Math 533 - Spring 2025 Practice Final Examination

1. For a measurable function $f : (0,\infty) \to [-\infty,\infty]$ define

$$Ef(x) := \frac{1}{\sqrt{x}} \int_{e^{-x}}^{1} e^{-t} f(e^{-t}) dt.$$

Show that if $f \in L^2((0,\infty))$ then $Ef : (0,\infty) \to \mathbb{R}$ is continuous and uniformly bounded.

2. Let

$$\Lambda f(x) := \frac{1}{\sqrt{x}} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt, \quad x \in (0,\infty).$$

Prove that $\Lambda : L^2(0,\infty) \to L^2(0,\infty)$ is continuous.

3. Let $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $A : L^p(\mathbb{R}^m) \to L^p(\mathbb{R}^m)$ is linear and continuous.

a. Show that there is a unique linear map $B : L^q(\mathbb{R}^m) \to L^q(\mathbb{R}^m)$ such that

$$\int_{\mathbb{R}^m} (Af) g dm = \int_{\mathbb{R}^m} f(Bg) dm \quad \text{for all } f \in L^p(\mathbb{R}^m), g \in L^q(\mathbb{R}^m).$$

b. Is B continuous?

c. Show that A is surjective if and only if $||Bg||_q \geq C||g||_q$ for some constant $C > 0$, and that in this case there is a continuous map $S : L^p(\mathbb{R}^m) \to L^p(\mathbb{R}^m)$ such that $ASg = g$ for all $g \in L^p(\mathbb{R}^m)$.

4. Let $f \in \mathscr{C}_c^k(\mathbb{R}^m)$ and let $g \in \mathscr{C}^\ell(\mathbb{R}^m)$. Prove that $f * g \in \mathscr{C}^{k+\ell}(\mathbb{R}^m)$ and that if $\mathrm{Support}(g)$ is compact then so is $\mathrm{Support}(f * g)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous 1-periodic function. Assume that

$$\int_0^1 f(x) dx = 0.$$

Show that

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{24} \int_0^1 |f'(x)|^2 dx.$$

6. Let ψ be a locally integrable function on $\mathbb{R}$ such that for some constant $C > 0$ the inequalities

$$\left| \int_{\mathbb{R}} \psi \varphi dm \right| \leq C \sup_{y \in K} \sum_{0 \leq j \leq m} \left| \varphi^{(j)}(y) \right|$$

hold for all $\varphi \in \mathcal{S}$, the Schwartz space on $\mathbb{R}$. (Here $\varphi^{(j)}$ denotes the j^{th} derivative of φ.) For each $y \in \mathbb{R}$, consider the initial value problem

$$\frac{d\phi}{dt} = -y^2 \phi, \quad \phi(0) = \hat{\psi}(y).$$

a. Show that every solution $\phi_y : \mathbb{R} \to \mathbb{R}$ of this equation satisfies

$$\lim_{|y| \to \infty} \phi_y(t) = 0$$

for all $t > 0$, and that this limit is locally uniform in $t \in (0,\infty]$.

b. Let

$$\Phi(x,t) := \int_{\mathbb{R}} \phi_y(t) e^{-2\pi\sqrt{-1}xy} dy.$$

Show that

i. Φ satisfies the partial differential equation $\dfrac{\partial \Phi}{\partial t} = \dfrac{1}{4\pi^2}\dfrac{\partial^2 \Phi}{\partial x^2}$,

ii. for each $t > 0$ the function $x \mapsto \Phi(x,t)$ lies in the Schwartz space $\mathcal{S}$ of $\mathbb{R}$, and

iii. $\lim_{t\to\infty} \Phi(\cdot, t) = 0$.