ORDINARY DIFFERENTIAL EQUATIONS

DROR VAROLIN

CONTENTS

1. DEFINITION OF ODE

DEFINITION 1.1. Let $D \subset \mathbb{R}^n$ be a domain, i.e., an open connected set.

i. A time-dependent vector field on D is a pair consisting of a domain $\mathfrak{p}_1(V) = D$ such that $D \times \{0\} \subset V$ together with a Borel-measurable map $F : V \to \mathbb{R}^n$.
ii. The time-dependent vector field F is said to be autonomous if $V = D \times \mathbb{R}$ and for each $x \in D$, $F(x, \cdot)$ is constant. That is to say, there is a Borel measurable map $\xi : D \to \mathbb{R}^n$ such that $F(x,t) = \xi(x)$ for all $x \in D$ and all $t \in \mathbb{R}$. $\diamond$

Vector fields are the data of ordinary differential equations (ODE). From such data one wishes to produce so-called *integral curves*; finding these integral curves constitutes solving the ODE.

DEFINITION 1.2. Let F be a time-dependent vector field on a domain $V \subset D \times \mathbb{R}$. An integral curve through $x \in D$ with initial time s is an open set $I_{(x,s)} \subset \mathbb{R}$ containing s, together with an absolutely continuous curve $\gamma_{(x,s)} : I_{(x,s)} \to D$, such that

i. $\gamma_{(x,s)}(s) = x$,
ii. $(\gamma_{(x,s)}(t), t) \in V$ for all $t \in I_{(x,s)}$, and
iii. $\frac{d\gamma_{(x,s)}(t)}{dt} = F(\gamma_{(x,s)}(t), t)$ for almost every $t \in I_{(x,s)}$.

The central question of ODE is whether, for a given time-dependent vector field, integral curves exist and, if so, are unique. In the next section we shall establish an existence theorem under rather weak hypotheses on the time-dependent vector field. Later on we shall impose slightly stronger conditions and then simultaneously prove existence and uniqueness.

2. CAUCHY-PEANO EXISTENCE THEOREM FOR FIRST ORDER ODE

THEOREM 2.1 (Cauchy-Peano Existence Theorem). *Let $D \subset \mathbb{R}^n$ and let $V \subset D \times R$ be domains. If $F : V \to \mathbb{R}^n$ is a continuous time-dependent vector field then for each $(x, s) \in V$ there exists an integral curve $\gamma_{x,s} : I_{x,s} \to D$ of F passing through x at the initial time s.*

Before turning to the proof of Theorem 2.1, let us note that continuity of the time-dependent vector field F is too weak an assumption as to imply uniqueness of the integral curve. Among the simplest examples is the vector field $F(x, t) = 3x^{2/3}$, for which the curve

$$\gamma_{0,0}^{\langle c\rangle} : I_{0,0} = \mathbb{R} \ni t \mapsto (t - c)^3 \chi_{[c,\infty)}(t) \in \mathbb{R}$$

is an integral curve through 0, with initial time 0 (i.e., $\gamma_{0,0}^{\langle c\rangle}(0) = 0$) whenever $c \geq 0$. (The case $c = \infty$, which we can take to mean the identically zero solution, is also such an integral curve.)

Proof of Theorem 2.1. Fix $(x, s) \in V$ and $\delta > 0$ such that $D_\delta(x, s) := \overline{B_\delta(x)} \times \overline{I_\delta(s)} \subset V$, where $I_\delta(s) = (s - \delta, s + \delta)$. Choose some number $\varepsilon > 0$ such that

$$\varepsilon \cdot \max\left(1, \sup_{D_\delta(x,s)} |F|\right) < \delta.$$

We claim there exist absolutely continuous curves $\gamma_j : I_\varepsilon(s) \to D$, $j = 1, 2, ...,$, such that

$$(1) \qquad \gamma_j(s) = x, \quad \left|\gamma_j'(t) - F(\gamma_j(t), t)\right| \leq 1/j \quad \text{and} \quad |\gamma_j(\tau_1) - \gamma_j(\tau_2)| \leq |\tau_1 - \tau_2| \sup_{D_\delta(x,s)} |F|$$

for all $\tau_1, \tau_2 \in I_\delta(s)$ and almost all $t \in I_\delta(s)$. To define γ_j let $N > 0$ be an integer (which will soon be taken very large), let $t_o := s$, let $t_m := s + m\varepsilon/N$, $m \in \mathbb{Z} \cap (-N, N)$, and define

- $\gamma_j(s) = x$,
- for $m \geq 0$ and $t \in (t_m, t_{m+1}]$, $\gamma_j(t) := \gamma_j(t_m) + (t - t_m)F(\gamma_j(t_m), t_m)$, and
- for $m < 0$ and $t \in [t_m, t_{m+1})$, $\gamma_j(t) := \gamma_j(t_{m+1}) + (t - t_{m+1})F(\gamma_j(t_{m+1}), t_{m+1})$.

First note that if $t \in (t_0, t_1]$ then

$$|\gamma_j(t_1) - x| = |t_1 - t_0||F(x, s)| \leq \frac{\varepsilon}{N} \sup_{D_\delta(x,s)} |F| \leq \delta/N.$$

Continuing inductively, we see that if $m > 0$ and $t \in (t_m, t_{m+1}]$ then

$$|\gamma_j(t_{m+1}) - \gamma_j(t_m)| = |t_{m+1} - t_m| \cdot |F(x, s)| \leq \frac{\varepsilon}{N} \sup_{D_\delta(x,s)} |F| \leq \delta/N.$$

Thus we see that $|\gamma_j(t_m) - x| \leq m\delta/N \leq \delta$. Similarly, if $m < 0$ then $|\gamma_j(t_m) - \gamma_j(t_{m+1})| \leq -m\delta/N < \delta$. Therefore $\gamma_j(t_m) \in B_\delta(x)$ for all $m \in (-N, N) \cap \mathbb{Z}$, and since $|t_m - s| \leq \varepsilon < \delta$, the curves γ_j are well-defined.

Next, by checking left and right hand limits, one easily sees that γ_j is continuous.

Now, if we set $m_* := m$ for $m \geq 0$ and $m_* = m + 1$ for $m < 0$ then

$$(2) \qquad |\gamma_j'(t) - F(\gamma_j(t), t)| \leq |F(\gamma_j(t_{m_*}), t_{m_*}) - F(\gamma_j(t), t)| \quad \text{for all } t \in (t_m, t_{m+1}).$$

Since $D_\delta(x, s)$ is compact, F is uniformly continuous on $D_\delta(x, s)$. Therefore, by taking N sufficiently large we can make the right hand side of (2) as small as we like.

Next, if $t_k < \tau_1 \leq t_{k+1}$ and $t_m < \tau_2 \leq t_{m+1}$ for some k, m satisfying $-N < k \leq m < N$ then

$$|\gamma_j(\tau_2) - \gamma_j(\tau_1)| = \left| \gamma_j(\tau_2) - \gamma_j(t_m) + \left(\sum_{\ell=k+1}^{m-1} \gamma_j(t_{\ell+1}) - \gamma_j(t_\ell) \right) + \gamma_j(t_{k+1}) - \gamma_j(\tau_1) \right|$$

$$= \left| (\tau - t_m) F(\gamma_j(t_{m_*}), t_{m_*}) + \left(\sum_{\ell=k+1}^{m-1} (t_{\ell+1} - t_\ell) F(\gamma_j(t_{\ell_*}), t_{\ell_*}) \right) + (t_{k+1} - \tau_1) F(\gamma_j(t_{k_*}), t_{k_*}) \right|$$

$$\leq (\tau - t_m) |F(\gamma_j(t_{m_*}), t_{m_*})| + \left(\sum_{\ell=k+1}^{m-1} (t_{\ell+1} - t_\ell) |F(\gamma_j(t_{\ell_*}), t_{\ell_*})| \right) + (t_{k+1} - \tau_1) |F(\gamma_j(t_{k_*}), t_{k_*})|$$

$$\leq \left(\sup_{D_\delta(x,s)} |F| \right) \left(\tau_2 - t_m + \left(\sum_{\ell=k+1}^{m} t_{\ell+1} - t_\ell \right) + t_{k+1} - \tau_1 \right) = |\tau_2 - \tau_1| \sup_{D_\delta(x,s)} |F| \leq \frac{\delta}{\varepsilon} |\tau_2 - \tau_1|.$$

Thus (1) is proved.

By (1) the sequence $\{\gamma_j\}$ is uniformly bounded and equicontinuous. Since $D_\delta(x, s)$ is compact, the theorem of Ascoli-Arzela yields a subsequence γ_{j_ℓ} converging uniformly to $\gamma : I_\varepsilon(s) \to B_\delta(s)$. The first estimate in (1) implies that $\gamma' = \lim_\ell \gamma'_{j_\ell}$ exists and equals $F(\gamma(\cdot), \cdot)$ almost everywhere. Thus γ is the integral curve we seek, and the proof is complete. □

REMARK 2.2. In fact, in Theorem 2.1 the integral curve γ obtained in the proof is differentiable. To see that this is the case, observe that since γ is absolutely continuous,

$$\gamma_j(t) = x + \int_s^t \gamma_j'(\tau) d\tau = x + \int_s^t \big(F(\gamma_j(\tau), \tau) + \big(\gamma_j'(\tau) - F(\gamma_j(\tau), \tau)\big) \big) \, d\tau.$$

By the first estimate in (1), we may pass to the limit as $j \to \infty$, and since $\lim_j (\gamma_j' - F(\gamma_j(\cdot), \cdot)) = 0$ a.e., we have

$$\gamma(t) = x + \int_s^t F(\gamma(\tau), \tau) d\tau.$$

Thus $\gamma(s) = x$, γ is differentiable, and $\gamma'(t) = F(\gamma(t), t)$, as desired. ◇

3. CONTRACTING MAPS

In the proof of the existence and uniqueness theorem to be stated in the next section, we will need to make use of an iteration scheme due to Picard. The convergence of this iteration scheme depends on the concept of contracting map, which we now define.

DEFINITION 3.1. Let X be a subset of a metric space. A map $S : X \to X$ is said to be *contracting* if there exists some $r \in (0, 1)$ such that

$$d(Sx, Sy) \leq r \cdot d(x, y)$$

for all $x, y \in X$. ◇

The basic fact about contracting maps is the following result.

PROPOSITION 3.2. *Let X be a complete metric space. If $S : X \to X$ is a contracting map then S has a unique fixed point.*

Proof. Let $x_o \in X$ be any point, and set

$$x_j := S^{(j)} x_o, \quad j = 1, 2, ...$$

where $S^{(1)} = S$ and $S^{(j)} := S \circ S^{(j-1)}$. For $j, k \in \mathbb{N}$ satisfying $j < k$,

$$d(x_j, x_k) \le \sum_{\ell=j}^{k-1} d(x_\ell, x_{\ell+1}) \le \sum_{\ell=j}^{k-1} r^\ell d(x, Sx) = \frac{r^j(1 - r^{k-j})}{1 - r} d(x, Sx) \le \frac{r^j}{1 - r} d(x, Sx).$$

Thus $\{x_j\}$ is a Cauchy sequence, and since X is complete the limit

$$x_* := \lim x_j$$

exists. Since S is evidently continuous,

$$x_* = \lim S^{(j)} x_* = \lim S \circ S^{(j-1)} x_* = S(\lim S^{(j)} x_*) = S x_*.$$

Thus x_* is a fixed point of S. Finally, if y is another fixed point of S then

$$0 \le (1 - r) d(x_*, y) = d(Sx_*, Sy) - r d(x_*, y) \le (r - r) d(x_*, y) = 0.$$

Thus $y = x_*$, and the proof is complete. □

4. The Existence and Uniqueness Theorem for First Order ODE

Definition 4.1. Let $f : U \to \mathbb{R}^n$ be a function defined on a domain $U \subset \mathbb{R}^m$.

i. We say that f is *locally Lipschitz* if for each $p \in U$ and each $\varepsilon \in (0, \mathrm{dist}(p, U^c))$ there exists a constant $K = K_{\varepsilon,p}$ such that

$$|f(x) - f(y)| \le K|x - y|$$

for all $x, y \in B(p, \varepsilon) := \{z \in \mathbb{R}^m \, ; \, |z - p| < \varepsilon\}$.

ii. We say that f is *globally Lipschitz* if there is a constant K such that

$$|f(x) - f(y)| \le K|x - y|$$

for all $x, y \in U$. ◇

Remark 4.2. The notion of Lipschitz makes sense on any metric space. ◇

Example 4.3. Any differentiable function is locally Lipschitz. On the other hand, the function $f : \mathbb{R} \ni x \mapsto |x| \in \mathbb{R}$ is (globally) Lipschitz but not differentiable. ◇

Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains. For each $t \in \mathbb{R}$, we write

$$V_t = \{x \in D \, ; \, (x, t) \in V\}.$$

(It may happen that $V_t = \emptyset$ for some t.)

Definition 4.4. Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains, let $F : V \to \mathbb{R}^n$ be a time-dependent vector field. We say that F is *uniformly locally Lipschitz* if for each $t \in \mathfrak{p}_2(V) \subset \mathbb{R}$ the function $F_t : V_t \to \mathbb{R}^n$ is locally Lipschitz and moreover the Lipschitz constant can be taken locally uniform with respect to t. In other words, for each $(x, t) \in V$ there is a neighborhood $U \subset V$ containing (x, t) and a constant $K > 0$ such that $|F_s(x_1) - F_s(x_2)| \le K|x_1 - x_2|$ for all $x_1, x_2 \in D$ such that $(x_1, s), (x_2, s) \in U$. ◇

THEOREM 4.5 (Existence and Uniqueness Theorem for Ordinary Differential Equations). *Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains and let $F : V \to \mathbb{R}^n$ be a continuous and locally uniformly Lipschitz time-dependent vector field. For each $(x,s) \in V$ there exists an integral curve $\gamma_{(x,s)} : I_{(x,s)} \to D$ for F. Moreover, the set of integral curves possesses the following uniqueness property: if $\gamma_{(x,s)} : I_{(x,s)} \to D$ and $\tilde{\gamma}_{(x,s)} : \tilde{I}_{(x,s)} \to D$ are two integral curves through x at time s, then $\gamma_{(x,s)}(t) = \tilde{\gamma}_{(x,s)}(t)$ for all $t \in I_{(x,s)} \cap \tilde{I}_{(x,s)}$.*

Proof. Let $(x_o, t_o) \in V$ and choose $\varepsilon > 0$ such that F is continuous in $B(x_o, \varepsilon) \times (t_o - \varepsilon, t_o + \varepsilon)$ and Lipschitz in the first variable with Lipschitz constant K, i.e.,

$$|F(x,t) - F(y,t)| \le K|x-y|$$

for all $(x,t),(y,t) \in B(x_o,\varepsilon) \times (t_o - \varepsilon, t_o + \varepsilon)$. By continuity, if $\varepsilon > 0$ is small enough then there exists a constant $M > 0$ such that

$$|F(x,t)| \le M$$

for all $(x,t) \in B(x_o,\varepsilon) \times (t_o - \varepsilon, t_o + \varepsilon)$.

Choose positive constants $\alpha < \varepsilon$ and $\beta < \varepsilon$ such that $\alpha < M\beta$ and $\alpha K < 1$. Note that

$$B_\beta(x_o) \times (t_o - \alpha, t_o + \alpha) \subset B(x_o,\varepsilon) \times (t_o - \varepsilon, t_o + \varepsilon).$$

Let $\mathscr{A}$ denote the set of continuous maps $\phi : I_\alpha \to \mathbb{R}^n$ such that

$$|\phi(t) - x_o| \le \beta \quad \text{for all } t \in I_\alpha.$$

Equip $\mathscr{A}$ with the norm

$$||\phi||_\infty := \sup_{I_\alpha} |\phi|.$$

Since uniform limits of continuous functions are continuous, $\mathscr{A}$ is a closed bounded subset of the Banach (and hence complete metric) space $(L^\infty(I_\alpha))^n$. Thus $\mathscr{A}$ is itself a complete metric space with respect to the metric

$$d(\phi, \tilde{\phi}) := ||\phi - \tilde{\phi}||_\infty.$$

Consider the operator T defined by

$$T\phi(t) := x_o + \int_{t_o}^{t} F(\phi(s), s)ds.$$

Observe first that if $\phi \in \mathscr{A}$ then clearly $T\phi$ is continuous and defined on all of I_α. Moreover, for $t \in I_\alpha$ one has

$$|T\phi(t) - x_o| \le M|t - t_o| \le M\alpha < \beta.$$

Thus $T\phi \in \mathscr{A}$, which is to say,

$$T : \mathscr{A} \to \mathscr{A}.$$

Next, observe that if $\phi_1, \phi_2 \in \mathscr{A}$ then

$$\begin{aligned}
|T\phi_1(t) - T\phi_2(t)| &= \left| \int_{t_o}^{t} \left(F(\phi_1(s), s) - F(\phi_2(s), s) \right) ds \right| \\
&\le \int_{t_o}^{t} K \left| \phi_1(s) - \phi_2(s) \right| ds \\
&\le K\alpha \sup_{I_\alpha} \left| \phi_1 - \phi_2 \right|.
\end{aligned}$$

Thus $T : \mathscr{A} \to \mathscr{A}$ is a contracting map, so by Proposition 3.2 T has a unique fixed point $\phi_* \in \mathscr{A}$.

Being a fixed point of T, ϕ_* satisfies the equation

(3)
$$\phi_*(t) = x_o + \int_{t_o}^{t} F(\phi_*(s), s)ds,$$

and therefore

$$\frac{\phi_*(t+h) - \phi_*(t)}{h} = \frac{1}{h}\int_t^{t+h} F(\phi_*(s), s)ds \overset{h\to 0}{\longrightarrow} F(\phi_*(t), t).$$

Since $\phi_* \in \mathscr{A}$, the latter limit is continuous, and thus the fixed point ϕ_* of T is continuously differentiable, and satisfies the equation

$$\phi_*'(t) = F(\phi_*(t), t).$$

Since $\phi_*(t_o) = x_o$, we see that $\gamma_{(x_o,t_o)}(t) := \phi_*(t)$ is an integral curve of F through x_o at time t_o.

Conversely, any integral curve of F satisfies the equation (3), and is therefore a fixed point of T. Since contracting maps have a unique fixed point, any two integral curves whose domain contains I_α must agree on I_α. By carrying out the same proof in small intervals centered at all points of the intersection of the open set $I_{(x,s)} \cap \tilde{I}_{(x,s)}$, we obtain the uniqueness statement claimed in the theorem. The proof is therefore complete. □

COROLLARY 4.6 (Linear Equations). *Let $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ be a time dependent vector field of the form*

$$F(y,t) = A(t)y$$

for some continuous matrix-valued function $A : I \to M_n(\mathbb{R})$, where $I \subset \mathbb{R}$ is an interval. Then the time-dependent flow $\Phi_F : \mathbb{R}^n \times I \times I \to \mathbb{R}^n$ is of the form

$$\Phi_F(y,s,t) = \alpha(s,t)y,$$

for some function $\alpha : I \times I \to GL_n(\mathbb{R})$.

Proof. Consider the matrix-valued function

$$a(s,t) := \int_s^t A(\tau)d\tau.$$

Define its exponential

$$\alpha(s,t) := e^{a(s,t)} := \mathrm{Id} + \sum_{j=1}^{\infty} \frac{(a(s,t))^j}{j!}$$

which converges locally uniformly on I. Moreover,

$$\begin{aligned}
\frac{d}{dt}\frac{(a(s,t))^j}{j!} &= \lim_{h\to 0} \frac{(a(s,t+h))^j - (a(s,t))^j}{j!h} \\
&= \lim_{h\to 0} \frac{a(s,t+h) - a(s,t)}{h} \cdot \frac{1}{j!}\left(\sum_{j=0}^{n-1} a(s,t+h)^{n-1-j} a(s,t)^j\right) \\
&= \lim_{h\to 0} \left(\frac{1}{h}\int_t^{t+h} A(\tau)d\tau A(t)\right) \cdot \frac{1}{j!}\left(\sum_{k=0}^{j-1} a(s,t+h)^{j-1-k} a(s,t)^k\right) \\
&= A(t)\frac{(a_s(t))^{j-1}}{(j-1)!}.
\end{aligned}$$

In this computation, we use the fact that

$$(A-B)\sum_{k=0}^{j-1} A^{j-1-k}B^k = A^k - B^k + \sum_{j=0}^{k-1}$$

Since the series

$$\sum_{j=1}^{\infty} \frac{(a(s,t))^{j-1}}{(j-1)!}$$

converges to $e^{a(s,t)}$ locally uniformly in t, we have

$$\frac{d}{dt}\alpha(s,t) = A(t)\alpha(s,t).$$

It follows that the map

$$\Psi : (y,s,t) \mapsto \alpha(s,t)y$$

satisfies

$$\frac{\partial \Psi}{\partial t}(y,s,t) = A(t)\Psi(y,s,t).$$

By the uniqueness part of Theorem 4.5, we must have $\Psi = \Phi_F$. □

5. Maximal Integral Curves, Fundamental Domains, and Flows

Our next goal is to 'glue together' the integral curves of a time-dependent vector field. The first task is to maximally extend integral curves.

Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains, and let $F : V \to \mathbb{R}^n$ be a continuous, uniformly locally Lipschitz time-dependent vector field. Fix an initial condition $(x,s) \in V$. By Theorem 4.5, F has an integral curve through x with initial time s.

PROPOSITION 5.1. *With the notation above, there exists a unique integral curve $\gamma_{(x,s)} : I_{(x,s)} \to D$ for F passing through x with initial time s such that if $\phi : I \to D$ is any other integral curve for F passing through x at time s then $I \subset I_{(x,s)}$ and $\phi = \gamma_{(x,s)}|_I$.*

Proof. The set $\mathscr{I}_{(x,s)}$ of all integral curves for F passing through x with initial time s is partially ordered with respect to inclusion of graphs. Moreover, given two such integral curves $\phi_i : I_i \to D$, $i = 1,2$, Theorem 4.5 implies that the function

$$\phi(t) := \begin{cases} \phi_1(t)\ , & t \in I_1 \\ \phi_2(t)\ , & t \in I_2 \end{cases}$$

is well-defined, and therefore $\phi : I_1 \cup I_2 \to D$ is also an integral curve for F passing through x with initial time s. It follows that $\mathscr{I}_{(x,s)}$ is a directed set. We have to show that it has a maximal element, which is then of course unique.

Toward this end, let $\{\phi_i : I_i \to D\}_{i\in A}$ be a maximal linearly ordered subset of $\mathscr{I}_{(x,s)}$. Then the set $I := \bigcup_{i\in A} I_i$ is open, and the curve $\phi : I \to D$ defined by

$$\phi(t) = \phi_i(t), \quad t \in I_i$$

is well-defined by the uniqueness part of Theorem 4.5, therefore in $\mathscr{I}_{(x,s)}$. Thus $\mathscr{I}_{(x,s)}$ has a unique maximal element in $\mathscr{I}_{(x,s)}$. □

DEFINITION 5.2. The unique maximal element of the set $\mathscr{I}_{(x,s)}$ defined in the proof of the previous proposition is called the *maximal integral curve for F through (x,s)*. We shall denote the maximal integral curve for F through (x,s) by

$$\Gamma_{(x,s)} : \mathcal{I}_{(x,s)} \to D.$$

One can also consider the unions of the graphs of the maximal integral curves.

DEFINITION 5.3. The set

$$\mathscr{U}_F := \{(x,s,t)\ ;\ (x,s) \in V, t \in \mathcal{I}_{(x,s)}\} \subset V \times \mathbb{R}$$

is called the fundamental domain of the time-dependent vector field F, and the map

$$\Phi_F : \mathscr{U}_F \to D$$

defined by $\Phi_F(x,s,t) := \Gamma_{(x,s)}(t)$ is called the time-dependent flow of F. ◇

DEFINITION 5.4. The map $\Phi_s^t : U_s^t \to D$

$$\Phi_s^t(x) := \Gamma_{(x,s)}(t) = \Phi_F(x,s,t) \tag{4}$$

is called the time-t map for the initial time s. ◇

The uniqueness part of Theorem 4.5 implies a symmetry appearing in the composition law for the maps (4), stated in the following result.

PROPOSITION 5.5. *For each $s \in \mathbb{R}$ one has*

$$\Phi_s^s(x) = x \quad \textit{for all } x \in V_s.$$

Moreover, if $(x,s,t) \in \mathscr{U}_F$ and $(\Phi_s^t(x), t, r) \in \mathscr{U}_F$, we have the pseudo-group law

$$\Phi_t^r \circ \Phi_s^t(x) = \Phi_s^r(x).$$

6. AUTONOMOUS VECTOR FIELDS

From the point of view of classical mechanics, the general setting of time-dependent vector fields corresponds to physical systems in which the laws of physics change with time. Such situations can happen, but in nature we mostly find them when the particular physical system we are studying is not *closed*, i.e., it is part of a larger physical system.

By definition, the vector field representing a closed physical system is autonomous. That is to say, for each $x \in D$

$$t \mapsto F(x,t)$$

is constant. In this case, we choose the convention of always taking initial value problems to start at time $s = 0$.

The fundamental domain and the flow are defined only slightly differently, so as to eliminate the initial time. Let us make the definitions precise.

DEFINITION 6.1. Let $\xi : D \to \mathbb{R}^n$ be a locally Lipschitz vector field on a domain $D \subset \mathbb{R}^n$.

(i) The *maximal integral curve* for ξ through $x \in D$ is the maximal integral curve

$$\Gamma_x : \mathcal{I}_x \to D$$

where $\Gamma_x := \Gamma_{(x,0)}$ and $\mathcal{I}_x := \mathcal{I}_{(x,0)}$.

(ii) The *fundamental domain* of ξ is the domain

$$\mathscr{U}_\xi^0 := \{(x,t)\ ;\ t \in \mathcal{I}_x\} \subset D \times \mathbb{R}.$$

(iii) The flow of ξ is the map $\Phi_\xi : \mathscr{U}^0_\xi \to D$ defined by

$$\Phi_\xi(x,t) = \Gamma_x(t).$$

(iv) The time-t map is the map Φ^t_ξ defined by

$$\Phi^t_\xi(x) = \Phi_\xi(x,t).$$

$\diamond$

Note that $\mathscr{U}^0_\xi$ always contains $D \times \{0\}$. Note as well that the time-t maps define the pseudo-group law

$$\Phi^t_\xi \circ \Phi^s_\xi = \Phi^{t+s}_\xi. \tag{5}$$

The link between the autonomous and time-dependent scenarios is the identity

$$\Phi^t_s = \Phi^{t-s}_\xi.$$

An additional feature afforded to autonomous vector fields is the fact the any two maximal integral curves that meet are in fact identical, i.e., distinct maximal integral curves never meet. This separation of the integral curves is the content of the uniqueness aspect of Theorem 4.5 in the case of autonomous equations.

7. Suspension

Autonomous vector fields are special cases of time-dependent vector fields. In this section, we note that in a sense the converse is also true. Toward this end, let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains and let $F : V \to \mathbb{R}^n$ be a time-dependent vector field. Define $\xi_F : V \to \mathbb{R}^n \times \mathbb{R}$ by the formula

$$\xi_F(x,s) := (F(x,s), 1)$$

The vector field ξ_F is then autonomous, and its flow is given by the time-t maps

$$\Phi^t_{\xi_F}(x,s) = (\Phi^{s+t}_s(x), s+t).$$

It is therefore possible to extract the flow of F from that of ξ_F. If one can find the latter flow, this is of course possible. In fact, if F is continuous and locally uniformly Lipschitz on V then ξ_F is locally Lipschitz on V, so Theorem 4.5 applies to ξ_F.

In view of the suspension construction, it suffices to focus attention on autonomous time-dependent vector fields, which we shall simply call *vector fields* from here on.

8. Regularity of Solutions

The flow of a vector field is constructed by gluing together integral curves. In this process, the regularity of the time-t maps and of the flows is far from clear. As it turns out, the behavior of the flow is remarkably good.

Before beginning our study, we establish the following lemma, whose usefulness in the study of regularity of solutions to ODE cannot be overstated.

Lemma 8.1 (Gronwall's Inequality). *Let $f, g : [a,b) \to [0,\infty)$ be continuous functions, and assume there is a constant $A \geq 0$ such that*

$$f(t) \leq A + \int_a^t f(s)g(s)ds.$$

Then

$$f(t) \le A \exp\left(\int_a^t g(s)ds\right) \quad \textit{for all } t \in [a,b).$$

Proof. Let $\varepsilon > 0$. The function $h(t) := (A+\varepsilon) + \int_a^t f(s)g(s)ds$ is positive and satisfies $h'(t) = f(t)g(t) \le h(t)g(t)$. Hence $\frac{d}{dt}\log h(t) \le f(t)$, so $\log h(t) \le \int_a^t f(s)ds + \log h(a) = \log\left(h(a)\exp\left(\int_a^t g(s)ds\right)\right)$. Since $h(a) = A + \varepsilon$ we have

$$f(t) \le h(t) \le (A+\varepsilon)\exp\left(\int_a^t g(s)ds\right).$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Next we define the required notion of regularity.

DEFINITION 8.2. Let $D \subset \mathbb{R}^n$ be an open set, let $k \in \mathbb{N}$ and let $\alpha \in (0,1]$. A function $f : D \to \mathbb{R}$ is said said to be $\mathscr{C}^{k,\alpha}_{\ell oc}$— one writes $f \in \mathscr{C}^{k,\alpha}_{\ell oc}(D)$— if $f \in \mathscr{C}^k(D)$ and for every $x \in D$ and every $\varepsilon \in (0, \operatorname{dist}(x, D^c))$ there is a positive constant $K = K(x,\varepsilon)$ such that every k^{th} order partial derivative $g_I := \frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_n}}$ of f (i.e., $I = (i_1, ..., i_n) \in \mathbb{N}^n$ is a multiindex of order $|I| := i_1 + \cdots + i_n = k$) satisfies

$$|g_I(x_1) - g_I(x_2)| \le K|x_1 - x_2|^\alpha \quad \text{for all } x_1, x_2 \in D_\varepsilon(x).$$

In particular, $\mathscr{C}^{0,1}_{\ell oc}(D)$ is the set of locally Lipschitz functions on D.

For a map $F = (f^1, ..., f^m) : D \to R \subset \mathbb{R}^m$, $F \in \mathscr{C}^{k,\alpha}_{\ell oc}(D,R)$ if $f^1, ..., f^m \in \mathscr{C}^{k,\alpha}_{\ell oc}(D)$, i.e., F is $\mathscr{C}^{k,\alpha}_{\ell oc}$ if and only if each component f^j of F is $\mathscr{C}^{k,\alpha}_{\ell oc}$. ◇

The central result about regularity of the time-t maps of a vector field $\xi \in \mathscr{C}^{k,\alpha}_{\ell oc}(D)$ on a domain D is the following theorem.

THEOREM 8.3 (Smooth dependence on Initial Conditions). *Let $D \subset \mathbb{R}^n$ be a domain and let $\xi : D \to \mathbb{R}^n$ be a $\mathscr{C}^{k,1}_{\ell oc}$ vector field. Denote by $\Phi_\xi : \mathscr{U}^o_\xi \to D$ the flow of ξ.*

a. *For any open set $U \subset\subset D$ and each $t \in \mathbb{R}$ such that the time-t map Φ^t_ξ is defined on U, $\Phi^t_\xi \in \mathscr{C}^{k,1}_{\ell oc}(U)$.*
b. *For each $x \in D$ the integral curve $\gamma_x : \mathcal{I}_x \ni t \mapsto \Phi^t_\xi(x)$ is in $\mathscr{C}^{k+1}(\mathcal{I}_x)$.*
c. *The flow $\Phi_\xi : \mathscr{U}^o_\xi \to D$ is $\mathscr{C}^{k,1}$.*

Proof. We begin with the case $k = 0$. In this case the fact that $\gamma_x \in \mathscr{C}^1(\mathcal{I}_x)$ is a part of Theorem 4.5, so we need only show that Φ_ξ is locally Lipschitz. We begin by showing that Φ^t_ξ is locally Lipschitz on its domain of definition. By the pseudogroup law it suffices to assume that $t \in \overline{[-\varepsilon, \varepsilon]}$ for some sufficiently small ε. Let $x \in D$ and let $\varepsilon > 0$ be so small that $\Phi^t_\xi(y) \in D$ if $y \in \overline{B_\varepsilon(x)}$ and $t \in [-\varepsilon, \varepsilon]$. For any $x_1, x_2 \in B_\varepsilon(x)$ consider the function $f(t) := ||\Phi^t_\xi(x_1) - \Phi^t_\xi(x_2)||$. Then

$$f(t) = \left|\left|\int_0^t \left(\xi(\Phi^s_\xi(x_1)) - \xi(\Phi^s_\xi(x_2))\right)ds + x_1 - x_2\right|\right| \le ||x_1 - x_2|| + K\int_0^t f(s)ds,$$

where K is the local Lipschitz constant of ξ on $B_\varepsilon(x)$. By Gronwall's Inequality

$$||\Phi^t_\xi(x_1) - \Phi^t_\xi(x_2)|| \le e^{K|t|}||x_1 - x_2|| \le e^{\varepsilon K}||x_1 - x_2||, \tag{6}$$

which proves a. We already know from Theorem 4.5 that the integral curve γ_x is $\mathscr{C}^1$, i.e., that b holds. Finally, if $t_1, t_2 \in I_\varepsilon(t) := (t-\varepsilon, t+\varepsilon)$ and $U \subset\subset D$ is such that $(t-\varepsilon, t+\varepsilon) \times U \subset \mathscr{U}_\xi^o$ then

$$\begin{aligned} ||\Phi_\xi^{t_1}(x_1) - \Phi_\xi^{t_2}(x_2)|| &\leq ||\Phi_\xi^{t_1}(x_1) - \Phi_\xi^{t_2}(x_1)|| + ||\Phi_\xi^{t_2}(x_1) - \Phi_\xi^{t_2}(x_2)|| \\ &\leq \left(\sup_{(\tau,x)\in I_\varepsilon(t)\times D} ||\xi(\Phi_\xi^\tau(x))|| \right) |t_1 - t_2| + ||\Phi_\xi^{t_2}(x_1) - \Phi_\xi^{t_2}(x_2)|| \\ &\leq \left(\sup_{(\tau,x)\in I_\varepsilon(t)\times D} ||\xi(\Phi_\xi^\tau(x))|| \right) |t_1 - t_2| + e^{K\varepsilon}||x_1 - x_2|| \end{aligned}$$

where the second inequality follows from the Mean Value Theorem and the third inequality is (6). Thus c holds, and the case $k=0$ is proved.

Let us now turn to the case $k=1$, i.e., assume $\xi \in \mathscr{C}^{1,1}_{\ell oc}(D)$. For fixed $x \in D$ consider the linear time-dependent vector field $F_x(y,t) := d\xi(\Phi_\xi^t(x))y$. Let us write $\Psi(x,t)y := \Phi_0^t(y)$, where Φ_o^t is the time-dependent flow of $F_x(y,t)$. By Corollary 4.6 $\Phi_0^t(y)$ depends linearly on y, which is to say, $\Psi(x,t)$ lies in the space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of linear maps of $\mathbb{R}^n$ to itself. Moreover, $\Psi(x,t)$ is invertible because $\Phi_t^0 \circ \Phi_0^t = \Phi_t^t = \mathrm{Id}$. We can therefore think of $d\xi(\Phi_\xi^t(x))$ as a vector field on the linear space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$.

By its definition, the curve $t \mapsto \Psi(x,t) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the differential equation

$$\frac{\partial}{\partial t}\Psi(x,t) = d\xi(\Phi_\xi^t(x))\Psi(x,t)$$

with the initial condition $\Psi(x,0) = \mathrm{Id}$. We claim that the map $(x,t) \mapsto \Psi(x,t)$ is continuous. Indeed, since $\xi \in \mathscr{C}^{1,1}_{\ell oc}(D)$, $d\xi$ is locally Lipschitz, and by the first part of the proof we have already seen that Φ_ξ is locally Lipschitz. Therefore

$$||\Psi(x,t)|| = \left|\left| \mathrm{Id} + \int_0^t d\xi(\Phi_\xi^s(x))\Psi(x,s)ds \right|\right| \leq 1 + \int_0^t ||d\xi(\Phi_\xi^s(x))|| \cdot ||\Psi(x,s)||ds$$

and Gronwall's Inequality yields $||\Psi(x,t)|| \leq \exp\left(\int_0^t ||d\xi(\Phi_\xi^s(x))||ds\right)$. In particular, $||\Psi(x,t)||$ is locally uniformly bounded in x and t.

Now, if x_1, x_2 are sufficiently close to x then, since $\Psi(x_1, 0) = \Psi(x_2, 0) = \mathrm{Id}$,

$$\begin{aligned} &\Psi(x_1,t) - \Psi(x_2,t) \\ &= \int_0^t \left(d\xi(\Phi_\xi^s(x_1))\Psi(x_1,s) - d\xi(\Phi_\xi^s(x_2))\Psi(x_2,s) \right) ds \\ &= \int_0^t \left((d\xi(\Phi_\xi^s(x_1)) - d\xi(\Phi_\xi^s(x_2)))\Psi(x_1,s) + d\xi(\Phi_\xi^s(x_2))(\Psi(x_1,s) - \Psi(x_2,s)) \right) ds, \end{aligned}$$

and hence, since we observed that $d\xi \circ \Phi_\xi^t$ is locally Lipschitz uniformly in t and we've just shown that $||\Psi(x,t)||$ is locally uniformly bounded in x and t,

$$||\Psi(x_1,t) - \Psi(x_2,t)|| \leq A||x_1 - x_2|| + K\int_0^t ||\Psi(x_1,s) - \Psi(x_2,s)||ds.$$

Thus by Gronwall's Inequality again,

$$||\Psi(x_1,t) - \Psi(x_2,t)|| \leq A||x_1 - x_2||e^{Kt}.$$

Moreover, another application of the Mean Value Theorem gives

$$\begin{aligned}||\Psi(x_1,t_1)-\Psi(x_2,t_2)|| &\leq ||\Psi(x_1,t_1)-\Psi(x_1,t_2)||+||\Psi(x_1,t_2)-\Psi(x_2,t_2)|| \\ &\leq \left(\sup_{t_1\leq t\leq t_2, x} ||d\xi(\Phi^t_\xi(x))\Psi(x,t)||\right)|t_1-t_2|+A||x_1-x_2||e^{K\varepsilon}\end{aligned}$$

which shows that Ψ is Lipschitz.

Finally, observe that the map $\widehat{\Psi}:(x,t)\mapsto d\Phi^t_\xi(x)$ satisfies $\widehat{\Psi}(x,0)=d\Phi^0_\xi(x)=\mathrm{Id}$ and

$$\begin{aligned}\frac{d}{dt}\widehat{\Psi}(x,t)y=\frac{d}{dt}d\Phi^t_\xi(x)y &= \frac{d}{dt}\left.\frac{d}{ds}\right|_{s=0}\Phi^t_\xi(x+sy)=\left.\frac{d}{ds}\right|_{s=0}\xi(\Phi^t_\xi(x+sy)) \\ &= d\xi(\Phi^t_\xi(x))d\Phi^t_\xi(x)y=d\xi(\Phi^t_\xi(x))\widehat{\Psi}(x,t)y\end{aligned}$$

for all $y\in\mathbb{R}^n$. By the uniqueness part of Theorem 4.5 $\widehat{\Psi}=\Psi$. Thus we have shown that the flow Φ_ξ is $\mathscr{C}^{1,1}$ when ξ is $\mathscr{C}^{1,1}$. Moreover,

$$\frac{d^2}{dt^2}\Phi^t_\xi(x)=\frac{d}{dt}\xi\circ\Phi^t_\xi(x)=d\xi(\Phi^t_\xi(x))\xi(\Phi^t_\xi(x))$$

which shows that the flow Φ_ξ is then $\mathscr{C}^2$ in t. This completes the proof of the case $k=1$.

Now suppose the result has been proved up to $k-1$, i.e., we have shown that, for any vector field η, if $\eta\in\mathscr{C}^{k-1,1}_{\ell oc}(D)$ then Φ_η is $\mathscr{C}^{k-1,1}_{\ell oc}$ in x and $\mathscr{C}^k$ in t. We have already computed that

$$\frac{d}{dt}d\Phi^t_\xi(x)=d\xi(\Phi^t_\xi(x))d\Phi^t_\xi(x) \quad \text{and} \quad \frac{d^2}{dt^2}\Phi^t_\xi(x)=d\xi(\Phi^t_\xi(x))\xi(\Phi^t_\xi(x)).$$

As one can verify by repeated application of the chain rule, the right hand sides of both equations are $\mathscr{C}^{k-1,1}_{\ell oc}$. Therefore, by our induction hypothesis, so are the solutions. Hence we see that Φ_ξ is $\mathscr{C}^{k,1}_{\ell oc}$ in x and $\mathscr{C}^{k+1}$ in t. The proof is therefore complete. □

COROLLARY 8.4. *If $\xi:D\to\mathbb{R}^n$ is a $\mathscr{C}^\infty$ vector field then $\Phi_\xi:\mathscr{U}^o_\xi\to D$ is $\mathscr{C}^\infty$.*

If $D\subset\mathbb{R}^n$ and $\xi:D\to\mathbb{R}^n$ is a real-analytic vector field, it is not immediately clear from Theorem 8.3 that Φ_ξ is real-analytic. Nevertheless this is indeed the case.

THEOREM 8.5. *If $\xi:D\to\mathbb{C}^n$ is a real-analytic vector field then the flow $\Phi_\xi:\mathscr{U}^o_\xi\to D$ is real-analytic.*

We shall omit the proof of Theorem 8.5. The reader is invited to check that the estimates obtained in the proof of Theorem 8.3 are strong enough to prove that the solution is real-analytic when the vector field ξ is real-analytic.

9. DEPENDENCE ON PARAMETERS

THEOREM 9.1 (Continuous Dependence on Parameters). *Let P be a compact topological space and let ξ_p be a locally Lipschitz vector field for each $p\in P$, with local Lipschitz constant independent of p. Assume, moreover, that the map*

$$P\times D\ni(p,x)\mapsto\xi_p(x)\in\mathbb{R}^n$$

is continuous. Then the flow Φ_{ξ_p} of ξ_p depends continuously on p, in the sense that for each relatively compact open set $U\subset\subset D$ and each $\varepsilon>0$ such that $\mathscr{U}^o_{\xi_p}$ contains $U\times(-\varepsilon,\varepsilon)$ for all $p\in P$ the map

$$P\times U\times(-\varepsilon,\varepsilon)\ni(p,x,t)\mapsto\Phi^t_{\xi_p}(x)\in D$$

is continuous.

Proof. We know that the map $f_{x,p} : t \mapsto \Phi^t_{\xi_p}(x)$ is the unique solution of the integral equation

$$f_{x,p}(t) := x + \int_0^t \xi_p(f_{x,p}(s))ds,$$

or equivalently, the unique fixed point of the contraction mapping

$$T_{x,p}(\phi) : t \mapsto x + \int_0^t \xi_p(\phi(s))ds.$$

Now, for fixed $(p, x) \in P \times D$ we have

$$|T_{x,p}(\phi)(t) - T_{x,p}(\phi')(t)| \le K\mathrm{sgn}(t) \int_0^t |\phi(s) - \phi'(s)|ds$$

while for $(p', x') \in P \times D$

$$|T_{x,p}(\phi) - T_{x',p'}(\phi)| \le |x - x'| + \mathrm{sgn}(t) \int_0^t |\xi_p(\phi(s)) - \xi_{p'}(\phi(s))|ds.$$

Thus if (p, x) and (p', x') are sufficiently close and $|t|$ is sufficiently small then

$$\begin{aligned} |f_{x,p}(t) - f_{x',p'}(t)| &= |T_{x,p}(f_{x,p}(t)) - T_{x',p'}(f_{x',p'}(t))| \\ &= |T_{x,p}(f_{x,p}(t)) - T_{x',p'}(f_{x,p}(t))| + |T_{x',p'}(f_{x,p}(t)) - T_{x',p'}(f_{x',p'}(t))| \\ &\le |x - x'| + \mathrm{sgn}(t) \int_0^t |\xi_p(f_{x,p}(s)) - \xi_{p'}(f_{x,p}(s))|ds \\ &\qquad + K\mathrm{sgn}(t) \int_0^t |f_{x,p}(s) - f_{k',p'}(s)|ds. \end{aligned}$$

By Gronwall's Inequality

$$|f_{x,p}(t) - f_{x',p'}(t)| \le \left(|x - x'| + \mathrm{sgn}(t) \int_0^t |\xi_p(f_{x,p}(s)) - \xi_{p'}(f_{x,p}(s))|ds \right) e^{K|t|}.$$

Now, since the integral curve $\{f_{x,p}(s) \ ; \ s \in [-|t|, |t|]\}$ is a compact set, $p' \mapsto \xi_{p'}$ is uniformly continuous on this set. Finally,

$$\begin{aligned} &|f_{x,p}(t) - f_{x',p'}(t')| \\ &\le |f_{x,p}(t) - f_{x',p'}(t)| + |f_{x',p'}(t) - f_{x',p'}(t')| \\ &\le \left(|x - x'| + \mathrm{sgn}(t) \int_0^t |\xi_p(f_{x,p}(s)) - \xi_{p'}(f_{x,p}(s))|ds \right) e^{K|t|} + \left| \int_t^{t'} \xi_{p'}(f_{x',p'}(s))ds \right| \end{aligned}$$

The desired continuity easily follows. □

If we are willing to allow our parameter space P to be an open set in Euclidean space then Theorem 9.1 has a much stronger generalization, which can be proved by a simple application of our work in Section 8.

COROLLARY 9.2. *Let $P \subset \mathbb{R}^m$ be a domain and let $\{\xi_p \ ; \ p \in P\}$ be a family of $\mathscr{C}^{k,1}_{\ell oc}$ vector fields whose local Lipschitz constant is locally uniform in* p*, such that the map*

$$P \times D \ni (p, x) \mapsto \xi_p(x) \in \mathbb{R}^n$$

is $\mathscr{C}^k$. Then the map

$$P \times D \times \mathbb{R} \ni (p, x, t) \mapsto \Phi^t_{\xi_p}(x) \in D,$$

wherever it is defined, is $\mathscr{C}^k$.

Proof. Consider the vector field $\tilde{\xi}$ on $D \times P$ defined by $\tilde{\xi}(x, p) := (\xi_p(x), 0)$. By hypothesis this vector field in $\mathscr{C}^{k,1}_{\ell oc}$, and hence by Theorem 8.3 its flow $\Phi_{\tilde{\xi}} : \mathscr{U}^o_{\tilde{\xi}} \to D \times P$ is $\mathscr{C}^{k,1}_{\ell oc}$. But this flow is uniquely determined by the differential equation, and one can check directly that the map

$$D \times P \ni (x, p) \mapsto (\Phi^t_{\xi_p}(x), p)$$

solves the equation. Therefore $\Phi^t_{\tilde{\xi}}(x, p) \equiv (\Phi^t_{\xi_p}(x), p)$, and the desired smoothness follows from Theorem 8.3. □

10. Complete Vector Fields

The pseudo-group law (5) is not a group law only because integral curves are not defined for a long enough time, i.e., even if t and s both lie in the domains of their respective integral curves, $t+s$ may not. The situation in which this failure does not happen is therefore particularly important, and we study it in more detail now.

Definition 10.1. A vector field $\xi : D \to \mathbb{R}^n$ is said to be *complete* (sometimes also called *completely integrable*) if the domain of every maximal integral curve is $\mathbb{R}$. ◇

We have the following simple Proposition.

Proposition 10.2. *Let $\xi : D \to \mathbb{R}^n$ be a $\mathscr{C}^{k,1}_{\ell oc}$ vector field defined on a domain $D \subset \mathbb{R}^n$. Then the following are equivalent.*

(i) ξ is complete.
(ii) There exists a positive number ε such that for each $x \in D$, $\mathcal{I}_x \supset (-\varepsilon, \varepsilon)$.
(iii) For each $t \in \mathbb{R}$, the map Φ^t_ξ is a $\mathscr{C}^{k,1}_{\ell oc}$-diffeomorphism of D: $\Phi^t_\xi \in \mathrm{Diff}^k(D) \cap \mathscr{C}^{k,1}_{\ell oc}(D)$.
(iv) For some $t \in \mathbb{R} - \{0\}$, $\Phi^t_\xi \in \mathrm{Diff}^k(D) \cap \mathscr{C}^{k,1}_{\ell oc}(D)$.
(v) The set of maps $\{\Phi^t_\xi\}_{t \in \mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{Diff}^k(D) \cap \mathscr{C}^{k,1}_{\ell oc}(D)$.
(vi) The fundamental domain of ξ is $D \times \mathbb{R}$.

The proof is left to the reader as an exercise.

11. Approximation

In this section we study a technique, initiated by Euler, for the approximation of integral curves and more generally flows. We confine ourselves to autonomous vector fields for the time being.

Definition 11.1. Let $\xi : D \to \mathbb{R}^n$ be a vector field on a domain $D \subset \mathbb{R}^n$ and let $I \subset \mathbb{R}$ be an open interval containing 0. An *algorithm for ξ* is a map $H : D \times I \to D$ such that, with $H_t(x) := H(x, t)$,

(i) $H_0 = \mathrm{Id}$,
(ii) $H(x, \cdot)$ is $\mathscr{C}^1$ and its derivative is continuous in $D \times I$, and
(iii) $\left.\frac{\partial H}{\partial t}\right|_{t=0} = \xi$.

The basic approximation theorem is the following result.

THEOREM 11.2. *Let H be an algorithm for a Lipschitz vector field ξ. If $(t,x) \in \mathscr{U}_\xi^0$ then for all $N >> 0$, $H_{t/N}^{(N)}(x)$ is defined, and converges to $\Phi_\xi^t(x)$. Conversely, if $H_{t/N}^{(N)}(x)$ is defined and converges for $t \in [0,T]$ then $(T,x) \in \mathscr{U}_\xi^0$ and*

$$\lim_{N\to\infty} H_{t/N}^{(N)}(x) = \Phi_\xi^t(x).$$

In both statements, the converges is locally uniform on $D \times I$.

Proof. We begin by showing that the convergence holds locally. Toward this end, let $x_o \in D$. Then

$$H_t(x) = x + O(t) \quad \text{and} \quad \Phi_\xi^t(x) - H_t(x) = o(t). \tag{7}$$

If $H_{t/j}^{(j)}(x)$ is well-defined for x in a small neighborhood of x_o, for $j = 1, 2, ..., N-1$, then the semi-group law for time-t maps and the first estimate in (7) shows that

$$\begin{aligned} H_{t/N}^{(N)}(x) - x &= H_{t/N}^{(N)}(x) - H_{t/N}^{(N-1)}(x) + H_{t/N}^{(N-1)}(x) - H_{t/N}^{(N-2)}(x) \\ &\quad + ... + H_{t/N}(x) - x \\ &= NO(t/N) = O(t), \end{aligned}$$

which is small independently of N, for t sufficiently small. Thus for x sufficiently close to x_o and t sufficiently small, $H_{t/N}^{(N)}(x)$ remains close to x_o for all N. In other words, with

$$x_j = H_{t/j}^{(j)}(x),$$

$||x_j - x_o|| < \varepsilon$ for x sufficiently close to x_o and t sufficiently small. From the semi-group law for Φ_ξ^t, we also have

$$\begin{aligned} \Phi_\xi^t(x) - H_{t/N}^{(N)}(x) &= (\Phi_\xi^{t/N})^{(N)}(x) - H_{t/N}^{(N)}(x) \\ &= (\Phi_\xi^{t/N})^{(N-1)}(\Phi_\xi^{t/N}(x)) - (\Phi_\xi^{t/N})^{(N-1)}(H_{t/N}(x)) \\ &\quad + \sum_{j=2}^{N} (\Phi_\xi^{t/N})^{(N-j)}(\Phi_\xi^{t/N}(x_j)) - (\Phi_\xi^{t/N})^{(N-j)}(H_{t/N}(x_j)), \end{aligned}$$

Now, the hypotheses on ξ imply the estimate (6), as was shown in the beginning of the proof of Theorem 8.3. Repeated application of (6) yields the estimate

$$\begin{aligned} ||\Phi_\xi^t(x) - H_{t/N}^{(N)}(x)|| &\le \sum_{k=1}^{N} e^{K|t|(N-k)/N} ||\Phi_\xi^{t/N}(x_{N-k-1}) - H_{t/N}(x_{N-k-1})|| \\ &\le N e^{K|t|} o(t/N), \end{aligned}$$

and the last quantity converges, as $N \to \infty$, to 0 uniformly on a small ball centered at x_o and for all sufficiently small t. The final estimate uses the second estimate of (7).

Having handled the case of short times, we now proceed to longer times. Toward this end, suppose first that $\Phi_\xi^t(x)$ is defined for all $t \in [0,T]$. By what we have just done, if k is sufficiently large then

$$\Phi_\xi^{t/k}(y) = \lim_{k\to\infty} H_{t/k}^{(k)}(y)$$

holds uniformly for $t \in [0, T]$ and y in a bounded neighborhood of the curve $\{\Phi^t_\xi(x)\ ;\ t \in [0, T]\}$. Thus

$$\Phi^t_\xi(x) = (\Phi^{t/k}_\xi)^{(k)}(x) = \lim_{N\to\infty} (H^{(N)}_{t/(kN)})^{(k)}(x) = \lim_{N\to\infty} H^{(Nk)}_{t/(kN)}(x) = \lim_{N\to\infty} H^{(N)}_{t/N}(x).$$

Conversely, suppose $t \mapsto H^{(N)}_{t/N}(x)$ converges to a curve $c : [o, T] \to D$. Let

$$S = \{t \in [0, T]\ ;\ \Phi^t_\xi(x) \text{ is defined and equal to } c(t)\}.$$

Clearly $0 \in S$, and from the local result S is relatively open. Let $\{t_k\} \subset S$ and suppose $t_k \to t$. Then $\Phi^{t_k}_\xi(x) \to c(t)$ so by Theorem 4.5 $\Phi^t_\xi(x)$ is defined, and by continuity, $\Phi^t_\xi(x) = c(t)$. Thus S is closed, and hence $S = [0, T]$.

Finally, observe that by existence and uniqueness, $\Phi^{-t}_\xi = \Phi^t_{-\xi}$, so the above proof applies to negative times as well. □