MAT 533 S25
PROBLEM SET 9

1. (Folland, Exercise 7.17) Let X be a locally compact Hausdorff space and let μ be a positive Radon measure on X such that $\mu(X) = \infty$.
a. Show that there exists $f \in \mathscr{C}_0(X)$ such that $\int_X f d\mu = \infty$.
b. Show that every positive linear functional on $C_0(X)$ is bounded.

2. (Folland, Exercise 7.21) Let X be a locally compact Hausdorff space, let $\{f_\alpha\}_{\alpha \in A} \subset \mathscr{C}_0(X)$ and let $\{c_\alpha\}_{\alpha \in A} \subset \mathbb{C}$. Suppose that for every finite subset $B \subset A$ there exists a finite complex Radon measure $\mu_B \in \mathscr{M}(X)$ such that

$$||\mu_B|| \leq 1 \quad \text{and} \quad \int_X f_\alpha d\mu_B = c_\alpha \text{ for all } \alpha \in B.$$

Show that there exists a measure $\mu \in \mathscr{M}(X)$ such that

$$||\mu|| \leq 1 \quad \text{and} \quad \int_X f_\alpha d\mu = c_\alpha \text{ for all } \alpha \in A.$$

3. (Folland, Exercise 7.22) Let X be a locally compact Hausdorff space. Show that a sequence $\{f_n\} \subset \mathscr{C}_0(X)$ converges weakly to $f \in \mathscr{C}_0(X)$ if and only if $\sup_n ||f_n|| < +\infty$ and $f_n \to f$ pointwise.

4. (Folland, Exercise 7.27) Let k be a positive integer and let $\mathscr{C}^k([0,1])$ denote the space of k-times continuously differentiable functions on $[0,1]$, with one-sided derivatives at the endpoints. Define

$$||f|| := ||f||_u + \sum_{j=1}^{k} ||f^{(j)}||_u,$$

where $f^{(j)}$ denotes the j-th derivative of f (one-sided at the endpoints). Show that if $I \in \mathscr{C}^k([0,1])^*$ then there exist a unique measure $\mu \in \mathscr{M}([0,1])$ and constants $c_0, ..., c_{k-1} \in \mathbb{C}$ such that

$$I(f) = \int_X f^{(k)} d\mu + \sum_{j=0}^{k=1} c_j f^{(j)}(0).$$