MAT 533 S25
PROBLEM SET 8

1. Let M be a compact submanifold of $\mathbb{R}^n$ and let
$$I(f) := \int_M f|_M d\mu, \quad f \in \mathscr{C}_c(\mathbb{R}^n).$$
Show that I is a positive linear functional, and compute the associated Radon measure.

2. (Folland, Exercise 7.2) Let μ be a Radon measure on a locally compact Hausdorff space X.
a. Show the union $\Omega(\mu)$ of all open subsets $U \subset X$ such that $\mu(U) = 0$ is an open set.

b. Show that $x \notin \Omega(\mu)$ if and only $\int f d\mu > 0$ if for every $f \in \mathscr{C}_c(X, [0,1])$ such that $f(x) > 0$.

3. (Folland, Exercise 7.8) Show that if μ is a Radon measure on a locally compact Hausdorff space X and $\phi \in L^1(\mu)$ is a non-negative function then $\nu(E) := \int_E \phi d\mu$ is a Radon measure.

4. (Folland, Exercise 7.11.) Let X be a locally compact Hausdorff space. Suppose μ is a Radon measure on X such that $\mu(\{x\}) = 0$ for all $x \in X$. Let $A \in B_X$ be a Borel set such that $0 < \mu(A) < \infty$. Show that for every $t \in (0, \mu(A))$ there exists $B \subset A$ such that $\mu(B) = t$.