MAT 533 S25
PROBLEM SET 4

1. Let $\mathscr{H}$ be a complex Hilbert space.

a. Prove the Polarization Identity: for every $x, y \in \mathscr{H}$,

$$\langle x, y\rangle = \frac{1}{4}\sum_{m=0}^{3} ||x + i^m y||^2.$$

b. Show that if $\mathscr{H}'$ is another complex Hilbert space then a linear map $T : \mathscr{H} \to \mathscr{H}'$ is unitary if and only if T is isometric and surjective.

2. (Folland, Exercise 57) Let $\mathscr{H}$ be a complex Hilbert space and let $T : \mathscr{H} \to \mathscr{H}$ be a continuous linear map.

a. Show that there is a unique continuous map $T^* : \mathscr{H} \to \mathscr{H}$ such that $\langle T^*x, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathscr{H}$.

b. Show that $||T^*|| = ||T||$, $||T^*T|| = ||T||^2$, $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$, $(ST)^* = T^*S^*$ and $T^{**} = T$.

c. Show that $T(\mathscr{H})^\perp = \mathrm{Kernel}(T^*)$ and $\mathrm{Kernel}(T)^\perp$ is the closure of $T(\mathscr{H})$.

d. Show that T is unitary if and only if T is invertible and $T^{-1} = T^*$.

3. (Folland, Exercise 58) Let $\mathscr{M}$ be a closed subspace of a Hilbert space $\mathscr{H}$, and let $P : \mathscr{H} \to \mathscr{H}$ be the map sending $x \in \mathscr{H}$ to the unique element $Px \in \mathscr{M}$ such that $x - Px \in \mathscr{M}^\perp$. (The map P is called the *orthogonal projection onto* $\mathscr{M}$.)

a. Show that P is continuous, that $P = P^* = P^2$ and that $\mathrm{Kernel}(P) = \mathscr{M}^\perp$.

b. Conversely, suppose $T : \mathscr{H} \to \mathscr{H}$ is a continuous linear map that satisfies $T = T^* = T^2$. Show that $T(\mathscr{H})$ is closed and that T is the orthogonal projection onto $T(\mathscr{H})$.

c. If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal basis, show that $Px = \sum_{\alpha \in A} \langle x, u_\alpha\rangle\, u_\alpha$.

4. (Folland, Exercise 59) Show that every closed convex subset in a Hilbert space has an element of minimal norm.

5. (Folland, Exercise 67) Let $\mathscr{H}$ be a complex Hilbert space and let $U : \mathscr{H} \to \mathscr{H}$ be a unitary operator. Let $\mathscr{M} := \{x \in \mathscr{H}\ ;\ Ux = x\}$, let $P : \mathscr{H} \to \mathscr{H}$ denote the orthogonal projection onto $\mathscr{M}$, and let

$$S_n := \frac{1}{n}\sum_{j=0}^{n-1} U^j.$$

Show that $S_n \to P$ in the strong operator topology. (Hint: If $x \in \mathscr{M}$ then $S_n x = x$, and if $x = y - Uy$ compute $S_n x$ and show that $S_n x \to 0$. Use Exercise 2d to show that $\mathscr{M} = \{x \in \mathscr{H}\ ;\ U^*x = x\}$, and then apply Exercise 2c with $T = I - U$.)