MAT 533 S25
PROBLEM SET 3

1. (Folland, Exercise 5.22) Let $\mathscr{X}$ and $\mathscr{Y}$ be normed vector spaces and let $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$.

a. Define $T^\dagger : \mathscr{Y}^* \to \mathscr{X}^*$ by $T^\dagger f := f \circ T$. Then $T^\dagger \in \mathscr{L}(\mathscr{Y}^*, \mathscr{X}^*)$ and $||T^\dagger|| = ||T||$.

b. Show that $(T^\dagger)^\dagger(\hat{x}) = Tx$ for all $x \in \mathscr{X}$.

c. Show that $T^\dagger$ is injective if and only if the image of T is dense in $\mathscr{Y}$.

d. Show that if the image of $T^\dagger$ is dense in $\mathscr{X}^*$ then T is injective, and that conversely, if $\mathscr{X}$ is reflexive and T is injective then $T^\dagger$ has dense image in $\mathscr{X}^*$.

2. (Folland, Exercise 5.39) Let $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ be Banach spaces and let $B : \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ be a bilinear map that is separately continuous, i.e., for each $x \in \mathscr{X}$ and each $y \in \mathscr{Y}$ the linear maps $B(x, \cdot) : \mathscr{Y} \to \mathscr{Z}$ and $B(\cdot, y) : \mathscr{X} \to \mathscr{Z}$ are continuous. Show that B is continuous. (Hint: Reduce the problem to proving that $||B(x, y)|| \leq C||x|| \cdot ||y||$ for some $C > 0$.)

3. (Folland, Exercise 5.44) Show that if $\mathscr{X}$ is a first countable topological vector space such that every Cauchy sequence converges then every Cauchy net converges.

4. (Folland, Exercise 5.49) Let $\mathscr{X}$ be an infinte-dimensional Banach space.

a. Show that every non-empty weakl-open set in $\mathscr{X}$ and every non-empty weak*-open set in $\mathscr{X}^*$ is unbounded.

b. Show that every bounded subset of $\mathscr{X}$ is nowhere-dense in the weak topology, and that every bounded subset of $\mathscr{X}^*$ is nowhere-dense in the weak* topology.

c. Show that $\mathscr{X}$ is meager in itself with respect to the weak topology, and that $\mathscr{X}^*$ is meager in itself with respect to the weak* topology.

d. Show that the weak* topology on $\mathscr{X}^*$ is not defined by any translation-invariant metric. (You may use Exercise 5.48**d**, which you solved last semester.)