

# A PRIMER ON ORDINARY DIFFERENTIAL EQUATIONS

## NOTES FOR MAT 402 S26

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# 1. VECTOR FIELDS AND ODE

## 1.1. Static vector fields and autonomous ODE.

DEFINITION 1.1. Let  $D \subset \mathbb{R}^n$  be a domain, i.e., an open connected set. A *vector field* on  $D$  is a map  $\xi : D \rightarrow \mathbb{R}^n$ . ◇

Vector fields are the data of ordinary differential equations (ODE). From such data one wishes to produce so-called *integral curves*; finding these integral curves constitutes solving the ODE. More precisely, we have the following definition.

DEFINITION 1.2. Let  $\xi : D \rightarrow \mathbb{R}^n$  be a vector field on a domain  $D \subset \mathbb{R}^n$ . An *integral curve* of  $\xi$  through  $x \in D$  is a map  $\gamma_x : (-a, b) \rightarrow D$ , where  $a, b > 0$ , such that

$$\gamma_x(0) = x \quad \text{and} \quad \dot{\gamma}_x(t) = \xi \circ \gamma_x(t). \quad \diamond$$

**1.2. Time-dependent vector fields and non-autonomous ODE.** Sometimes we want to allow the coefficients of the ODE to depend on the time parameter of the integral curve we seek. That is to say, we want to consider time-dependent vector fields.

EXAMPLE 1.3. Consider the differential equation

$$\frac{dx}{dt} = tx + \log(x + e^t)$$

on the set  $D = \mathbb{R}$ . The vector field

$$\xi_t(x) := tx + \log(x + e^t)$$

is defined for all  $t \in \mathbb{R}$ , but if  $x < 0$  then  $\xi_t(x)$  is defined only for  $t \in ((\log(-x), +\infty))$ . Therefore the map

$$(x, t) \mapsto \xi_t(x) \in \mathbb{R}$$

is defined on the domain

$$V = \{(x, t) \in \mathbb{R}^2 ; x + e^t > 0\},$$

which is not a Cartesian product. ◇

## 1.3. A link between autonomous and non-autonomous ODE.

DEFINITION 1.4. Let  $D \subset \mathbb{R}^n$  be a domain. A *time dependent vector field* on  $D$  is

- (a) a subset  $\text{Domain}(F) \subset \mathbb{R} \times D$ , called the *domain of  $F$* , such that  $\text{p}_2(\text{Domain}(F)) = D$ , where  $\text{p}_2 : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the Cartesian projection to the second factor, and
- (b) a map  $F : \text{Domain}(F) \rightarrow \mathbb{R}^n$ . ◇

DEFINITION 1.5. Let  $F$  be a time-dependent vector field on  $D$  and let  $(s, x) \in \text{Domain}(F)$ . An *integral curve of  $F$  through  $x \in D$  with initial time  $s$*  is an open set  $I_{(x,s)} \subset \mathbb{R}$  containing  $s$ , together with an absolutely continuous curve  $\gamma_{(x,s)} : I_{(x,s)} \rightarrow D$ , such that

- i.  $\gamma_{(x,s)}(s) = x$ ,
- ii.  $(\gamma_{(x,s)}(t), t) \in V$  for all  $t \in I_{(x,s)}$ , and
- iii.  $\frac{d\gamma_{(x,s)}(t)}{dt} = F(\gamma_{(x,s)}(t), t)$  for almost every  $t \in I_{(x,s)}$ .

The central question of ODE is whether, for a given time-dependent vector field, integral curves exist and, if so, are unique. In the next section we shall establish an existence theorem under rather weak hypotheses on the time-dependent vector field. Later on we shall impose slightly stronger conditions and then simultaneously prove existence and uniqueness.

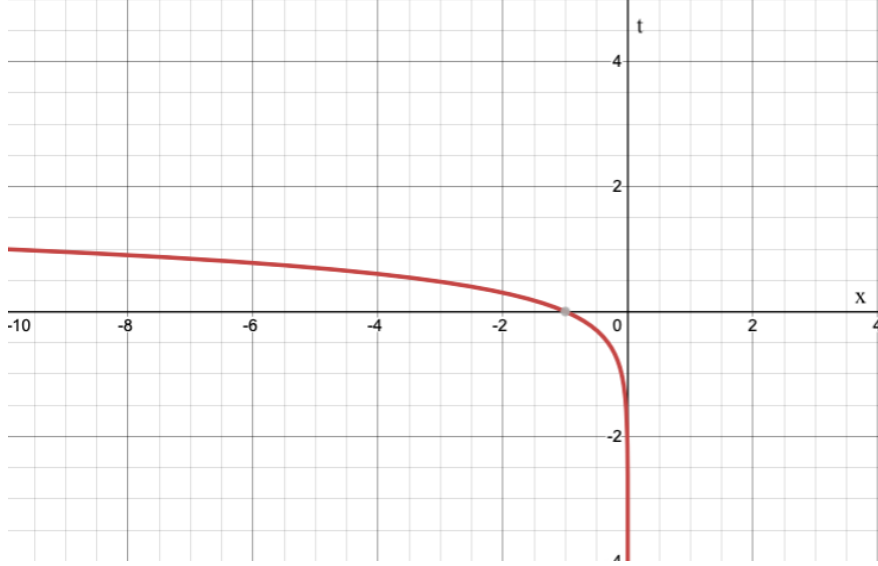


FIGURE 1. The domain of  $(x, t) \mapsto \xi_t(x)$  consists of all points the right of the red curve.

## 2. CAUCHY-PEANO EXISTENCE THEOREM FOR FIRST ORDER ODE

**THEOREM 2.1 (Cauchy-Peano Existence Theorem).** *Let  $D \subset \mathbb{R}^n$  and let  $F$  be a continuous time-dependent vector field on  $D$ . For each  $(x, s) \in \text{Domain}(F)$  there exists an integral curve of  $F$  through  $x$  at the initial time  $s$ .*

**REMARK.** Continuity of the time-dependent vector field  $F$  is too weak an assumption as to imply uniqueness of the integral curve. Among the simplest examples is the time-dependent vector field on  $\mathbb{R}$  given by  $F(t, x) = 3x^{2/3}$ . For each  $c \in [0, \infty]$  the curve

$$\gamma_c : \mathbb{R} \ni t \mapsto (t - c)^3 \chi_{[c, \infty)}(t) \in \mathbb{R}$$

is an integral curve through 0 with initial time 0, i.e.,  $\gamma_c(0) = 0$ . (By  $\gamma_\infty$  we mean the constant function 0.)  $\diamond$

*Proof of Theorem 2.1.* Fix  $(s, x) \in \text{Domain}(F)$  and let  $\delta > 0$  be so small that the Cartesian product  $D_\delta(s, x) := I_\delta(s) \times B_\delta(x)$  lies in  $\text{Domain}(F)$ , where  $I_\delta(s) = (s - \delta, s + \delta)$ . Choose some number  $\varepsilon > 0$  such that

$$\varepsilon \cdot \max \left( 1, \sup_{D_\delta(s, x)} |F| \right) < \delta.$$

There exist absolutely continuous curves  $\gamma_j : I_\varepsilon(s) \rightarrow D$ ,  $j = 1, 2, \dots$ , such that

$$(1) \quad \gamma_j(s) = x, \quad |\gamma_j'(t) - F(t, \gamma_j(t))| \leq 1/j \quad \text{and} \quad |\gamma_j(\tau_1) - \gamma_j(\tau_2)| \leq |\tau_1 - \tau_2| \sup_{D_\delta(s, x)} |F|$$

for all  $\tau_1, \tau_2 \in I_\varepsilon(s)$  and almost all  $t \in I_\varepsilon(s)$ . Indeed, define  $\gamma_j$  as follows. Let  $N > 0$  be an integer, soon to be specified, let  $t_o := s$ , let  $t_m := s + m\varepsilon/N$ ,  $m \in \mathbb{Z} \cap (-N, N)$ , and define

- $\gamma_j(s) = x$ ,
- for  $m \geq 0$  and  $t \in (t_m, t_{m+1}]$ ,  $\gamma_j(t) := \gamma_j(t_m) + (t - t_m)F(t_m, \gamma_j(t_m))$ , and
- for  $m < 0$  and  $t \in [t_m, t_{m+1})$ ,  $\gamma_j(t) := \gamma_j(t_{m+1}) + (t - t_{m+1})F(t_{m+1}, \gamma_j(t_{m+1}))$ .

If  $t \in (t_0, t_1]$  then

$$|\gamma_j(t_1) - x| = |t_1 - t_0| |F(s, x)| \leq \frac{\varepsilon}{N} \sup_{D_\delta(s, x)} |F| \leq \delta/N.$$

Continuing inductively, we see that if  $m > 0$  and  $t \in (t_m, t_{m+1}]$  then

$$|\gamma_j(t_{m+1}) - \gamma_j(t_m)| = |t_{m+1} - t_m| \cdot |F(s, x)| \leq \frac{\varepsilon}{N} \sup_{D_\delta(s, x)} |F| \leq \delta/N.$$

Thus  $|\gamma_j(t_m) - x| \leq m\delta/N \leq \delta$ . Similarly, if  $m < 0$  then  $|\gamma_j(t_m) - \gamma_j(t_{m+1})| \leq -m\delta/N < \delta$ . Therefore  $\gamma_j(t_m) \in B_\delta(x)$  for all  $m \in (-N, N) \cap \mathbb{Z}$ , and since  $|t_m - s| \leq \varepsilon < \delta$ , the curves  $\gamma_j$  are well-defined, piecewise linear and evidently continuous.

Now, if we set  $m_* := m$  for  $m \geq 0$  and  $m_* = m + 1$  for  $m < 0$  then

$$(2) \quad |\gamma'_j(t) - F(t, \gamma_j(t))| \leq |F(t_{m_*}, \gamma_j(t_{m_*})) - F(t, \gamma_j(t))| \quad \text{for all } t \in (t_m, t_{m+1}).$$

Since  $D_\delta(s, x)$  is compact,  $F$  is uniformly continuous on  $D_\delta(s, x)$ . Therefore if  $N$  is sufficiently large then the right hand side of (2) as small as needed.

Next, if  $t_k < \tau_1 \leq t_{k+1}$  and  $t_m < \tau_2 \leq t_{m+1}$  for some  $k, m$  satisfying  $-N < k \leq m < N$  then

$$\begin{aligned} |\gamma_j(\tau_2) - \gamma_j(\tau_1)| &= \left| \gamma_j(\tau_2) - \gamma_j(t_m) + \left( \sum_{\ell=k+1}^{m-1} \gamma_j(t_{\ell+1}) - \gamma_j(t_\ell) \right) + \gamma_j(t_{k+1}) - \gamma_j(\tau_1) \right| \\ &= \left| (\tau - t_m)F(t_{m_*}, \gamma_j(t_{m_*})) + \left( \sum_{\ell=k+1}^{m-1} (t_{\ell+1} - t_\ell)F(t_{\ell_*}, \gamma_j(t_{\ell_*})) \right) + (t_{k+1} - \tau_1)F(t_{k_*}, \gamma_j(t_{k_*})) \right| \\ &\leq (\tau - t_m)|F(t_{m_*}, \gamma_j(t_{m_*}))| + \left( \sum_{\ell=k+1}^{m-1} (t_{\ell+1} - t_\ell)|F(t_{\ell_*}, \gamma_j(t_{\ell_*}))| \right) + (t_{k+1} - \tau_1)|F(t_{k_*}, \gamma_j(t_{k_*}))| \\ &\leq \left( \sup_{D_\delta(s, x)} |F| \right) \left( \tau_2 - t_m + \left( \sum_{\ell=k+1}^m t_{\ell+1} - t_\ell \right) + t_{k+1} - \tau_1 \right) = |\tau_2 - \tau_1| \sup_{D_\delta(s, x)} |F| \leq \frac{\delta}{\varepsilon} |\tau_2 - \tau_1|. \end{aligned}$$

Thus (1) is proved.

By (1) the sequence  $\{\gamma_j\}$  is uniformly bounded and equicontinuous. Since  $D_\delta(s, x)$  is compact, the theorem of Ascoli-Arzelà yields a subsequence  $\gamma_{j_\ell}$  converging uniformly to  $\gamma : I_\varepsilon(s) \rightarrow B_\delta(s)$ . The first estimate in (1) implies that  $\gamma' = \lim_{\ell} \gamma'_{j_\ell}$  exists and equals  $F(\cdot, \gamma(\cdot))$  almost everywhere. Since  $F$  is continuous,  $\gamma'$  has a continuous extension to  $I_\varepsilon(s)$ , and hence to a continuous curve on  $I_\varepsilon(s)$ , which we continue to denote  $\gamma$ . This curve  $\gamma$  is the integral curve we seek, and the proof is complete.  $\square$

**REMARK 2.2.** In fact, in Theorem 2.1 the integral curve  $\gamma$  obtained in the proof is already differentiable. To see that this is the case, observe that since  $\gamma$  is absolutely continuous,

$$\gamma_j(t) = x + \int_s^t \gamma'_j(\tau) d\tau = x + \int_s^t (F(\gamma_j(\tau), \tau) + (\gamma'_j(\tau) - F(\gamma_j(\tau), \tau))) d\tau.$$

By the first estimate in (1) we may pass to the limit as  $j \rightarrow \infty$ , and since  $\lim_j (\gamma'_j - F(\gamma_j(\cdot), \cdot)) = 0$  a.e., we have

$$\gamma(t) = x + \int_s^t F(\gamma(\tau), \tau) d\tau.$$

Thus  $\gamma(s) = x$ ,  $\gamma$  is differentiable, and  $\gamma'(t) = F(\gamma(t), t)$ , as desired.  $\diamond$

### 3. CONTRACTING MAPS

In the proof of the existence and uniqueness theorem to be stated in the next section, we will need to make use of an iteration scheme due to Picard. The convergence of this iteration scheme depends on the concept of contracting map, which we now define.

**DEFINITION 3.1.** Let  $X$  be a subset of a metric space. A map  $S : X \rightarrow X$  is said to be *contracting* if there exists some  $r \in (0, 1)$  such that

$$d(Sx, Sy) \leq r \cdot d(x, y)$$

for all  $x, y \in X$ . ◇

The basic fact about contracting maps is the following result.

**PROPOSITION 3.2.** *Let  $X$  be a complete metric space. If  $S : X \rightarrow X$  is a contracting map then  $S$  has a unique fixed point.*

*Proof.* Let  $x_o \in X$  be any point, and set

$$x_j := S^{(j)}x_o, \quad j = 1, 2, \dots$$

where  $S^{(1)} = S$  and  $S^{(j)} := S \circ S^{(j-1)}$ . For  $j, k \in \mathbb{N}$  satisfying  $j < k$ ,

$$d(x_j, x_k) \leq \sum_{\ell=j}^{k-1} d(x_\ell, x_{\ell+1}) \leq \sum_{\ell=j}^{k-1} r^\ell d(x, Sx) = \frac{r^j(1 - r^{k-j})}{1 - r} d(x, Sx) \leq \frac{r^j}{1 - r} d(x, Sx).$$

Thus  $\{x_j\}$  is a Cauchy sequence, and since  $X$  is complete the limit

$$x_* := \lim x_j$$

exists. Since  $S$  is evidently continuous,

$$x_* = \lim S^{(j)}x_* = \lim S \circ S^{(j-1)}x_* = S(\lim S^{(j)}x_*) = Sx_*.$$

Thus  $x_*$  is a fixed point of  $S$ . Finally, if  $y$  is another fixed point of  $S$  then

$$0 \leq (1 - r)d(x_*, y) = d(Sx_*, Sy) - rd(x_*, y) \leq (r - r)d(x_*, y) = 0.$$

Thus  $y = x_*$ , and the proof is complete. □

### 4. THE EXISTENCE AND UNIQUENESS THEOREM FOR FIRST ORDER ODE

**DEFINITION 4.1.** Let  $f : U \rightarrow \mathbb{R}^n$  be a function defined on a domain  $U \subset \mathbb{R}^m$ .

- i. We say that  $f$  is *locally Lipschitz* if for each  $p \in U$  and each  $\varepsilon \in (0, \text{dist}(p, U^c))$  there exists a constant  $K = K_{\varepsilon, p}$  such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all  $x, y \in B(p, \varepsilon) := \{z \in \mathbb{R}^m ; |z - p| < \varepsilon\}$ .

- ii. We say that  $f$  is *globally Lipschitz* if there is a constant  $K$  such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all  $x, y \in U$ . ◇

**REMARK 4.2.** The notion of Lipschitz makes sense on any metric space. ◇

EXAMPLE 4.3. Any differentiable function is locally Lipschitz. On the other hand, the function  $f : \mathbb{R} \ni x \mapsto |x| \in \mathbb{R}$  is (globally) Lipschitz but not differentiable.  $\diamond$

Let  $D \subset \mathbb{R}^n$  and  $V \subset D \times \mathbb{R}$  be domains. For each  $t \in \mathbb{R}$ , we write

$$V_t = \{x \in D ; (x, t) \in V\}.$$

(It may happen that  $V_t = \emptyset$  for some  $t$ .)

DEFINITION 4.4. Let  $D \subset \mathbb{R}^n$  and let  $F$  be a time-dependent vector field on  $D$ . We say that  $F$  is *uniformly locally Lipschitz* if for each  $t \in \text{p}_1\text{Domain}(F) \subset \mathbb{R}$  the function  $F_t : \text{Domain}(F)_t \rightarrow \mathbb{R}^n$  is locally Lipschitz and moreover the Lipschitz constant can be taken locally uniform with respect to  $t$ . In other words, for each  $(t, x) \in \text{Domain}(F)$  there is a neighborhood  $U \subset \text{Domain}(F)$  containing  $(t, x)$  and a constant  $K > 0$  such that  $|F_s(x_1) - F_s(x_2)| \leq K|x_1 - x_2|$  for all  $x_1, x_2 \in D$  such that  $(s, x_1), (s, x_2) \in U$ .  $\diamond$

THEOREM 4.5 (Existence and Uniqueness Theorem for Ordinary Differential Equations).

Let  $D \subset \mathbb{R}^n$  and let  $F$  be a continuous and locally uniformly Lipschitz time-dependent vector field on  $D$ . For each  $(s, x) \in \text{Domain}(F)$  there exists an integral curve  $\gamma_{(s,x)} : I_{(s,x)} \rightarrow D$  for  $F$  with initial time  $s$ . Moreover, the set of integral curves possesses the following uniqueness property: if  $\gamma_{(s,x)} : I_{(s,x)} \rightarrow D$  and  $\tilde{\gamma}_{(s,x)} : \tilde{I}_{(s,x)} \rightarrow D$  are two integral curves through  $x$  with initial time  $s$  then  $\gamma_{(s,x)}(t) = \tilde{\gamma}_{(s,x)}(t)$  for all  $t \in I_{(s,x)} \cap \tilde{I}_{(s,x)}$ .

*Proof.* Let  $(t_o, x_o) \in \text{Domain}(F)$  and choose  $\varepsilon > 0$  such that  $F$  is continuous in  $(t_o - \varepsilon, t_o + \varepsilon) \times B(x_o, \varepsilon)$  and Lipschitz in the first variable with Lipschitz constant  $K$ , i.e.,

$$|F(t, x) - F(t, y)| \leq K|x - y|$$

for all  $(t, x), (t, y) \in (t_o - \varepsilon, t_o + \varepsilon) \times B(x_o, \varepsilon)$ . The continuity of  $F$  implies that if  $\varepsilon > 0$  is small enough then there exists a constant  $M > 0$  such that

$$|F(t, x)| \leq M$$

for all  $(t, x) \in (t_o - \varepsilon, t_o + \varepsilon) \times B(x_o, \varepsilon)$ .

Choose positive constants  $\alpha < \varepsilon$  and  $\beta < \varepsilon$  such that  $\alpha < M\beta$  and  $\alpha K < 1$ . Note that

$$(t_o - \alpha, t_o + \alpha) \times B(x_o, \beta) \subset (t_o - \varepsilon, t_o + \varepsilon) \times B(x_o, \varepsilon)$$

Let  $\mathcal{A}$  denote the set of continuous maps  $\phi : I_\alpha \rightarrow \mathbb{R}^n$  such that

$$|\phi(t) - x_o| \leq \beta \quad \text{for all } t \in I_\alpha.$$

Equip  $\mathcal{A}$  with the norm

$$\|\phi\|_\infty := \sup_{I_\alpha} |\phi|.$$

Since uniform limits of continuous functions are continuous,  $\mathcal{A}$  is a closed bounded subset of the Banach space  $(L^\infty(I_\alpha))^n$ . Thus  $\mathcal{A}$  is a complete metric space with respect to the metric

$$d(\phi, \tilde{\phi}) := \|\phi - \tilde{\phi}\|_\infty.$$

Consider the operator  $T$  defined by

$$T\phi(t) := x_o + \int_{t_o}^t F(s, \phi(s))ds.$$

Observe first that if  $\phi \in \mathcal{A}$  then clearly  $T\phi$  is continuous and defined on all of  $I_\alpha$ . Moreover, for  $t \in I_\alpha$  one has

$$|T\phi(t) - x_o| \leq M|t - t_o| \leq M\alpha < \beta.$$

Thus  $T\phi \in \mathcal{A}$ , which is to say,

$$T : \mathcal{A} \rightarrow \mathcal{A}.$$

Next, observe that if  $\phi_1, \phi_2 \in \mathcal{A}$  then

$$\begin{aligned} |T\phi_1(t) - T\phi_2(t)| &= \left| \int_{t_o}^t (F(s, \phi_1(s)) - F(s, \phi_2(s))) ds \right| \\ &\leq \int_{t_o}^t K |\phi_1(s) - \phi_2(s)| ds \\ &\leq K\alpha \sup_{I_\alpha} |\phi_1 - \phi_2|. \end{aligned}$$

Thus  $T : \mathcal{A} \rightarrow \mathcal{A}$  is a contracting map, so by Proposition 3.2  $T$  has a unique fixed point  $\phi_* \in \mathcal{A}$ . Being a fixed point of  $T$ ,  $\phi_*$  satisfies the equation

$$(3) \quad \phi_*(t) = x_o + \int_{t_o}^t F(s, \phi_*(s)) ds,$$

and therefore

$$\frac{\phi_*(t+h) - \phi_*(t)}{h} = \frac{1}{h} \int_t^{t+h} F(s, \phi_*(s)) ds \xrightarrow{h \rightarrow 0} F(t, \phi_*(t)).$$

Since  $\phi_* \in \mathcal{A}$ , the latter limit is continuous, and thus the fixed point  $\phi_*$  of  $T$  is a continuously differentiable curve satisfying the equation

$$\phi_*'(t) = F(t, \phi_*(t)).$$

Since  $\phi_*(t_o) = x_o$ , the curve  $\gamma_{(x_o, t_o)}(t) := \phi_*(t)$  is an integral curve of  $F$  through  $x_o$  at time  $t_o$ .

Conversely, any integral curve of  $F$  satisfies the equation (3), and is therefore a fixed point of  $T$ . Since contracting maps have a unique fixed point, any two integral curves whose domain contains  $I_\alpha$  must agree on  $I_\alpha$ . By carrying out the same proof in small intervals centered at all points of the intersection of the open set  $I_{(x,s)} \cap \tilde{I}_{(x,s)}$ , we obtain the uniqueness statement claimed in the theorem. The proof is therefore complete.  $\square$

**COROLLARY 4.6 (Linear Equations).** *Let  $I \subset \mathbb{R}$  be an interval and let  $A : I \rightarrow M_n(\mathbb{R})$  be a continuous matrix-valued function. Let  $F$  be the time-dependent vector field on  $\mathbb{R}^n$  defined by*

$$F(t, y) = A(t)y.$$

*Then there is a function  $\alpha : I \times I \rightarrow GL_n(\mathbb{R})$  such that for each  $y \in \mathbb{R}^n$  the curve*

$$\gamma_s(t) := \alpha(s, t)y, \quad t \in I$$

*is an integral curve of  $F$  through  $y \in \mathbb{R}^n$  with initial time  $s$ .*

*Proof.* Consider the matrix-valued function

$$a(s, t) := \int_s^t A(\tau) d\tau.$$

Define its exponential

$$\alpha(s, t) := e^{a(s, t)} := \text{Id} + \sum_{j=1}^{\infty} \frac{(a(s, t))^j}{j!}$$

which converges locally uniformly on  $I$ . Moreover,

$$\begin{aligned} \frac{d}{dt} \frac{(a(s, t))^j}{j!} &= \lim_{h \rightarrow 0} \frac{(a(s, t+h))^j - (a(s, t))^j}{j!h} \\ &= \lim_{h \rightarrow 0} \frac{a(s, t+h) - a(s, t)}{h} \cdot \frac{1}{j!} \left( \sum_{j=0}^{n-1} a(s, t+h)^{n-1-j} a(s, t)^j \right) \\ &= \lim_{h \rightarrow 0} \left( \frac{1}{h} \int_t^{t+h} A(\tau) d\tau A(t) \right) \cdot \frac{1}{j!} \left( \sum_{k=0}^{j-1} a(s, t+h)^{j-1-k} a(s, t)^k \right) \\ &= A(t) \frac{(a_s(t))^{j-1}}{(j-1)!}. \end{aligned}$$

In this computation, we use the fact that

$$(A - B) \sum_{k=0}^{j-1} A^{j-1-k} B^k = A^j - B^j + \sum_{j=0}^{k-1}$$

Since the series

$$\sum_{j=1}^{\infty} \frac{(a(s, t))^{j-1}}{(j-1)!}$$

converges to  $e^{a(s, t)}$  locally uniformly in  $t$ , we have

$$\alpha(s, s) = \text{Id} \quad \text{and} \quad \frac{\partial}{\partial t} \alpha(s, t) = A(t) \alpha(s, t).$$

It follows that the curve  $\gamma : I \ni t \mapsto \alpha(s, t)y \in \mathbb{R}^n$  satisfies

$$\gamma(s) = y \quad \text{and} \quad \gamma'(t) = A(t)\gamma(t),$$

as required. □

## 5. MAXIMAL INTEGRAL CURVE, FUNDAMENTAL DOMAIN AND TIME-DEPENDENT FLOW

Our next goal is to ‘glue together’ the integral curves of a time-dependent vector field. The first task is to maximally extend integral curves.

Let  $F$  be a continuous, uniformly locally Lipschitz time-dependent vector field on a domain  $D \subset \mathbb{R}^n$ . Fix an initial condition  $(x, s) \in \text{Domain}(F)$ . By Theorem 4.5,  $F$  has an integral curve through  $x$  with initial time  $s$ .

**PROPOSITION 5.1.** *With the notation above, there exists a unique integral curve  $\gamma_{(s, x)} : I_{(s, x)} \rightarrow D$  for  $F$  passing through  $x$  with initial time  $s$  such that if  $\phi : I \rightarrow D$  is any other integral curve for  $F$  passing through  $x$  at time  $s$  then  $I \subset I_{(s, x)}$  and  $\phi = \gamma_{(s, x)}|_I$ .*

*Proof.* The set  $\mathcal{I}_{(s,x)}$  of all integral curves for  $F$  passing through  $x$  with initial time  $s$  is partially ordered with respect to inclusion of graphs. Moreover, given two such integral curves  $\phi_i : I_i \rightarrow D$ ,  $i = 1, 2$ , Theorem 4.5 implies that the function

$$\phi(t) := \begin{cases} \phi_1(t), & t \in I_1 \\ \phi_2(t), & t \in I_2 \end{cases}$$

is well-defined, and therefore  $\phi : I_1 \cup I_2 \rightarrow D$  is also an integral curve for  $F$  passing through  $x$  with initial time  $s$ . It follows that  $\mathcal{I}_{(s,x)}$  is a directed set, i.e., for any two elements  $\phi_1, \phi_2 \in \mathcal{I}_{(s,x)}$  there exists  $\phi_3 \in \mathcal{I}_{(s,x)}$  such that  $\phi_3 \geq \phi_1$  and  $\phi_3 \geq \phi_2$ . We have to show that it has a maximal element, which is then of course unique.

Toward this end, let  $\{\phi_i : I_i \rightarrow D\}_{i \in A}$  be a maximal linearly ordered subset of  $\mathcal{I}_{(s,x)}$ . Then the set  $I := \bigcup_{i \in A} I_i$  is open, and the curve  $\phi : I \rightarrow D$  defined by

$$\phi(t) = \phi_i(t), \quad t \in I_i$$

is well-defined by the uniqueness part of Theorem 4.5, therefore in  $\mathcal{I}_{(s,x)}$ . Thus  $\mathcal{I}_{(s,x)}$  has a unique maximal element in  $\mathcal{I}_{(s,x)}$ .  $\square$

DEFINITION 5.2. The unique maximal element of the set  $\mathcal{I}_{(s,x)}$  defined in the proof of the previous proposition is called the *maximal integral curve for  $F$  through  $x$  at time  $s$* , and is denoted

$$\Gamma_{(s,x)} : \mathcal{I}_{(s,x)} \rightarrow D. \quad \diamond$$

One can also consider the unions of the graphs of the maximal integral curves.

DEFINITION 5.3. Let  $F$  be a time-dependent vector field on a domain  $D \subset \mathbb{R}$ .

a. The set

$$\mathcal{U}_F := \{(t, s, x) ; (s, x) \in \text{Domain}(F), t \in \mathcal{I}_{(s,x)}\} \subset \mathbb{R} \times \text{Domain}(F) \subset \mathbb{R} \times \mathbb{R} \times D$$

is called the *fundamental domain* of the time-dependent vector field  $F$ .

b. The map

$$\Phi_F : \mathcal{U}_F \rightarrow D \quad \text{defined by} \quad \Phi_F(t, s, x) := \Gamma_{(s,x)}(t)$$

is called the time-dependent flow of  $F$ .

c. The map  $\Phi_F^{t,s} : \mathcal{U}_F^{t,s} \rightarrow D$  defined by

$$(4) \quad \Phi_F^{t,s}(x) := \Gamma_{(s,x)}(t) = \Phi_F(t, s, x), \quad \text{and} \quad \mathcal{U}_F^{t,s} := \{x \in D ; (t, s, x) \in \mathcal{U}_F\}$$

is called the time- $t$  map for the initial time  $s$ .  $\diamond$

There is also a composition law for the maps (4), stated in the following result.

PROPOSITION 5.4. For each  $s \in \mathbb{R}$  one has

$$\Phi_F^{s,s}(x) = x \quad \text{for all } x \in \mathcal{U}_F^{s,s} = D.$$

Moreover, if  $(r, s, x) \in \mathcal{U}_F$  and  $(t, r, \Phi_F^{r,s}(x)) \in \mathcal{U}_F$ , we have the pseudo-group law

$$(5) \quad \Phi_F^{t,r} \circ \Phi_F^{r,s}(x) = \Phi_F^{t,s}(x).$$

*Proof.* The reader can check that, as functions of  $t$ , both sides of (5) satisfy the initial value problem  $\dot{y}(t) = F(t, y(t))$ ,  $y(s) = x$ , so the result follows by the uniqueness part of Theorem 4.5.  $\square$

## 6. AUTONOMOUS ODE

An autonomous ODE on a domain  $D \subset \mathbb{R}^n$  is determined by a vector field on  $D$ , i.e., a map  $\xi : D \rightarrow \mathbb{R}^n$ . A vector field is a special case of a time-dependent vector field; the corresponding time-dependent vector field  $F_\xi$  has domain  $\text{Domain}(F_\xi) = \mathbb{R} \times D$ , and the map  $F : \mathbb{R} \times D \rightarrow \mathbb{R}^n$  is given by  $F_\xi(t, x) := \xi(x)$ .

In view of the time-independence of the corresponding ODE, we choose the convention of always taking initial value problems to start at time  $s = 0$ . The fundamental domain and the flow are also defined slightly differently, so as to eliminate the initial time. The definitions are as follows.

**DEFINITION 6.1.** Let  $\xi : D \rightarrow \mathbb{R}^n$  be a locally Lipschitz vector field on a domain  $D \subset \mathbb{R}^n$ .

- (i) The *maximal integral curve* for  $\xi$  through  $x \in D$  is the maximal integral curve

$$\Gamma_x : \mathcal{I}_x \rightarrow D$$

where  $\Gamma_x := \Gamma_{(x,0)}$  and  $\mathcal{I}_x := \mathcal{I}_{(x,0)}$ .

- (ii) The *fundamental domain* of  $\xi$  is the domain

$$\mathcal{U}_\xi^0 := \{(t, x) ; t \in \mathcal{I}_x\} \subset \mathbb{R} \times D.$$

- (iii) The flow of  $\xi$  is the map  $\Phi_\xi : \mathcal{U}_\xi^0 \rightarrow D$  defined by

$$\Phi_\xi(t, x) = \Gamma_x(t).$$

- (iv) The time- $t$  map is the map  $\Phi_\xi^t$  defined by

$$\Phi_\xi^t(x) = \Phi_\xi(t, x). \quad \diamond$$

**REMARK.** Vector fields that are not locally Lipschitz can also be considered, but without any regularity assumptions we have no general results about the existence or uniqueness of integral curves. In particular, the fundamental domain, the flow and the time- $t$  maps do not exist.  $\diamond$

Note that  $\mathcal{U}_\xi^0$  always contains  $D \times \{0\}$ , and that the time- $t$  maps define the pseudo-group law

$$(6) \quad \Phi_\xi^t \circ \Phi_\xi^s = \Phi_\xi^{t+s}.$$

The link between the autonomous and time-dependent scenarios is the identity

$$\Phi_{F_\xi}^{t,s} = \Phi_\xi^{t-s}.$$

An additional feature afforded to autonomous ODE is the fact the any two maximal integral curves that meet are in fact identical. Equivalently, distinct maximal integral curves never meet. This separation of the integral curves is the content of the uniqueness aspect of Theorem 4.5 in the case of autonomous equations.

**REMARK (Suspension).** As noted, autonomous vector fields are special cases of time-dependent vector fields. In a certain sense, however, the converse is also true. More precisely, let  $F$  be a time-dependent vector field on a domain  $D \subset \mathbb{R}^n$ . Define the vector field  $\xi^F : \text{Domain}(F) \rightarrow \mathbb{R} \times \mathbb{R}^n$  by the formula

$$\xi^F(s, x) := (1, F(s, x))$$

The flow of the vector field  $\xi^F$  is given by the formula

$$\Phi_{\xi^F}(t, (s, x)) = (t, \Phi_F(t, s, x))$$

and hence the time- $t$  maps are

$$\Phi_{\xi_F}^t(x, s) = (s + t, \Phi_F^{s+t, s}(x)).$$

It is therefore possible to extract the flow of  $F$  from that of  $\xi_F$ . If one can find the latter flow, this is of course possible.

One might wonder if time-dependent vector fields can be completely done away with. The slight annoyance is that in the proof of Theorem 4.5 the regularity required from time-dependent vector fields is slightly weaker; when applied to an autonomous ODE, it would require the entire vector field to be locally Lipschitz. Thus if we use the same method or proof to establish Theorem 4.5 for autonomous ODE then we cannot deduce this theorem for time-dependent ODE as a corollary.

Nevertheless, it suffices from here on to focus attention on vector fields from here on.  $\diamond$

## 7. REGULARITY OF SOLUTIONS

The flow of a vector field is constructed by gluing together integral curves. In this process, the regularity<sup>1</sup> of the time- $t$  maps and of the flows is far from clear. As it turns out, the behavior of the flow of a well-behaved vector field is remarkably good.

**7.1. Gronwall's Inequality.** Before beginning our study, we establish the following lemma, whose usefulness in the study of regularity of solutions to ODE cannot be overstated.

**LEMMA 7.1 (Gronwall's Inequality).** *Let  $f, g : [a, b) \rightarrow [0, \infty)$  be continuous functions, and assume there is a constant  $A \geq 0$  such that*

$$f(t) \leq A + \int_a^t f(s)g(s)ds.$$

*Then*

$$f(t) \leq A \exp\left(\int_a^t g(s)ds\right) \quad \text{for all } t \in [a, b).$$

*Proof.* Let  $\varepsilon > 0$ . The function  $h(t) := (A + \varepsilon) + \int_a^t f(s)g(s)ds$  is positive and satisfies  $h'(t) = f(t)g(t) \leq h(t)g(t)$ . Hence  $\frac{d}{dt} \log h(t) \leq f(t)$ , so  $\log h(t) \leq \int_a^t f(s)g(s)ds + \log h(a) = \log\left(h(a) \exp\left(\int_a^t g(s)ds\right)\right)$ . Since  $h(a) = A + \varepsilon$  we have

$$f(t) \leq h(t) \leq (A + \varepsilon) \exp\left(\int_a^t g(s)ds\right).$$

Since  $\varepsilon > 0$  is arbitrary, the proof is complete.  $\square$

Next we define the required notion of regularity.

**DEFINITION 7.2.** Let  $D \subset \mathbb{R}^n$  be an open set, let  $k \in \mathbb{N}$  and let  $\alpha \in (0, 1]$ . A function  $f : D \rightarrow \mathbb{R}$  is said to be  $\mathcal{C}_{loc}^{k, \alpha}$ —one writes  $f \in \mathcal{C}_{loc}^{k, \alpha}(D)$ —if  $f \in \mathcal{C}^k(D)$  and for every  $x \in D$  and every  $\varepsilon \in (0, \text{dist}(x, D^c))$  there is a positive constant  $K = K(x, \varepsilon)$  such that every  $k^{\text{th}}$  order

<sup>1</sup>The term *regularity* refers to the degree of differentiability, or continuity, etc., of the map in question. The more differentiable a map is, the more ‘regular’. This choice might seem odd to the person that knows that the vast majority of set-theoretic maps are very far from regular in this sense, but this terminology is common, and we shall not break with tradition on this point.

partial derivative  $g_I := \frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_n}}$  of  $f$  (i.e.,  $I = (i_1, \dots, i_n) \in \mathbb{N}^n$  is a multiindex of order  $|I| := i_1 + \dots + i_n = k$ ) satisfies

$$|g_I(x_1) - g_I(x_2)| \leq K|x_1 - x_2|^\alpha \quad \text{for all } x_1, x_2 \in D_\varepsilon(x).$$

In particular,  $\mathcal{C}_{loc}^{0,1}(D)$  is the set of locally Lipschitz functions on  $D$ .

For a map  $F = (f^1, \dots, f^m) : D \rightarrow R \subset \mathbb{R}^m$ ,  $F \in \mathcal{C}_{loc}^{k,\alpha}(D, R)$  if  $f^1, \dots, f^m \in \mathcal{C}_{loc}^{k,\alpha}(D)$ , i.e.,  $F$  is  $\mathcal{C}_{loc}^{k,\alpha}$  if and only if each component  $f^j$  of  $F$  is  $\mathcal{C}_{loc}^{k,\alpha}$ .  $\diamond$

**7.2. The result.** The central result about regularity of the time- $t$  maps of a vector field  $\xi \in \mathcal{C}_{loc}^{k,\alpha}(D)$  on a domain  $D$  is the following theorem.

**THEOREM 7.3 (Smooth dependence on Initial Conditions).** *Let  $D \subset \mathbb{R}^n$  be a domain and let  $\xi : D \rightarrow \mathbb{R}^n$  be a  $\mathcal{C}_{loc}^{k,1}$  vector field. Denote by  $\Phi_\xi : \mathcal{U}_\xi^o \rightarrow D$  the flow of  $\xi$ .*

- For any open set  $U \subset\subset D$  and each  $t \in \mathbb{R}$  such that the time- $t$  map  $\Phi_\xi^t$  is defined on  $U$ ,  $\Phi_\xi^t \in \mathcal{C}_{loc}^{k,1}(U)$ .
- For each  $x \in D$  the integral curve  $\gamma_x : \mathcal{I}_x \ni t \mapsto \Phi_\xi^t(x)$  is in  $\mathcal{C}^{k+1}(\mathcal{I}_x)$ .
- The flow  $\Phi_\xi : \mathcal{U}_\xi^o \rightarrow D$  is  $\mathcal{C}^{k,1}$ .

*Proof.* We begin with the case  $k = 0$ . In this case the fact that  $\gamma_x \in \mathcal{C}^1(\mathcal{I}_x)$  is a part of Theorem 4.5, so we need only show that  $\Phi_\xi$  is locally Lipschitz. We begin by showing that  $\Phi_\xi^t$  is locally Lipschitz on its domain of definition. By the pseudogroup law it suffices to assume that  $t \in [-\varepsilon, \varepsilon]$  for some sufficiently small  $\varepsilon$ . Let  $x \in D$  and let  $\varepsilon > 0$  be so small that  $\Phi_\xi^t(y) \in D$  if  $y \in B_\varepsilon(x)$  and  $t \in [-\varepsilon, \varepsilon]$ . For any  $x_1, x_2 \in B_\varepsilon(x)$  consider the function  $f(t) := \|\Phi_\xi^t(x_1) - \Phi_\xi^t(x_2)\|$ . Then

$$f(t) = \left\| \int_0^t (\xi(\Phi_\xi^s(x_1)) - \xi(\Phi_\xi^s(x_2))) ds + x_1 - x_2 \right\| \leq \|x_1 - x_2\| + K \int_0^t f(s) ds,$$

where  $K$  is the local Lipschitz constant of  $\xi$  on  $B_\varepsilon(x)$ . By Gronwall's Inequality

$$(7) \quad \|\Phi_\xi^t(x_1) - \Phi_\xi^t(x_2)\| \leq e^{K|t|} \|x_1 - x_2\| \leq e^{\varepsilon K} \|x_1 - x_2\|,$$

which proves a. We already know from Theorem 4.5 that the integral curve  $\gamma_x$  is  $\mathcal{C}^1$ , i.e., that b holds. Finally, if  $t_1, t_2 \in I_\varepsilon(t) := (t - \varepsilon, t + \varepsilon)$  and  $U \subset\subset D$  is such that  $(t - \varepsilon, t + \varepsilon) \times U \subset \mathcal{U}_\xi^o$  then

$$\begin{aligned} \|\Phi_\xi^{t_1}(x_1) - \Phi_\xi^{t_2}(x_2)\| &\leq \|\Phi_\xi^{t_1}(x_1) - \Phi_\xi^{t_2}(x_1)\| + \|\Phi_\xi^{t_2}(x_1) - \Phi_\xi^{t_2}(x_2)\| \\ &\leq \left( \sup_{(\tau, x) \in I_\varepsilon(t) \times D} \|\xi(\Phi_\xi^\tau(x))\| \right) |t_1 - t_2| + \|\Phi_\xi^{t_2}(x_1) - \Phi_\xi^{t_2}(x_2)\| \\ &\leq \left( \sup_{(\tau, x) \in I_\varepsilon(t) \times D} \|\xi(\Phi_\xi^\tau(x))\| \right) |t_1 - t_2| + e^{K\varepsilon} \|x_1 - x_2\| \end{aligned}$$

where the second inequality follows from the Mean Value Theorem and the third inequality is (7). Thus c holds, and the case  $k = 0$  is proved.

Let us now turn to the case  $k = 1$ , i.e., assume  $\xi \in \mathcal{C}_{loc}^{1,1}(D)$ . For fixed  $x \in D$  consider the linear time-dependent vector field  $F_x(t, y) := d\xi(\Phi_\xi^t(x))y$ . Let us write  $\Psi(t, x)y := \Phi_o^t(y)$ , where  $\Phi_o^t$  is the time-dependent flow of  $F_x(t, y)$ . By Corollary 4.6  $\Phi_o^t(y)$  depends linearly on  $y$ , which is to say,  $\Psi(t, x)$  lies in the space  $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$  of linear maps of  $\mathbb{R}^n$  to itself. Moreover,  $\Psi(t, x)$  is

invertible because  $\Phi_t^0 \circ \Phi_0^t = \Phi_t^t = \text{Id}$ . We can therefore think of  $d\xi(\Phi_\xi^t(x))$  as a vector field on the linear space  $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ .

By its definition, the curve  $t \mapsto \Psi(t, x) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$  satisfies the differential equation

$$\frac{\partial}{\partial t} \Psi(t, x) = d\xi(\Phi_\xi^t(x)) \Psi(t, x)$$

with the initial condition  $\Psi(0, x) = \text{Id}$ . We claim that the map  $(t, x) \mapsto \Psi(t, x)$  is continuous. Indeed, since  $\xi \in \mathcal{C}_{loc}^{1,1}(D)$ ,  $d\xi$  is locally Lipschitz, and by the first part of the proof we have already seen that  $\Phi_\xi$  is locally Lipschitz. Therefore

$$\|\Psi(t, x)\| = \left\| \text{Id} + \int_0^t d\xi(\Phi_\xi^s(x)) \Psi(s, x) ds \right\| \leq 1 + \int_0^t \|d\xi(\Phi_\xi^s(x))\| \cdot \|\Psi(s, x)\| ds$$

and Gronwall's Inequality yields  $\|\Psi(t, x)\| \leq \exp\left(\int_0^t \|d\xi(\Phi_\xi^s(x))\| ds\right)$ . In particular,  $\|\Psi(t, x)\|$  is locally uniformly bounded in  $x$  and  $t$ .

Now, if  $x_1, x_2$  are sufficiently close to  $x$  then, since  $\Psi(0, x_1) = \Psi(0, x_2) = \text{Id}$ ,

$$\begin{aligned} & \Psi(t, x_1) - \Psi(t, x_2) \\ &= \int_0^t (d\xi(\Phi_\xi^s(x_1)) \Psi(x_1, s) - d\xi(\Phi_\xi^s(x_2)) \Psi(s, x_2)) ds \\ &= \int_0^t ((d\xi(\Phi_\xi^s(x_1)) - d\xi(\Phi_\xi^s(x_2))) \Psi(s, x_1) + d\xi(\Phi_\xi^s(x_2)) (\Psi(s, x_1) - \Psi(s, x_2))) ds, \end{aligned}$$

and hence, since we observed that  $d\xi \circ \Phi_\xi^t$  is locally Lipschitz uniformly in  $t$  and we've just shown that  $\|\Psi(t, x)\|$  is locally uniformly bounded in  $x$  and  $t$ ,

$$\|\Psi(t, x_1) - \Psi(t, x_2)\| \leq A\|x_1 - x_2\| + K \int_0^t \|\Psi(s, x_1) - \Psi(s, x_2)\| ds.$$

Thus by Gronwall's Inequality again,

$$\|\Psi(t, x_1) - \Psi(t, x_2)\| \leq A\|x_1 - x_2\| e^{Kt}.$$

Moreover, another application of the Mean Value Theorem gives

$$\begin{aligned} \|\Psi(t_1, x_1) - \Psi(t_2, x_2)\| &\leq \|\Psi(t_1, x_1) - \Psi(t_2, x_1)\| + \|\Psi(t_2, x_1) - \Psi(t_2, x_2)\| \\ &\leq \left( \sup_{t_1 \leq t \leq t_2, x} \|d\xi(\Phi_\xi^t(x)) \Psi(t, x)\| \right) |t_1 - t_2| + A\|x_1 - x_2\| e^{K\varepsilon} \end{aligned}$$

which shows that  $\Psi$  is Lipschitz.

Finally, observe that the map  $\widehat{\Psi} : (t, x) \mapsto d\Phi_\xi^t(x)$  satisfies  $\widehat{\Psi}(0, x) = d\Phi_\xi^0(x) = \text{Id}$  and

$$\begin{aligned} \frac{d}{dt} \widehat{\Psi}(t, x) y &= \frac{d}{dt} d\Phi_\xi^t(x) y = \frac{d}{dt} \frac{d}{ds} \Big|_{s=0} \Phi_\xi^t(x + sy) = \frac{d}{ds} \Big|_{s=0} \xi(\Phi_\xi^t(x + sy)) \\ &= d\xi(\Phi_\xi^t(x)) d\Phi_\xi^t(x) y = d\xi(\Phi_\xi^t(x)) \widehat{\Psi}(t, x) y \end{aligned}$$

for all  $y \in \mathbb{R}^n$ . By the uniqueness part of Theorem 4.5  $\widehat{\Psi} = \Psi$ . Thus we have shown that the flow  $\Phi_\xi$  is  $\mathcal{C}^{1,1}$  when  $\xi$  is  $\mathcal{C}^{1,1}$ . Moreover,

$$\frac{d^2}{dt^2} \Phi_\xi^t(x) = \frac{d}{dt} \xi \circ \Phi_\xi^t(x) = d\xi(\Phi_\xi^t(x)) \xi(\Phi_\xi^t(x))$$

which shows that the flow  $\Phi_\xi$  is then  $\mathcal{C}^2$  in  $t$ . This completes the proof of the case  $k = 1$ .

Now suppose the result has been proved up to  $k - 1$ , i.e., we have shown that, for any vector field  $\eta$ , if  $\eta \in \mathcal{C}_{loc}^{k-1,1}(D)$  then  $\Phi_\eta$  is  $\mathcal{C}_{loc}^{k-1,1}$  in  $x$  and  $\mathcal{C}^k$  in  $t$ . We have already computed that

$$\frac{d}{dt}d\Phi_\xi^t(x) = d\xi(\Phi_\xi^t(x))d\Phi_\xi^t(x) \quad \text{and} \quad \frac{d^2}{dt^2}\Phi_\xi^t(x) = d\xi(\Phi_\xi^t(x))\xi(\Phi_\xi^t(x)).$$

As one can verify by repeated application of the chain rule, the right hand sides of both equations are  $\mathcal{C}_{loc}^{k-1,1}$ . Therefore, by our induction hypothesis, so are the solutions. Hence  $\Phi_\xi$  is  $\mathcal{C}_{loc}^{k,1}$  in  $x$  and  $\mathcal{C}^{k+1}$  in  $t$ , and the proof is complete.  $\square$

**COROLLARY 7.4.** *If  $\xi : D \rightarrow \mathbb{R}^n$  is a  $\mathcal{C}^\infty$  vector field then  $\Phi_\xi : \mathcal{U}_\xi^o \rightarrow D$  is  $\mathcal{C}^\infty$ .*

If  $D \subset \mathbb{R}^n$  and  $\xi : D \rightarrow \mathbb{R}^n$  is a real-analytic vector field, it is not immediately clear from Theorem 7.3 that  $\Phi_\xi$  is real-analytic. Nevertheless this is indeed the case.

**THEOREM 7.5.** *If  $\xi : D \rightarrow \mathbb{C}^n$  is a real-analytic vector field then the flow  $\Phi_\xi : \mathcal{U}_\xi^o \rightarrow D$  is real-analytic.*

We shall omit the proof of Theorem 7.5. The interested reader is invited to check that the estimates obtained in the proof of Theorem 7.3 are strong enough to prove that the solution is real-analytic when the vector field  $\xi$  is real-analytic. There are also simplified proofs, using complex analysis, that are beyond the scope of these notes.

## 8. DEPENDENCE ON PARAMETERS

**THEOREM 8.1 (Continuous Dependence on Parameters).** *Let  $P$  be a compact topological space and let  $\xi_p$  be a locally Lipschitz vector field for each  $p \in P$ , with local Lipschitz constant independent of  $p$ . Assume, moreover, that the map*

$$P \times D \ni (p, x) \mapsto \xi_p(x) \in \mathbb{R}^n$$

*is continuous. Then the flow  $\Phi_{\xi_p}$  of  $\xi_p$  depends continuously on  $p$ , in the sense that for each relatively compact open set  $U \subset\subset D$  and each  $\varepsilon > 0$  such that  $\mathcal{U}_{\xi_p}^o$  contains  $U \times (-\varepsilon, \varepsilon)$  for all  $p \in P$  the map*

$$P \times U \times (-\varepsilon, \varepsilon) \ni (p, x, t) \mapsto \Phi_{\xi_p}^t(x) \in D$$

*is continuous.*

*Proof.* We know that the map  $f_{x,p} : t \mapsto \Phi_{\xi_p}^t(x)$  is the unique solution of the integral equation

$$f_{x,p}(t) := x + \int_0^t \xi_p(f_{x,p}(s))ds,$$

or equivalently, the unique fixed point of the contraction mapping

$$T_{x,p}(\phi) : t \mapsto x + \int_0^t \xi_p(\phi(s))ds.$$

Now, for fixed  $(p, x) \in P \times D$  we have

$$|T_{x,p}(\phi)(t) - T_{x,p}(\phi')(t)| \leq K \operatorname{sgn}(t) \int_0^t |\phi(s) - \phi'(s)|ds$$

while for  $(p', x') \in P \times D$

$$|T_{x,p}(\phi) - T_{x',p'}(\phi)| \leq |x - x'| + \operatorname{sgn}(t) \int_0^t |\xi_p(\phi(s)) - \xi_{p'}(\phi(s))| ds.$$

Thus if  $(p, x)$  and  $(p', x')$  are sufficiently close and  $|t|$  is sufficiently small then

$$\begin{aligned} |f_{x,p}(t) - f_{x',p'}(t)| &= |T_{x,p}(f_{x,p}(t)) - T_{x',p'}(f_{x',p'}(t))| \\ &= |T_{x,p}(f_{x,p}(t)) - T_{x',p'}(f_{x,p}(t))| + |T_{x',p'}(f_{x,p}(t)) - T_{x',p'}(f_{x',p'}(t))| \\ &\leq |x - x'| + \operatorname{sgn}(t) \int_0^t |\xi_p(f_{x,p}(s)) - \xi_{p'}(f_{x,p}(s))| ds \\ &\quad + K \operatorname{sgn}(t) \int_0^t |f_{x,p}(s) - f_{x',p'}(s)| ds. \end{aligned}$$

By Gronwall's Inequality

$$|f_{x,p}(t) - f_{x',p'}(t)| \leq \left( |x - x'| + \operatorname{sgn}(t) \int_0^t |\xi_p(f_{x,p}(s)) - \xi_{p'}(f_{x,p}(s))| ds \right) e^{K|t|}.$$

Now, since the integral curve  $\{f_{x,p}(s) ; s \in [-|t|, |t|]\}$  is a compact set,  $p' \mapsto \xi_{p'}$  is uniformly continuous on this set. Finally,

$$\begin{aligned} &|f_{x,p}(t) - f_{x',p'}(t')| \\ &\leq |f_{x,p}(t) - f_{x',p'}(t)| + |f_{x',p'}(t) - f_{x',p'}(t')| \\ &\leq \left( |x - x'| + \operatorname{sgn}(t) \int_0^t |\xi_p(f_{x,p}(s)) - \xi_{p'}(f_{x,p}(s))| ds \right) e^{K|t|} + \left| \int_t^{t'} \xi_{p'}(f_{x',p'}(s)) ds \right| \end{aligned}$$

The desired continuity easily follows.  $\square$

If we are willing to allow our parameter space  $P$  to be an open set in Euclidean space then Theorem 8.1 has a much stronger generalization, which can be proved by a simple application of our work in Section 7.

**COROLLARY 8.2.** *Let  $P \subset \mathbb{R}^m$  be a domain and let  $\{\xi_p ; p \in P\}$  be a family of  $\mathcal{C}_{loc}^{k,1}$  vector fields whose local Lipschitz constant is locally uniform in  $p$ , such that the map*

$$P \times D \ni (p, x) \mapsto \xi_p(x) \in \mathbb{R}^n$$

*is  $\mathcal{C}^k$ . Then the map*

$$P \times D \times \mathbb{R} \ni (p, x, t) \mapsto \Phi_{\xi_p}^t(x) \in D,$$

*wherever it is defined, is  $\mathcal{C}^k$ .*

*Proof.* Consider the vector field  $\tilde{\xi}$  on  $D \times P$  defined by  $\tilde{\xi}(x, p) := (\xi_p(x), 0)$ . By hypothesis this vector field is in  $\mathcal{C}_{loc}^{k,1}$ , and hence by Theorem 7.3 its flow  $\Phi_{\tilde{\xi}} : \mathcal{U}_{\tilde{\xi}}^o \rightarrow D \times P$  is  $\mathcal{C}_{loc}^{k,1}$ . But this flow is uniquely determined by the differential equation, and one can check directly that the map

$$D \times P \ni (x, p) \mapsto (\Phi_{\xi_p}^t(x), p)$$

solves the equation. Therefore  $\Phi_{\tilde{\xi}}^t(x, p) \equiv (\Phi_{\xi_p}^t(x), p)$ , and the desired smoothness follows from Theorem 7.3.  $\square$

## 9. COMPLETE VECTOR FIELDS

The pseudo-group law (6) is not a group law only because integral curves are not defined for a long enough time, i.e., even if  $t$  and  $s$  both lie in the domains of their respective integral curves,  $t+s$  may not. The situation in which this failure does not happen is therefore particularly important, and we study it in more detail now.

**DEFINITION 9.1.** A vector field  $\xi : D \rightarrow \mathbb{R}^n$  is said to be *complete* (sometimes also called *completely integrable*) if the domain of every maximal integral curve is  $\mathbb{R}$ .  $\diamond$

We have the following simple Proposition.

**PROPOSITION 9.2.** Let  $\xi : D \rightarrow \mathbb{R}^n$  be a  $\mathcal{C}_{loc}^{k,1}$  vector field defined on a domain  $D \subset \mathbb{R}^n$ . Then the following are equivalent.

- (i)  $\xi$  is complete.
- (ii) There exists a positive number  $\varepsilon$  such that for each  $x \in D$ ,  $\mathcal{I}_x \supset (-\varepsilon, \varepsilon)$ .
- (iii) For each  $t \in \mathbb{R}$ , the map  $\Phi_\xi^t$  is a  $\mathcal{C}_{loc}^{k,1}$ -diffeomorphism of  $D$ :  $\Phi_\xi^t \in \text{Diff}^k(D) \cap \mathcal{C}_{loc}^{k,1}(D)$ .
- (iv) For some  $t \in \mathbb{R} - \{0\}$ ,  $\Phi_\xi^t \in \text{Diff}^k(D) \cap \mathcal{C}_{loc}^{k,1}(D)$ .
- (v) The set of maps  $\{\Phi_\xi^t\}_{t \in \mathbb{R}}$  is a 1-parameter subgroup of  $\text{Diff}^k(D) \cap \mathcal{C}_{loc}^{k,1}(D)$ .
- (vi) The fundamental domain of  $\xi$  is  $D \times \mathbb{R}$ .

The proof is left to the reader as an exercise.

## 10. APPROXIMATION

In this section we study a technique, initiated by Euler, for the approximation of integral curves and more generally flows. We confine ourselves to autonomous vector fields for the time being.

**DEFINITION 10.1.** Let  $\xi : D \rightarrow \mathbb{R}^n$  be a vector field on a domain  $D \subset \mathbb{R}^n$  and let  $I \subset \mathbb{R}$  be an open interval containing 0. An *algorithm* for  $\xi$  is a map  $H : D \times I \rightarrow D$  such that, with  $H_t(x) := H(x, t)$ ,

- (i)  $H_0 = \text{Id}$ ,
- (ii)  $H(x, \cdot)$  is  $\mathcal{C}^1$  and its derivative is continuous in  $D \times I$ , and
- (iii)  $\frac{\partial H}{\partial t} \Big|_{t=0} = \xi$ .

The basic approximation theorem is the following result.

**THEOREM 10.2.** Let  $H$  be an algorithm for a Lipschitz vector field  $\xi$ . If  $(t, x) \in \mathcal{U}_\xi^0$  then for all  $N \gg 0$ ,  $H_{t/N}^{(N)}(x)$  is defined, and converges to  $\Phi_\xi^t(x)$ . Conversely, if  $H_{t/N}^{(N)}(x)$  is defined and converges for  $t \in [0, T]$  then  $(T, x) \in \mathcal{U}_\xi^0$  and

$$\lim_{N \rightarrow \infty} H_{t/N}^{(N)}(x) = \Phi_\xi^t(x).$$

In both statements, the convergence is locally uniform on  $D \times I$ .

*Proof.* We begin by showing that the convergence holds locally. Toward this end, let  $x_o \in D$ . Then

$$(8) \quad H_t(x) = x + O(t) \quad \text{and} \quad \Phi_\xi^t(x) - H_t(x) = o(t).$$

If  $H_{t/j}^{(j)}(x)$  is well-defined for  $x$  in a small neighborhood of  $x_o$ , for  $j = 1, 2, \dots, N-1$ , then the semi-group law for time- $t$  maps and the first estimate in (8) shows that

$$\begin{aligned} H_{t/N}^{(N)}(x) - x &= H_{t/N}^{(N)}(x) - H_{t/N}^{(N-1)}(x) + H_{t/N}^{(N-1)}(x) - H_{t/N}^{(N-2)}(x) \\ &\quad + \dots + H_{t/N}(x) - x \\ &= NO(t/N) = O(t), \end{aligned}$$

which is small independently of  $N$ , for  $t$  sufficiently small. Thus for  $x$  sufficiently close to  $x_o$  and  $t$  sufficiently small,  $H_{t/N}^{(N)}(x)$  remains close to  $x_o$  for all  $N$ . In other words, with

$$x_j = H_{t/j}^{(j)}(x),$$

$\|x_j - x_o\| < \varepsilon$  for  $x$  sufficiently close to  $x_o$  and  $t$  sufficiently small. From the semi-group law for  $\Phi_\xi^t$ , we also have

$$\begin{aligned} \Phi_\xi^t(x) - H_{t/N}^{(N)}(x) &= (\Phi_\xi^{t/N})^{(N)}(x) - H_{t/N}^{(N)}(x) \\ &= (\Phi_\xi^{t/N})^{(N-1)}(\Phi_\xi^{t/N}(x)) - (\Phi_\xi^{t/N})^{(N-1)}(H_{t/N}(x)) \\ &\quad + \sum_{j=2}^N (\Phi_\xi^{t/N})^{(N-j)}(\Phi_\xi^{t/N}(x_j)) - (\Phi_\xi^{t/N})^{(N-j)}(H_{t/N}(x_j)), \end{aligned}$$

Now, the hypotheses on  $\xi$  imply the estimate (7), as was shown in the beginning of the proof of Theorem 7.3. Repeated application of (7) yields the estimate

$$\begin{aligned} \|\Phi_\xi^t(x) - H_{t/N}^{(N)}(x)\| &\leq \sum_{k=1}^N e^{K|t|(N-k)/N} \|\Phi_\xi^{t/N}(x_{N-k-1}) - H_{t/N}(x_{N-k-1})\| \\ &\leq Ne^{K|t|} o(t/N), \end{aligned}$$

and the last quantity converges, as  $N \rightarrow \infty$ , to 0 uniformly on a small ball centered at  $x_o$  and for all sufficiently small  $t$ . The final estimate uses the second estimate of (8).

Having handled the case of short times, we now proceed to longer times. Toward this end, suppose first that  $\Phi_\xi^t(x)$  is defined for all  $t \in [0, T]$ . By what we have just done, if  $k$  is sufficiently large then

$$\Phi_\xi^{t/k}(y) = \lim_{k \rightarrow \infty} H_{t/k}^{(k)}(y)$$

holds uniformly for  $t \in [0, T]$  and  $y$  in a bounded neighborhood of the curve  $\{\Phi_\xi^t(x); t \in [0, T]\}$ .

Thus

$$\Phi_\xi^t(x) = (\Phi_\xi^{t/k})^{(k)}(x) = \lim_{N \rightarrow \infty} (H_{t/(kN)}^{(N)})^{(k)}(x) = \lim_{N \rightarrow \infty} H_{t/(kN)}^{(Nk)}(x) = \lim_{N \rightarrow \infty} H_{t/N}^{(N)}(x).$$

Conversely, suppose  $t \mapsto H_{t/N}^{(N)}(x)$  converges to a curve  $c : [0, T] \rightarrow D$ . Let

$$S = \{t \in [0, T]; \Phi_\xi^t(x) \text{ is defined and equal to } c(t)\}.$$

Clearly  $0 \in S$ , and from the local result  $S$  is relatively open. Let  $\{t_k\} \subset S$  and suppose  $t_k \rightarrow t$ . Then  $\Phi_\xi^{t_k}(x) \rightarrow c(t)$  so by Theorem 4.5  $\Phi_\xi^t(x)$  is defined, and by continuity,  $\Phi_\xi^t(x) = c(t)$ . Thus  $S$  is closed, and hence  $S = [0, T]$ .

Finally, observe that by existence and uniqueness,  $\Phi_\xi^{-t} = \Phi_{-\xi}^t$ , so the above proof applies to negative times as well.  $\square$