Math 550 - Spring 2025 Practice Midterm Examination

1. Let $S := \{x = (x_1, ..., x_n) \in \mathbb{R}^n \; ; \; ||x|| = 1\}$ denote the unit sphere in $\mathbb{R}^n$ and let $\mathscr{A} \subset \mathscr{C}(X, \mathbb{R})$ be the algebra generated by the functions

$$f_i(x) := e^{-|x_i|^2} x_i + e^{-|x_{i+1}|^2} x_{i+1}, \; 1 \leq i \leq n-1, \; f_n(x) := e^{-|x_n|^2} x_n + e^{-|x_1|^2} x_1 - e^{-1}.$$

a. Show that for any $\varepsilon > 0$ there exists a polynomial $P_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ such that $P_\varepsilon(0) = 0$ and if $x \in S$ then

$$|\sin(x_1 \cdot x_2 \cdot ... \cdot x_n) - P_\varepsilon(f_1(x), f_2(x), ..., f_n(x))| < \varepsilon.$$

b. Show that there is no polynomial P such that $P(0) = 0$ and

$$\sup_{x \in S} \left| e^{\sin(x_1 \cdot x_2 \cdot ... \cdot x_n)} - P(f_1(x), f_2(x), ..., f_n(x)) \right| < \frac{3}{4}.$$

2. Consider the set of functions $f_n : [0, 1] \to [0, \infty]$ defined by

$$f_n(x) := x^{-\frac{1}{2}} e^{\sin(nx)}; \quad n \in \mathbb{N}.$$

Show that there exist a function $g \in L^{3/2}([0, 1])$ and an increasing map $\phi : \mathbb{N} \to \mathbb{N}$ such that for all $h \in L^3([0, 1])$

$$\lim_{n \to \infty} \int_0^1 f_{\phi(n)}(x) h(x) dx = \int_0^1 g(x) h(x) dx.$$

3. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field such that

$$|\mathbf{F}(\mathbf{x})| \leq A(1 + |\mathbf{x}|)$$

for some constant $A > 0$. Show that $\mathbf{F}$ is a complete vector field.

4. Let $\mathscr{Y} = \ell^2(\mathbb{Z})$, equipped with the usual ℓ^2-norm

$$||f|| := \left(\sum_{n \in \mathbb{Z}} |f(n)|^2 \right)^{1/2},$$

and let $\mathscr{X} = \{f \in \mathscr{Y} \; ; \; \sum_{n \in \mathbb{Z}} n^4 |f(n)|^2 < +\infty\}$.

Consider the map $\Phi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ defined by $\Phi(f)(n) := 4nf(n), \quad n \in \mathbb{Z}$.

a. Show that $\mathscr{X}$ is a subspace of $\mathscr{Y}$.
b. Show that Φ is linear and that $\Phi(\mathscr{X}) \subset \mathscr{Y}$.
c. Show that the graph of $\Phi|_{\mathscr{X}} : \mathscr{X} \to \mathscr{Y}$ is a closed subspace of $\mathscr{X} \times \mathscr{Y}$.
d. Show that $\Phi|_{\mathscr{X}} : \mathscr{X} \to \mathscr{Y}$ not continuous.
e. Use the Closed Graph Theorem to prove that $\mathscr{X}$ is not a closed subspace of $\mathscr{Y}$.
f. Show that $\mathscr{X}$ is dense in $\mathscr{Y}$.
g. Find an explicit $f \in \mathscr{Y} \setminus \mathscr{X}$.
h. Define $|| \cdot ||_\sigma : \mathscr{X} \to [0, \infty]$ by $||f||_\sigma = \sqrt{|f(0)|^2 + \sum_{n \in \mathbb{Z}} |n^2 f(n)|^2}$.
Show that $|| \cdot ||_\sigma$ is a norm and that $\mathscr{X}_\sigma := (\mathscr{X}, || \cdot ||_\sigma)$ is a Hilbert space.
i. Does the Closed Graph Theorem imply that $\Phi : \mathscr{X}_\sigma \to \mathscr{Y}$ is continuous?
j. Compute the norm of the linear map $\Phi : \mathscr{X}_\sigma \to \mathscr{Y}$.