\bar\partial u = \theta$ and

$$\int_{\mathbb{B}} \frac{|u(\zeta)|^2 e^{-(2\varphi(\zeta) - \delta\rho_o(\zeta))}}{|\zeta|^{2n}} dV_{\Theta}(\zeta) < +\infty.$$

In particular, $u(0) = 0$, and therefore the function

$$f(\zeta) = \chi(\zeta)e^{F(\zeta)+\varphi(0)} - u(\zeta)$$

is holomorphic, satisfies $f(0)e^{-\varphi(0)} = 1$, and has L^2-estimate

$$\int_{\mathbb{B}} |f|^2 e^{-2\varphi+\delta\rho_o} dV_{\Theta} \lesssim \int_{\mathbb{B}} |\chi|^2 e^{\delta\rho_o} e^{-2(\varphi-\varphi(0)-\mathrm{Re}\, F)} dV_{\Theta} + \int_{\mathbb{B}} \frac{|u(\zeta)|^2 e^{-(2\varphi(\zeta)-\delta\rho_o(\zeta))}}{|\zeta|^{2n}} dV_{\Theta}(\zeta),$$

and the right hand side is finite by the construction of u and the fact that χ has compact support. Since $e^{\delta\rho_o} \ge 1$, our assertion is proved for $p = 2$.

Now, by (9) of Proposition 4.4 we have the estimate

$$|f(z)|^2 e^{-(2\varphi(z) - (\frac{2n}{p}+\delta)\rho_o(z))} \lesssim \int_{\mathbb{E}(z,1/4)} |f(\zeta)|^2 e^{-(2\varphi(\zeta)-\delta\rho_o(\zeta))} dV_{\Theta}(\zeta) \le \tilde{C}, \tag{16}$$

and again since $|f(\zeta)|^2 e^{-2\varphi(\zeta)} \le |f(\zeta)|^2 e^{-(2\varphi(\zeta)-(\frac{2n}{p}+\delta)\rho_o(\zeta))}$, we have shown that $||f||_{\infty,\varphi} \le C$ for some uniform constant $C > 0$. This establishes the case $p = \infty$.

Observe, however, that (16) also yields

$$\int_{\mathbb{B}} |f|^p e^{-p\varphi} dV_\Theta = \int_{\mathbb{B}} \left(|f|^2 e^{-(2\varphi - (\frac{2n}{p}+\delta)\rho_o)} \right)^{p/2} e^{-\frac{p\delta}{2}\rho_o} dV_\Theta \le C^p \int_{\mathbb{B}} (1-|\zeta|^2)^{\frac{p\delta}{2}-1} d\lambda(\zeta) < +\infty.$$

The proof is finished. □

DEFINITION 5.7. A positive measure μ is Carleson for $\mathscr{F}^p(dV_\Theta, \varphi)$ if the inclusion

$$\iota_\mu : \mathscr{F}^p(dV_\Theta, \varphi) \hookrightarrow \mathscr{F}^p(\mu, \varphi)$$

is bounded. ◇

In this section, our goal is to prove the following ball analog of a theorem of Ortega-Cerdà [O-1998].

THEOREM 5.8. *Suppose $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfies $0 < (n+c)\Theta \le 2dd^c\varphi \le C\Theta$ for some constants $0 < c < C$. Let $p \ge 1$ be a real number and let μ be a positive measure in $\mathbb{B}$. The following are equivalent.*

(a) *The measure μ is Carleson for $\mathscr{F}^p(dV_\Theta, \varphi)$.*
(b) *There exists $C > 0$ and $r \in (0,1)$ such that $\mu(\mathbb{E}(z,r)) \le C$ for any $z \in \mathbb{B}$.*
(c) *For each $r \in (0,1)$ there exists $C_r > 0$ such that $\mu(\mathbb{E}(z,r)) \le C_r$ for any $z \in \mathbb{B}$.*

Proof. It is clear that $(b) \iff (c)$. To prove that $(b) \Rightarrow (a)$, cover $\mathbb{B}$ by a countable collection of pseudo-hyperbolic balls of radius r such that each point of $\mathbb{B}$ is contained in at most N balls, for some fixed number $N \sim 2^n$. On each such ball $\mathbb{E}(z,r)$, we have

$$\int_{\mathbb{E}(z,r)} |f|^p e^{-p\varphi} d\mu \lesssim \sup_{\mathbb{E}(z,r)} |f|^p e^{-p\varphi} \lesssim \int_{\mathbb{E}(z,2r)} |f|^p e^{-p\varphi} dV_\Theta,$$

(here and below the standard notation $\lesssim$ means $\le C$ for some C, and usually the constant C is also independent of certain data; in this case the data f and z) and summing over the countable collection of centers z, we have

$$\begin{aligned} \int_{\mathbb{B}} |f|^p e^{-p\varphi} d\mu &\lesssim \sum_j \int_{\mathbb{E}(z_j,r)} |f|^p e^{-p\varphi} d\mu \\ &\lesssim \sum_j \int_{\mathbb{E}(z_j,2r)} |f|^p e^{-p\varphi} dV_\Theta \\ &\lesssim \int_{\mathbb{B}} |f|^p e^{-p\varphi} dV_\Theta. \end{aligned}$$

Finally we prove $(a) \Rightarrow (b)$. By Proposition 5.6 there exists $f \in \mathscr{F}^p(dV_\Theta, \varphi)$ such that $f(z) = e^{\varphi(z)}$ and $||f||_p \le C$ for some $C > 0$ independent of z. By the estimate (10) of Proposition 4.4 there exists $r > 0$ sufficiently small such that for all $w \in \mathbb{E}(z,r)$, $|f(w)|e^{-\varphi(w)} \ge 1/2$. It follows that

$$\mu(\mathbb{E}(z,r)) \lesssim 2^p \int_{\mathbb{E}(z,r)} |f|^p e^{-p\varphi} d\mu \le 2^p \int_{\mathbb{B}} |f|^p e^{-p\varphi} d\mu \lesssim 2^p \int_{\mathbb{B}} |f|^p e^{-p\varphi} dV_\Theta \lesssim 2^p,$$

where the Carleson condition is used in the third inequality. The proof is therefore finished. □

REMARK 5.9. There is also a notion of *vanishing Carleson measures*, and a corresponding geometric characterization. The analogous result in $\mathbb{C}^n$ was stated and proved in [SV-2011, Theorem 5.2]. Since we are not going to use Carleson measures in the present paper, we leave the details to the interested reader. ◇

5.4. **Berndtsson's Theorem on the minimal L^2 solution of $\bar\partial$.**

THEOREM 5.10 ([B-1997]). *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let $\tau : \Omega \to (0,\infty)$ be a $\mathscr{C}^2$ function and let A be a symmetric matrix whose entries are functions in Ω such that at each point $z \in \Omega$, $A(z)$ is positive definite. Assume furthermore that the matrix*

$$\left(\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar z^j} - \frac{\partial^2 \tau}{\partial z^i \partial \bar z^j} - \tau A_{i\bar j}\right)$$

is positive-semi-definite at each point of Ω. Then for any $\bar\partial$-closed $(0,1)$-form θ, the solution u of $\bar\partial u = \theta$ having minimal norm in $L^2(e^{-\psi}d\lambda)$ satisfies the estimate

$$\int_\Omega \tau |u|^2 e^{-\psi} d\lambda \le \int_\Omega \tau |\theta|^2_A e^{-\psi} d\lambda,$$

where

$$|\theta|^2_A = \sum_{i,j} (A^{-1})_{i\bar j} \theta_{\bar i} \overline{\theta_{\bar j}}.$$

Proof. As is well-known, the minimal solution u is $\bar\partial^* \beta$ where β is the (unique) solution of the equation $\Box\beta = \theta$, and furthermore $\bar\partial\beta = 0$. Indeed, $\bar\partial u = \bar\partial\bar\partial^*\beta = \Box\beta = \theta$ and since $0 = (\bar\partial\theta, \bar\partial\beta) = (\bar\partial\bar\partial^*\bar\partial\theta, \bar\partial\beta) = ||\bar\partial^*\bar\partial\beta||^2$, we obtain $0 = (\bar\partial^*\bar\partial\beta, \beta) = ||\bar\partial\beta||^2$.

Applying the identity (14) to the form β, one obtains

$$2\mathrm{Re} \int_\Omega \tau \left\langle \theta, \beta \right\rangle e^{-\psi} d\lambda = ||\sqrt{\tau} u||^2 + \int_\Omega \sum_{i,j} \left(\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar z^j} - \frac{\partial^2 \tau}{\partial z^i \partial \bar z^j}\right) \beta_{\bar i} \overline{\beta_{\bar j}} e^{-\psi} d\lambda$$
$$+ ||\sqrt{\tau}\nabla\beta||^2 + \int_{\partial\Omega} \sum_{i,j} \frac{\partial^2 \rho}{\partial z^i \partial \bar z^j} \beta_{\bar i} \overline{\beta_{\bar j}} e^{-\psi} d\lambda. \tag{17}$$

From the identity (17), the inequality $2\mathrm{Re}\ \langle\theta, \beta\rangle \le \sum_{i,j} A_{i\bar j} \beta_{\bar i} \overline{\beta_{\bar j}} + |\theta|^2_A$ and the pseudoconvexity of Ω imply the estimate

$$\int_\Omega \tau |\theta|^2_A e^{-\psi} d\lambda \ge \int_\Omega \tau |u|^2 + \int_\Omega \sum_{i,j} \left(\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar z^j} - \frac{\partial^2 \tau}{\partial z^i \partial \bar z^j} - \tau A_{i\bar j}\right) \beta_{\bar i} \overline{\beta_{\bar j}} e^{-\psi} d\lambda.$$

The hypothesis implies that the right-most integral is non-negative, and thus the proof is complete. □

EXAMPLE 5.11. Let $A_{i\bar j} = c\Theta_{i\bar j} = c\frac{\partial^2}{\partial z^i \partial \bar z^j}\rho_o$ for some $c > 0$, and let $\tau = e^{-\alpha\rho_o}$ for some $\alpha > 0$. Then

$$dd^c\tau = \frac{\sqrt{-1}}{2}\partial\left(-\alpha e^{-\alpha\rho_o}\bar\partial\rho_o\right) = \left(-\alpha\Theta + \alpha^2 \tfrac{\sqrt{-1}}{2}\partial\rho_o \wedge \bar\partial\rho_o\right)\tau \le \left(\alpha^2 - \alpha\right)\tau\Theta,$$

since

$$\tfrac{\sqrt{-1}}{2}\partial\rho_o \wedge \bar\partial\rho_o = \frac{\bar z \cdot dz \wedge z \cdot d\bar z}{(1-|z|^2)^2} \le \Theta.$$

Let us take the weight

$$\psi = 2\varphi - \rho_o,$$

where φ satisfies the lower bound $dd^c\varphi \ge \sigma\Theta$. Then

$$\tau dd^c\psi - dd^c\tau - c\tau\Theta \ge \tau(-\alpha^2 + \alpha + 2\sigma - c - 1)\Theta = \tau\left(2\sigma - c - \tfrac{3}{4} - \left(\alpha - \tfrac{1}{2}\right)^2\right)\Theta,$$

and the latter can be non-negative precisely when the graph of the quadratic $\alpha \mapsto 2\sigma - c - \frac{3}{4} - (\alpha - 1/2)^2$ meets the α-axis. Thus in order to apply Theorem 5.10 we must require that $\sigma > 3/8$. This puts a limitation on the sorts of weights to which we might expect to apply the theorem. Ideally, we would like to consider all weights for which $\sigma > 0$. We are still unable to do so, but the double-twisted method introduced in the next section will allow us to take things down to $\sigma > 1/4$. ◇

6. DOUBLE-TWISTED TECHNIQUES

6.1. **The twisted local complex and Neuman problem.** For functions u and $(0,1)$-forms α, define the twisted operators

$$Tu := \bar{\partial}(\sqrt{\tau} u) \qquad \text{and} \qquad S\alpha := \sqrt{\tau}\bar{\partial}\alpha.$$

The operators T and S satisfy the relation

$$ST = 0.$$

Therefore in order to solve the equation $Tu = \theta$ for a given θ, we must require that $S\theta = 0$.

Given a $(0,1)$-form θ such that $S\theta = 0$, we seek a function u such that $Tu = \theta$ and

$$\int_\Omega |u|^2 e^{-\psi} d\lambda < +\infty.$$

Naturally, one would like to find the solution u for which

$$\int_\Omega |u|^2 e^{-\psi} d\lambda$$

is minimized. A standard argument shows that u is the minimal solution if and only if u is orthogonal to $\mathrm{Ker}(T)$.

Since $\mathrm{Ker}(T) \perp \mathrm{Image}(T^*)$, we could seek a solution u of the form $u = T^*\beta$. (If T^* has closed range then the minimal solution is necessarily in the image of T^*.) The form β is not unique since we may add to it any form in $\mathrm{Ker}(T^*)$. At first blush, it may seem as though this matter should not concern us, since we are interested in $T^*\beta$ and not β itself. But as it turns out, we obtain estimates for $T^*\beta$ from estimates for β, and therefore we should find the $(0,1)$-form β of minimal norm for which $TT^*\beta = \theta$. Therefore we should find a form β that is orthogonal to the kernel of T^*. Since the relation $ST = 0$ implies that the image of S^* is contained in the kernel of T^*, the solution β of minimal norm is orthogonal to the image of S^*, and therefore satisfies the pair of equations

$$TT^*\beta = \theta \quad \text{and} \quad S\beta = 0.$$

Therefore β satisfies

$$(S^*S + TT^*)\beta = \theta.$$

On the other hand, suppose we can solve the equation $(S^*S + TT^*)\beta = \theta$. Since $S\theta = 0$, we find that

$$0 = (S(S^*S + TT^*)\beta, S\beta) = (SS^*S\beta, S\beta) = ||S^*S\beta||^2,$$

and thus $0 = (S^*S\beta, \beta) = ||S\beta||^2$. Thus we can find a solution β such that both β and $T^*\beta$ have minimal norm, if and only if we can solve the equation $(S^*S + TT^*)\beta = \theta$.

We can also drop the condition that $S\theta = 0$ and ask whether or not the equation in question has a solution, and whether that solution is unique. The uniqueness of the solution holds if and only if the sequence

$$L^2_{0,2}(\lambda, \psi) \xrightarrow{S^*} L^2_{0,1}(\lambda, \psi) \xrightarrow{T^*} L^2(\lambda, \psi)$$

is exact at $L^2_{0,1}(\lambda, \psi)$, and as usual the exactness is measured by the vanishing of the cohomology, i.e., the kernel of $S^*S + TT^*$. If the sequence is not exact, we may try to find the unique solution orthogonal to the kernel of $S^*S + TT^*$. The most difficult part of this general problem is the issue of existence, which is intimately linked to the complex geometry of the domain and of the weight. In the unweighted case these matters were famously treated by Kohn in the case where $\tau = 1$ (so $T = \bar{\partial}$). Here we require a weighted analog. The deepest part of Kohn's work is the regularity, up to the boundary, of the $\bar{\partial}$-Neumann problem. Here we need the regularity on the boundary. Kohn's work in the weighted case [Ko-1973] can be used to deduce the necessary facts for establishing existence for these twisted operators, as we now explain.

Morrey (for the case of $(0,1)$-forms) and Kohn (in general) showed that for any smooth forms in the domain of $\bar{\partial}^*$ one has the so-called basic identity (11). It was then proved that when the domain is pseudoconvex, the smooth forms in the domain of $\bar{\partial}^*$ are dense in the so-called *graph-norm* $||\alpha||^2 + ||\bar{\partial}\alpha||^2 + ||\bar{\partial}^*\alpha||^2$.

The proof of this density uses a subtle adaptation of the Friedrichs regularization method, which has to be modified to deal with the boundary geometry.

We observe that the $\bar\partial$-Neumann boundary condition for smooth forms, and the graph-norm regularization for $\bar\partial$, are exactly the same as their (S,T)-analogs. In the latter case, the graph norm is $||\alpha||^2 + ||S\alpha||^2 + ||T^*\alpha||^2$. Indeed, this is the case since both S and T^* are obtained from $\bar\partial$ and $\bar\partial^*$ by multiplication by $\sqrt{\tau}$ and the function τ is smooth and strictly positive. Thus the whole machinery of the $\bar\partial$-Neumann problem automatically goes over to the twisted case.

Finally, let us link the minimal solution of the T-equation with the minimal solution of the $\bar\partial$-equation.

PROPOSITION 6.1. *Fix a $\bar\partial$-closed smooth $(0,1)$-form θ. Let $u \in L^2(\lambda, \frac{1}{2}\psi)$ be the solution of $Tu = \theta$ having minimal norm, and let $U \in L^2(\lambda, \frac{1}{2}(\psi + \log\tau))$ be the solution of $\bar\partial U = \theta$ having minimal norm. Assume that one of u or U exists. Then the other also exists, and $U = \sqrt{\tau}u$.*

Proof. By the obvious symmetry of the problem, it suffices to assume u exists. Let $\tilde U = \sqrt{\tau}u$. Then

$$\int_\Omega |\tilde U|^2 \frac{e^{-\psi}}{\tau} d\lambda = \int_\Omega |u|^2 e^{-\psi} d\lambda \le \int_\Omega |U/\sqrt{\tau}|^2 e^{-\psi} d\lambda = \int_\Omega |U|^2 \frac{e^{-\psi}}{\tau} d\lambda,$$

where the inequality follows from the minimality of u and the fact that $T(U/\sqrt{\tau}) = \theta$. By the minimality of U, $||\tilde U|| = ||U||$. But since $f = \tilde U - U$ is holomorphic and thus orthogonal to U, we calculate that $||\tilde U||^2 = ||f||^2 + ||U||^2$, and thus $f = 0$, as desired. □

6.2. **The double-twisted basic identity and the double-twisted basic estimate.** In the twisted basic identity (12), let us replace τ by the product $\gamma\tau$, thus rewriting the identity as

$$\begin{aligned}
(18)\qquad & ||\sqrt{\gamma}T_\psi^*\alpha||^2 + ||\sqrt{\gamma}S\alpha||^2 \\
&= \int_\Omega \sum_{i,j}\left(\gamma\tau\frac{\partial^2\psi}{\partial z^i\partial\bar z^j} - \gamma\frac{\partial^2\tau}{\partial z^i\partial\bar z^j} - \tau\frac{\partial^2\gamma}{\partial z^i\partial\bar z^j} - 2\mathrm{Re}\left(\frac{\partial\gamma}{\partial z^i}\frac{\partial\tau}{\partial\bar z^j}\right)\right)\alpha_{\bar i}\overline{\alpha_{\bar j}}e^{-\psi}d\lambda \\
&\quad +2\mathrm{Re}\left(\int_\Omega(\bar\partial_\psi^*\alpha)\gamma\overline{\sum_i\frac{\partial\tau}{\partial z^i}\alpha_{\bar i}}e^{-\psi}d\lambda + \int_\Omega(\bar\partial_\psi^*\alpha)\tau\overline{\sum_i\frac{\partial\gamma}{\partial z^i}\alpha_{\bar i}}e^{-\psi}d\lambda\right) \\
&\quad +||\sqrt{\tau\gamma}\overline{\nabla}\alpha||^2 + \int_{\partial\Omega}\tau\gamma\sum_{i,j}\frac{\partial^2\rho}{\partial z^i\partial\bar z^j}\alpha_{\bar i}\overline{\alpha_{\bar j}}e^{-\psi}d\lambda,
\end{aligned}$$

where as before

$$T := \bar\partial \circ \sqrt{\tau} \qquad \text{and} \qquad S = \sqrt{\tau}\circ\bar\partial.$$

Now, using integration-by-parts and some simple algebraic manipulation,

$$\begin{aligned}
\int_\Omega(\bar\partial_\psi^*\alpha)\tau\overline{\sum_i\frac{\partial\gamma}{\partial z^i}\alpha_{\bar i}}e^{-\psi}d\lambda &= \int_\Omega\sqrt{\tau}(T_\psi^*\alpha)\sum_i\overline{e^{-\psi}\alpha_{\bar i}}\frac{\partial\gamma}{\partial\bar z^i}d\lambda \\
&= \int_\Omega\gamma|T_\psi^*\alpha|^2e^{-\psi}d\lambda - \int_\Omega\sum_i\gamma(TT_\psi^*\alpha)_{\bar i}\overline{\alpha_{\bar i}}e^{-\psi}d\lambda.
\end{aligned}$$

We therefore have the identity

$$
\begin{aligned}
(19)\qquad & 2\mathrm{Re} \int_\Omega \sum_i \gamma (TT^*_\psi \alpha)_{\bar i} \overline{\alpha_{\bar i}} e^{-\psi} d\lambda + ||\sqrt{\gamma} S\alpha||^2 \\
&= ||\sqrt{\gamma} T^*_\psi \alpha||^2 + \int_\Omega \sum_{i,j} \left(\gamma\tau \frac{\partial^2 \psi}{\partial z^i \partial \bar z^j} - \gamma \frac{\partial^2 \tau}{\partial z^i \partial \bar z^j} - \tau \frac{\partial^2 \gamma}{\partial z^i \partial \bar z^j} - 2\mathrm{Re}\left(\frac{\partial \gamma}{\partial z^i} \frac{\partial \tau}{\partial \bar z^j} \right) \right) \alpha_{\bar i} \overline{\alpha_{\bar j}} e^{-\psi} d\lambda \\
&\quad + 2\mathrm{Re} \int_\Omega (\bar\partial^*_\psi \alpha) \gamma \overline{\sum_i \frac{\partial \tau}{\partial z^i} \alpha_{\bar i}} e^{-\psi} d\lambda \\
&\quad + ||\sqrt{\tau\gamma}\overline{\nabla}\alpha||^2 + \int_{\partial\Omega} \tau\gamma \sum_{i,j} \frac{\partial^2 \rho}{\partial z^i \partial \bar z^j} \alpha_{\bar i} \overline{\alpha_{\bar j}} e^{-\psi} d\lambda,
\end{aligned}
$$

Applying the Cauchy-Schwarz and big-constant-small-constant inequalities to the term found on the third line of (19), we obtain the following result.

THEOREM 6.2. *Define the* $(1,1)$*-form*

$$
\Xi_\delta(\psi|\gamma;\tau) := \gamma\tau\partial\bar\partial\psi - \tau\partial\bar\partial\gamma - \gamma\partial\bar\partial\tau - \partial\tau\wedge\bar\partial\gamma - \partial\gamma\wedge\bar\partial\tau - (1+\delta)\frac{\gamma}{\tau}\partial\tau\wedge\bar\partial\tau.
$$

Then for all smooth $(0,1)$*-forms* $\alpha \in \mathrm{Dom}(T^*_\psi) \cap \mathrm{Dom}(S)$*, one has the a priori estimate*

$$
\begin{aligned}
& 2\mathrm{Re} \int_\Omega \sum_i \gamma (TT^*_\psi \alpha)_{\bar i} \overline{\alpha_{\bar i}} e^{-\psi} d\lambda \\
(20)\qquad & \geq \frac{\delta}{1+\delta} ||\sqrt{\gamma} T^*_\psi \alpha||^2 + ||\sqrt{\gamma} S\alpha||^2 + \int_\Omega \sum_{i,j} \Xi_\delta(\psi|\gamma;\tau)_{i\bar j} \alpha_{\bar i} \overline{\alpha_{\bar j}} e^{-\psi} d\lambda \\
&\quad + ||\sqrt{\tau\gamma}\overline{\nabla}\alpha||^2 + \int_{\partial\Omega} \tau\gamma \sum_{i,j} \frac{\partial^2 \rho}{\partial z^i \partial \bar z^j} \alpha_{\bar i} \overline{\alpha_{\bar j}} e^{-\psi} d\lambda
\end{aligned}
$$

6.3. **A twisted form of Berndtsson's Theorem.** By analogy with the proof of Berndtsson's Theorem by use of the twisted basic identity, we now use Theorem 6.2 to establish the following result.

THEOREM 6.3. *Let* Ω *be a pseudoconvex domain in* $\mathbb{C}^n$ *and* $\psi \in \mathscr{C}^2(\Omega)$*. Let* $\tau, \gamma : \Omega \to (0,\infty)$ *be smooth functions and let* A *be a symmetric matrix whose entries are functions in* Ω *such that at each point* $z \in \Omega$*,* $A(z)$ *is positive definite. Assume furthermore that there is a positive number* δ *such that the Hermitian* $(1,1)$*-form*

$$
\Xi_\delta(\psi|\gamma;\tau) - \sqrt{-1}\gamma A_{i\bar j} dz^i \wedge d\bar z^j
$$

is positive-semi-definite at each point of Ω*. Then for any* S*-closed* $(0,1)$*-form* θ*, the solution* u *of* $Tu = \theta$ *having minimal norm in* $L^2(e^{-\psi}d\lambda)$ *satisfies the estimate*

$$
\int_\Omega \gamma |u|^2 e^{-\psi} d\lambda \leq \int_\Omega \gamma |\theta|^2_A e^{-\psi} d\lambda,
$$

where

$$
|\theta|^2_A = \sum_{i,j} (A^{-1})_{i\bar j} \theta_{\bar i} \overline{\theta_{\bar j}}.
$$

Proof. As was explained in Paragraph 6.1, the minimal solution u is of the form $u = T^*\beta$ for some β, and furthermore we may take β to be the solution of the equation $(S^*S + TT^*)\beta = \theta$, in which case (i) β satisfies the (S,T)-Neumann boundary condition, and (ii) $S\beta = 0$.

Applying the the estimate (20) to the form β and using the pseudoconvexity of Ω, we obtain

$$2\mathrm{Re}\int_\Omega \gamma \langle \theta, \beta \rangle e^{-\psi} d\lambda \geq \frac{\delta}{2}||\sqrt{\gamma} u||^2 + \int_\Omega \sum_{i,j} \Xi_\delta(\psi|\gamma;\tau)_{i\bar{j}} \beta_{\bar{i}} \overline{\beta_{\bar{j}}} e^{-\psi} d\lambda. \tag{21}$$

The inequality $2\mathrm{Re}\ \langle \theta, \beta \rangle \leq \sum_{i,j} A_{i\bar{j}} \beta_i \overline{\beta_j} + |\theta|^2_A$ implies the estimate

$$\int_\Omega \gamma |\theta|^2_A e^{-\psi} d\lambda \geq \frac{\delta}{2} \int_\Omega \gamma |u|^2 e^{-\psi} + \int_\Omega \sum_{i,j} \left(\Xi_\delta(\psi|\gamma;\tau)_{i\bar{j}} - \gamma A_{i\bar{j}} \right) \beta_i \overline{\beta_{\bar{j}}} e^{-\psi} d\lambda.$$

Since by hypothesis the right-most integral is non-negative, the proof is complete. □

As a corollary, we obtain the following result.

THEOREM 6.4. *Using the notation of Theorems 6.2 and 6.3, assume that A is again pointwise positive definite, and that there exists a positive number δ such that*

$$\Xi_\delta(\psi|\gamma;\tau) - \sqrt{-1}\gamma A_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

is positive-semi-definite at each point of Ω. Then for any $\bar\partial$-closed $(0,1)$-form θ, the solution U of $\bar\partial U = \theta$ having minimal norm in $L^2(e^{-\psi}d\lambda)$ satisfies the estimate

$$\int_\Omega \gamma |U|^2 \frac{e^{-\psi}}{\tau} d\lambda \leq C_\delta \int_\Omega \gamma |\theta|^2_A e^{-\psi} d\lambda,$$

where the constant C_δ is independent of θ, as well as ψ, γ, τ and A, so long as the hypotheses are satisfied.

Proof. Let u be the minimal solution of $Tu = \theta$. Then by proposition 6.1 $U = \sqrt{\tau} u$ is the minimal solution of $\bar\partial u = \theta$ with respect to the weight $e^{-\psi}/\tau$. Thus the result follows immediately from Theorem 6.3 □

EXAMPLE 6.5. Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy $dd^c\varphi \geq \sigma\Theta$ for some positive number σ, and let $w \in \mathbb{B}$. Letting

$$\psi = 2\varphi - (1 - m_\tau)\rho_w, \quad \tau := e^{-m_\tau \rho_w}, \quad \gamma := e^{-m_\gamma \rho_w}, \quad \text{and} \quad A := c\tau\Theta$$

for some real numbers $c > 0$, m_τ and m_γ, we find that

$$\begin{aligned} \Xi_\delta(\psi|\gamma;\tau) &= \gamma\tau \left(2dd^c\varphi + (2m_\tau + m_\gamma - 1)\Theta - \left((2+\delta)m_\tau^2 + m_\gamma^2 + 2m_\gamma m_\tau \right) \partial\rho_w \wedge \bar\partial\rho_w \right) \\ &\geq \gamma\tau \left(2\sigma - 1 + 2m_\tau + m_\gamma - (2+\delta)m_\tau^2 - m_\gamma^2 - 2m_\tau m_\gamma \right) \Theta. \end{aligned}$$

(Here we have used the fact that $\sqrt{-1}\partial\bar\partial\rho_w = \Theta$ and $\sqrt{-1}\partial\rho_w \wedge \bar\partial\rho_w = F_w^*(\sqrt{-1}\partial\rho_o \wedge \bar\partial\rho_o) \leq F_w^*\Theta = \Theta$.) It is evident from this formula that if we don't care about the absolute constant C_δ in the estimate of Theorem 6.4, then we gain the most by taking δ as small as possible. Also, a bit of a subtlety here is that if either m_τ or m_γ is negative, then we run into some trouble because the respective τ or γ is not smooth across the boundary of the ball, and this creates some trouble for the $\bar\partial$-Neumann problem and its twisted relative. However, one can get around this problem by working on balls of radius r with $r \nearrow 1$.

Suppose first that our goal is to admit the most liberal possible condition on the possible curvatures of the weights to which our theorem would apply. In this case, we wish to maximize the function $F(m_\tau, m_\gamma)$, where $F(x,y) = -1 + 2x + y - (2+\delta)x^2 - y^2 - 2xy$. Basic calculus shows that $F(x,y)$ has exactly one critical point, which is a maximum, located at $x = \frac{1}{2(1+\delta)}, y = \frac{\delta}{2(1+\delta)}$, and

$$F\left(\tfrac{1}{2(1+\delta)}, \tfrac{\delta}{2(1+\delta)}\right) = \frac{1}{4}\left(1 + \frac{1}{1+\delta}\right) - 1.$$

The latter can be made arbitrarily close to $-1/2$ by choosing $\delta > 0$ sufficiently small, and therefore as long as $\sigma > 1/4$, if we take $m_\gamma = \frac{\delta}{1+\delta}$ and $m_\tau = \frac{1}{2(1+\delta)}$ then we can choose δ and c so small that the quadratic form $\Xi - \gamma A$ is positive semi-definite. On the other hand, we cannot apply our theorem if $\sigma \leq 1/4$. At this point, it is worth remarking that, with the single twisted technique, one can only apply Berndtsson's

Theorem in the ball if $\sigma > 3/8$. From this point of view, the double-twisted technique is therefore an improvement.

Assume therefore, that $\sigma > 1/4$. Suppose now that we wish to maximize m_γ. (Indeed, in the example at hand, $e^{-\psi}/\tau d\lambda = e^{-2\varphi}d\mu$ is independent of τ, so it makes sense to want to optimize m_γ.) Therefore, we are interested in maximizing the function

$$G(x,y) = y$$

subject to the constraint $F(x,y) = -2\sigma$. We can try to solve this problem via Lagrange's method: we must have

$$(2+\delta)x + y = 1.$$

Straightforward calculation shows that the maximum is achieved at

$$m_\gamma = \frac{1 + \sqrt{\frac{4(2\sigma-1)(1+\delta)}{2+\delta}}}{2(1+\delta)} = \frac{1+\sqrt{4\sigma-1}}{2} + O(\delta),$$

as $\delta \to 0$. Note that if $\sigma < 1/4$ there is no solution. Let us observe that, in this case,

$$m_\tau = 1 - 2m_\gamma + O(\delta) = -\sqrt{4\sigma - 1} + O(\delta) < 0$$

for $\delta > 0$ sufficiently small, so that care must be taken to work on balls of radius $r \nearrow 1$, as previously mentioned. $\diamond$

6.4. **Improved minimal norm estimates for $\bar\partial$ in the unit ball.** We can now use Example 6.5 to obtain from the twisted version of Berndtsson's Theorem 5.10 the following improved estimates, in the unit ball, for the solution of the $\bar\partial$ equation having minimal norm.

THEOREM 6.6. *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ be a weight in the unit ball satisfying*

$$2dd^c\varphi \geq (n+2\sigma)\Theta$$

for some number $\sigma > 1/4$. Write

$$\alpha_\sigma := \frac{1+\sqrt{4\sigma-1}}{2}.$$

Let θ be a smooth $(0,1)$-form on $\mathbb{B}$ such that

$$\int_{\mathbb{B}} |\theta|^2_\Theta e^{-2\varphi} dV_\Theta < +\infty.$$

Let $w \in \mathbb{B}$ be any point. Then the solution U of the equation $\bar\partial U = \theta$ whose norm

$$\int_{\mathbb{B}} |U|^2 e^{-2\varphi} dV_\Theta$$

is minimal, also satisfies the estimate

$$\int_{\mathbb{B}} e^{-\alpha_\sigma \rho_w} |U|^2 e^{-2\varphi} dV_\Theta \leq C \int_{\mathbb{B}} e^{-\alpha_\sigma \rho_w} |\theta|^2_\Theta e^{-2\varphi} dV_\Theta,$$

where the constant C is independent of w, of θ and of φ.

In view of (7), Theorem 3 is now proved. $\square$

7. L^∞ ESTIMATES IN THE BALL: BERNDTSSON AND SLIGHTLY FURTHER

In [B-1997, Corollary $2'$], Berndtsson established the following result.

THEOREM 7.1 (Berndtsson). *Let $k > \frac{1}{2}(n+1)^2$ and suppose that $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfies $\varepsilon\Theta \le dd^c\varphi \le A\Theta$ for some positive constants $\varepsilon < A$. Let θ be a $\bar\partial$-closed $(0,1)$-form satisfying*

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^k|\theta(z)|_\Theta e^{-\varphi(z)} < +\infty.$$

Suppose $\bar\partial u = \theta$ and u is of minimal $L^2(dV_\Theta,\varphi)$-norm. Then

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^k|u(z)|e^{-\varphi(z)} \le C\sup_{z\in\mathbb{B}}(1-|z|^2)^k|\theta(z)|_\Theta e^{-\varphi(z)} < +\infty$$

for some constant C that is independent of such θ and φ.

The aim of this section is to establish the same result, but with a smaller lower bound for for k. In fact, we will prove the following theorem.

THEOREM 7.2. *Theorem 7.1 holds for any $k > \frac{n}{2}+\frac{1}{4}$.*

REMARK 7.3. In fact, Berndtsson established Theorem 7.1 as a corollary to a more general result about weights φ that don't necessarily satisfy the upper curvature bounds we have assumed. Our methods here can easily be adjusted to prove an analogous improvement for Berndtsson's more general theorem; as our sketch of the proof will show, we are only changing one part of the proof, namely we are applying Theorem 6.3 (or rather its corollary, Theorem 6.6) instead of theorem 5.10. $\diamond$

7.1. **Reduction to L^2 estimates.** The first step is the following lemma, which is a rephrasing of a slightly weaker version of Lemma 3.1 in [B-1997] that makes more explicit the role of the hyperbolic geometry of the ball.

LEMMA 7.4. *Let ψ be a smooth weight satisfying*

$$-M\Theta \le dd^c\psi \le M\Theta$$

for some positive constant M, and let u be a smooth function. Assume that

$$\int_{\mathbb{E}(w,1/2)}|u|^2e^{-2\psi}dV_\Theta \le 1 \quad \textit{and} \quad \sup_{z\in\mathbb{E}(w,1/2)}|\bar\partial u(z)|^2_\Theta e^{-2\psi(z)} \le 1.$$

Then

$$|u(w)|^2e^{-2\psi(w)} \le C$$

for some constant C that is independent of w, u and ψ.

Reduction to Lemma 3.1 in [B-1997]*, and sketch of the latter.* First we reduce to the case $w=0$. Indeed, suppose the result holds, with $w=0$, for all such data. Fix some non-zero $w\in\mathbb{B}$ and let

$$v(z) := u\circ F_w(z) \quad \text{and} \quad \varphi(z) := \psi\circ F_w(z).$$

Note that $dd^c\varphi = F_w^*dd^c\psi$ so that, since Θ is $\mathrm{Aut}(\mathbb{B})$-invariant, φ and ψ satisfy the same curvature conditions. We compute that

$$\int_{\mathbb{B}(0,1/2)}|v(z)|^2e^{-2\varphi}dV_\Theta = \int_{\mathbb{E}(w,1/2)}|u(\zeta)|^2e^{-2\psi(\zeta)}dV_\Theta,$$

and that

$$\begin{aligned}\sup_{z\in\mathbb{B}(0,1/2)} |\bar\partial v(z)|^2_\Theta e^{-2\varphi(z)} &= \sup_{z\in\mathbb{B}(0,1/2)} |F_w^*\bar\partial u(z)|^2_\Theta e^{-2\psi\circ F_w(z)}\\ &= \sup_{z\in\mathbb{B}(0,1/2)} F_w^*\left(|\bar\partial u|^2_\Theta e^{-2\psi}\right)(z)\\ &= \sup_{z\in\mathbb{E}(w,1/2)} |\bar\partial u(z)|^2_\Theta e^{-2\psi(z)}.\end{aligned}$$

By the assumption that the result holds for $w = 0$, we see that

$$|u(w)|^2 e^{-2\psi(w)} = |v(0)|^2 e^{-2\varphi(0)} \le C.$$

Thus it suffices to establish the case $w = 0$.

Observing that, on $\mathbb{B}(0, 1/2)$,

$$d\lambda \le dV_\Theta \le \frac{4}{3} d\lambda \quad \text{and } \frac{1}{2}|\cdot|^2 \le |\cdot|^2_\Theta \le (n+1)|\cdot|^2,$$

we see that our lemma for $w = 0$ is equivalent to Lemma 3.1 in [B-1997], which is the same as our lemma except with dV_Θ and $|\cdot|_\Theta$ replaced by $d\lambda$ and $|\cdot|$ respectively. For the reader's convenience, we sketch the proof of the latter.

If the weight is constant, one deduces the claim from the Bochner Martinelli formula. To reduce to the unweighted case, one replaces u by ue^F where F is holomorphic in $\mathbb{B}(0, 3/4)$ and $\mathrm{Re}\, F + \psi$ is uniformly bounded above and below in $\mathscr{C}^1$-norm; such F exists by Corollary 4.2. Using F in this way allows one to reduce to the unweighted case. This completes the sketch of proof. □

REMARK 7.5. Analogous to the situation alluded to in Remark 7.3, Berndtsson's lemma is a bit more general, and does not assume the boundedness of $dd^c\psi$, but his conclusions are slightly different. Under the above curvature assumptions, his conclusions imply the conclusions of the lemma we've stated here. ◇

7.2. **Conclusion of the proof of Theorem 7.2.** We define the weight

$$\tilde\varphi := \varphi + \tfrac{1}{4}\rho_o,$$

so that $dd^c\tilde\varphi \ge (\frac{1}{4} + \varepsilon)\Theta$ for some positive ε.

Fix $w \in \mathbb{B}$. Let θ satisfy

$$\sup_{z\in\mathbb{B}} |\theta|_\Theta e^{-\varphi} \le 1.$$

Then with $\psi = \tilde\varphi - \left(\frac{n}{2} + \varepsilon\right)\log(1 - |z|^2)$ we have

$$\int_{\mathbb{B}} |\theta|^2_\Theta e^{-2\psi} dV_\Theta = \int_{\mathbb{B}} |\theta|^2_\Theta e^{-2\varphi} \frac{d\lambda(z)}{(1-|z|^2)^{1-\varepsilon}} < +\infty.$$

As $dd^c\psi \ge (\frac{n}{2} + \frac{1}{4} + \varepsilon)\Theta$, Theorem 6.6 (with $\sigma = \frac{1}{4} + \varepsilon$) provides a function u such that $\bar\partial u = \theta$ and

$$\int_{\mathbb{B}} e^{-\alpha_\sigma\rho_w}|u|^2 e^{-2\psi} dV_\Theta \le C \int_{\mathbb{B}} e^{-\alpha_\sigma\rho_w}|\theta|^2_\Theta e^{-2\psi} dV_\Theta.$$

Therefore

$$\int_{\mathbb{E}(w,1/2)} |u|^2 e^{-2\psi} dV_\Theta \lesssim \int_{\mathbb{E}(w,1/2)} e^{-\alpha_\sigma\rho_w}|u|^2 e^{-2\psi} dV_\Theta \le C.$$

It follows from Lemma 7.4 that

$$(1-|w|^2)^{(\frac{n}{2}+\frac{1}{4}+\varepsilon)}|u(w)|e^{-\varphi(w)} = |u(w)|e^{-\psi(w)} \le C.$$

8. The Bergman kernel

8.1. Near-diagonal lower bounds.

Proposition 8.1. *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy $(n+\delta)\Theta \le 2dd^c\varphi \le M\Theta$ for positive constants M and δ. Denote by $K(z,\bar{w})$ the Bergman kernel of the orthogonal projection $P : L^2(dV_\Theta,\varphi) \to \mathscr{F}^2(dV_\Theta,\varphi)$. Then there exist positive constants ε, C_o and C_1 such that for each $z \in \mathbb{B}$ and each $w \in \mathbb{E}(z,\varepsilon)$,*

$$|K(z,\bar{w})|e^{-(\varphi(z)+\varphi(w))} \ge C_1|K(z,\bar{z})|e^{-2\varphi(z)} \ge C_o.$$

Proof. In view of the extremal characterization (8), Proposition 5.6 shows that

$$K(z,\bar{z})e^{-2\varphi(z)} \ge C_o$$

for some $C_o > 0$ independent of z.

Next, fix z and consider the function $F(w) = K(w,\bar{z})$. Then

$$F \in \mathscr{F}^2(dV_\Theta,\varphi) \quad \text{and} \quad |F(z)|^2 e^{-2\varphi(z)} \ge C_o.$$

By Corollary 4.5, there exist positive constants C and ε independent of z such that $|F(w)|e^{-\varphi(w)} \ge C$ for all $w \in B(z,\varepsilon)$. The proof is finished. □

8.2. Diagonal upper bounds for Bergman kernels.

Proposition 8.2. *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy $(n+\delta)\Theta \le 2dd^c\varphi \le M\Theta$ for positive constants M and δ. Denote by $K(z,\bar{w})$ the Bergman kernel for $P : L^2(dV_\Theta,\varphi) \to \mathscr{F}^2(dV_\Theta,\varphi)$. Then there is a positive constant C_1, depending on M and δ but otherwise independent of φ, such that*

$$K(z,\bar{z})e^{-2\varphi(z)} \le C,$$

for all $z \in \mathbb{B}$.

Proof. By Cauchy-Schwarz, $|K(z,\bar{w})|^2 e^{-2\varphi(z)-2\varphi(w)} \le K(z,\bar{z})e^{-2\varphi(z)}K(w,\bar{w})e^{-2\varphi(w)}$, and thus it suffices to prove the first estimate. By Proposition 3.1, it suffices to show that $K_{dV_\Theta,\varphi_z}(0,0)e^{-2\varphi_z(0)} \le C$. Since φ and φ_z have the same curvature bounds, the estimate is a consequence of (9) of Proposition 4.4. □

Combining the last two propositions, we see that, along the diagonal, the pointwise weighted norm of the Bergman kernel remains bounded above and below by positive constants.

Example 8.3. If one takes $\varphi_\alpha = (\frac{n}{2}+\alpha)\rho_o$ for a constant α, one obtains the classical weighted Bergman space of holomorphic functions on the unit ball, that are also square-integrable with respect to the measure $(1-|z|^2)^{2\alpha-1}d\lambda$. (Up to addition of a harmonic function, the weights φ_α are exactly the weights whose curvature is a negative constant multiple of Θ.) For this choice of φ, a computation shows that the Bergman kernel $K_\alpha := K_{dV_\Theta,\varphi_\alpha}$ is given by the formula

$$K_\alpha(z,\bar{w}) = C_\alpha(1 - z\cdot\bar{w})^{-2\alpha}.$$

We therefore have $K_\alpha(z,\bar{z})e^{-2\varphi_\alpha(z)} = C_\alpha$. ◇

Remark 8.4. By the Cauchy-Schwarz Inequality one has the estimate

$$|K(0,\bar{z})|^2 e^{-2\varphi(0)-2\varphi(z)} \le K(z,\bar{z})e^{-2\varphi(z)}K(0,0)e^{-2\varphi(0,0)} \le C^2, \tag{22}$$

where C is an in Proposition 8.2. (Such an upper bound also follows directly from Proposition 4.4.) This upper bound for $K(0,\bar{z})$ is not optimal, even though the bound for $K(z,\bar{z})$ is sharp. Indeed, for the classical weighted Bergman spaces mentioned in the previous remark, one has

$$|K_\alpha(0,\bar{z})|e^{-\varphi_\alpha(0)-\varphi_\alpha(z)} = (1-|z|^2)^{\frac{n}{2}+\alpha},$$

which decays as z approaches $\partial\mathbb{B}$. Of course, Theorem 1, which we are in the process of proving, says that for general weights whose curvature is uniformly bounded by sufficiently positive constants, there is, as well, decay off the diagonal. ◇

8.3. **Off diagonal decay.** In this section we prove the following result.

THEOREM 8.5. *Let $\varphi \in \mathscr{C}^2(\mathbb{B})$ satisfy the curvature bounds $(n+2\sigma)\,\Theta \le 2dd^c\varphi \le M\Theta$ for some $M>0$ and $\sigma > 1/4$. Let K denote the kernel for the Bergman projection $P : L^2(dV_\Theta, \varphi) \to \mathscr{F}^2(dV_\Theta, \varphi)$. Define as before*

$$\alpha_\sigma := \frac{1+\sqrt{4\sigma-1}}{2}.$$

Then there is a constant $C_* > 0$ such that for all $w \in B$,

$$|K(0,\bar{w})|e^{-\varphi(0)-\varphi(w)} \le C_* e^{\frac{-\alpha_\sigma}{2}\rho_o(w)}.$$

REMARK 8.6. Theorem 8.5 is an analog, in the unit ball, of a theorem about the Bergman kernel of generalized Bargmann-Fock spaces, due to Christ in $\mathbb{C}$ [C-1991] and Delin in $\mathbb{C}^n$ for $n \ge 2$ [D-1998]. (See also [SV-2011].) In fact, the proof is very similar to our proof of Theorem 3.2 in [SV-2011], except that it uses Theorem 6.6, which is a sharpened version of Berndtsson's improved L^2 estimates for the minimal solution of $\bar\partial$, and requires the double-twisted method developed above. $\diamond$

Proof of Theorem 8.5. Fix a smooth function $\chi \in \mathscr{C}_0^\infty(\mathbb{B}(0,1/4))$ such that $0 \le \chi \le 1$, $\chi|_{\mathbb{B}(0,1/8)} \equiv 1$ and $\bar\partial\chi$ is supported in $\mathbb{B}(0,1/4) - \mathbb{B}(0,1/8)$. Let $\chi_w := \chi \circ F_w$. Then we have

$$\chi_w \in \mathscr{C}_0^\infty(\mathbb{E}(w,1/2)), \quad 0 \le \chi_w \le 1, \quad \chi_w|_{\mathbb{E}(w,1/8)} \equiv 1,$$

$$\text{Support}(\bar\partial\chi_w) \subset \mathbb{E}(w,1/4) - \mathbb{E}(w,1/8), \quad \text{and} \quad |\bar\partial\chi_w|_\Theta \le Ce^{-\rho_w}$$

for some positive constant C that is independent of w. The last estimate follows from the length formula (4) for radial $(0,1)$-forms and the invariance formula (6).

Using (9) of Proposition 4.4, we have

$$\begin{aligned}
|K(0,\bar{w})|e^{-\varphi(0)-\varphi(w)} &\lesssim \left(\int_{\mathbb{E}(w,1/4)} |K(0,\bar\zeta)|^2 e^{-2\varphi(0)-2\varphi(\zeta)} dV_\Theta(\zeta)\right)^{1/2} \\
&\le \left(\int_{\mathbb{B}} |K(0,\bar\zeta)|^2 \chi_w(\zeta) e^{-2\varphi(0)-2\varphi(\zeta)} dV_\Theta(\zeta)\right)^{1/2} \\
&= \left(e^{-\varphi(0)}\int_{\mathbb{B}} \left(\chi_w(\zeta)K(\zeta,0)e^{-\varphi(0)}\right) K(0,\bar\zeta) e^{-2\varphi(\zeta)} dV_\Theta(\zeta)\right)^{1/2} \\
&= \left|e^{-\varphi(0)} P(\chi_w K(\cdot,0)e^{-\varphi(0)})(0)\right|^{1/2},
\end{aligned}$$

where P is the Bergman projection. Now, the function $\zeta \mapsto \chi(\zeta)K(\zeta,0)e^{-\varphi(0)}$ is smooth and compactly supported, and therefore lies in the domain of $\bar\partial$. It follows that, since $\chi_w(0) = 0$,

$$P(\chi K(\cdot,0)e^{-\varphi(0)})(0) = \chi_w(0)K(0,0)e^{-\varphi(0)} - u(0) = -u(0),$$

where u is the solution of the equation $\bar\partial u = \bar\partial(\chi K(\cdot,\bar z)e^{-\varphi(z)})$ having minimal L^2-norm. Moreover, since $\chi_w \equiv 0$ on $\mathbb{B}(0,1/2)$, $u \in \mathcal{O}(\overline{\mathbb{B}(0,1/2)})$ and therefore again by (9) of Proposition 4.4,

$$|u(0)|^2 e^{-2\varphi(0)} \lesssim \int_{\mathbb{B}(0,1/4)} |u|^2 e^{-2\varphi} dV_\Theta.$$

Let $\gamma = e^{-\alpha_\sigma \rho_o}$. Then by Theorem 6.6

$$\begin{aligned}
|u(0)|^2 e^{-2\varphi(0)} &\lesssim \int_{\mathbb{B}(0,1/4)} |u|^2 e^{-2\varphi} dV_\Theta \lesssim \int_{\mathbb{B}(0,1/4)} |u|^2 \gamma e^{-2\varphi} dV_\Theta \\
&\lesssim \int_{\mathbb{B}} \gamma |\bar\partial\chi_w|^2_\Theta |K(\zeta,0)|^2 e^{-2\varphi(0)-2\varphi(\zeta)} dV_\Theta(\zeta).
\end{aligned}$$

Observe finally that $\bar{\partial}\chi_w$ is supported on the ball $\mathbb{E}(w, 1/4)$. Therefore on $\mathbb{E}(w, 1/4)$,

$$\gamma \lesssim e^{-\alpha_\sigma \rho_o(w)} \quad \text{and} \quad |\bar{\partial}\chi_w|^2_\Theta \lesssim 1.$$

We thus have the estimate

$$\begin{aligned} (23) \qquad |K(w,0)|e^{-\varphi(w)-\varphi(0)} &\lesssim \sqrt{|u(0)|e^{-\varphi(0)}} \\ &\leq C_o e^{-\frac{\alpha_\sigma}{4}\rho_o(w)} \left(\int_{\mathbb{E}(w,1/4)} |K(\zeta,0)|^2 e^{-2\varphi(\zeta)-2\varphi(0)} dV_\Theta(\zeta) \right)^{1/4}. \end{aligned}$$

Let $\alpha_o = 0$ and $\alpha_{j+1} := \frac{\alpha_\sigma}{4} + \frac{\alpha_j}{2}$. Note that, for $j \geq 1$, $\alpha_j = \frac{\alpha_\sigma}{4}\left(1 + ... + 2^{-j}\right)$, so that α_j increases to $\alpha_\sigma/2$. We claim that

$$(24) \qquad |K(w,0)|e^{-\varphi(w)-\varphi(0)} \leq C^2 e^{-\alpha_j \rho_o(w)}$$

for some constant C that is independent of j. In fact, we can take

$$C := 1 + C_o + \int_{\mathbb{B}(0,1/4)} \left(\frac{1-|z|^2}{(1-|z|)^2} \right)^{2\alpha_\sigma} dV_\Theta + \sup_{z \in \mathbb{B}} |K(z,z)|e^{-2\varphi(z)}.$$

(The right-most summand is bounded by Proposition 8.2.) We prove (24) by induction. The base case $j = 0$ follows from the definition of C and the fact that $C > 1$. Assuming the result for j, one has

$$\begin{aligned} \int_{\mathbb{E}(w,1/4)} |K(\zeta,0)|^2 e^{-2\varphi(0)-2\varphi(\zeta)} dV_\Theta(\zeta) &\leq C^2 \int_{\mathbb{E}(w,1/4)} (1-|z|)^{2\alpha_j} dV_\Theta(z) \\ &= C^2 \int_{\mathbb{B}(0,1/4)} (1-|F_w(z)|^2)^{2\alpha_j} dV_\Theta \\ &= (1-|w|^2)^{2\alpha_j} C^2 \int_{\mathbb{B}(0,1/4)} \frac{(1-|z|^2)^{2\alpha_j}}{|1-\bar{w}\cdot z|^{4\alpha_j}} dV_\Theta(z) \\ &= (1-|w|^2)^{2\alpha_j} C^4, \end{aligned}$$

where in passing from the second to the third line, we used (5), and in the last line the exponent for C can be taken to be 4 instead of 3 because $C > 1$. Applying (23) proves the case of $j + 1$.

We therefore conclude that, with $C_* = C$ for example,

$$|K(w,0)|e^{-\varphi(w)-\varphi(0)} \leq C_* \exp\left(\frac{-\alpha_\sigma}{2} \rho_o(w) \right),$$

as claimed. □

REMARK 8.7. Note that our decay estimate suffers from the fact that our initial estimate for K is the crude estimate of Proposition 8.2, which is obtained from the diagonal behavior of the Bergman kernel, precisely where the kernel is most badly behaved. Moreover, since for any $w \in \mathbb{B}$ one has

$$\int_{\mathbb{B}} |K(w,z)|^2 e^{-2\varphi(z)-2\varphi(w)} dV_\Theta(z) = 1,$$

there is hope that $|K(z,0)|e^{-\varphi(z)-\varphi(0)}$ behaves better than just boundedly as $z \to \partial\mathbb{B}$. At present we don't have any idea how to improve the initial estimate. ◇

Conclusion of the proof of Theorem 1. Let $\varphi_z := \varphi \circ F_z$. As we have seen numerous times now, ψ_z and φ satisfy the same curvature bounds, and therefore by Theorem 8.5 we have

$$|K_{dV_\Theta,\varphi_z}(0,\bar{\zeta})|e^{-\varphi_z(0)-\varphi_z(\zeta)} \leq C_* e^{-\frac{\alpha_\sigma}{2}\rho_o(\zeta)}.$$

Letting $\zeta = F_z(w)$ and applying Proposition 3.1, we obtain

$$|K_{dV_{\Theta},\varphi}(z,\bar{w})|e^{-\varphi(z)-\varphi(w)} \leq C_* e^{-\frac{\alpha_\sigma}{2}\rho(z,w)} \leq 2^{\alpha_\sigma} C_* e^{-\alpha_\sigma d_\Theta(z,w)},$$

where the last inequality follows from (7). This completes the proof of Theorem 1. $\square$

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