MAT 533 S25
PROBLEM SET 2

1. Let $\mathscr{X} \subset \mathscr{C}([0,1])$ denote the vector subspace
$$\mathscr{X} = \{f_{N;a_1,\ldots,a_N,;n_1,\ldots,n_N} \ ; \ N \in \mathbb{N}_{>0},\ n_1,\ldots,n_N \in \mathbb{N}_{>0},\ a_1,\ldots,a_N \in \mathbb{C}\},$$
where
$$f_{N;a_1,\ldots,a_N,;n_1,\ldots,n_N}(x) := a_1 e^{n_1 x} + \ldots + a_N e^{n_N x}, \qquad x \in [0,1].$$
Compute the closure of $\mathscr{X}$ in $\mathscr{C}([0,1])$.

2. Consider the figure-8
$$E := \{z \in \mathbb{C}\ ;\ (|z-2i|^2 - 1)(|z|^2 - 1) = 0\}$$
with its relative topology as a subset of $\mathbb{C}$. Is the algebra generated by the functions $f_j : E \to \mathbb{C}$ defined by
$$f_o(z) = z^j, \quad j \in \mathbb{Z}$$
dense in $\mathscr{C}(E)$? Justify your answer rigorously.

3. Let $\mathscr{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathscr{X} \to \mathbb{R}$ be a sublinear functional.

a. Show that $p(0) = 0$ and that p is a convex function on $\mathscr{X}$.

b. Show that p need not be a semi-norm.

4. (Folland, Exercise 5.8) Let $(X, \mathscr{M})$ be a measurable space and let $\mathscr{X}$ be the space of all complex measures on X. Show that
$$||\mu|| := |\mu|(X), \quad \mu \in \mathscr{X}$$
is a norm on $\mathscr{X}$ and that $\mathscr{X}$ is a Banach space with respect to this norm.

5. (Folland, Exercise 5.12) Let $\mathscr{X}$ be a normed vector space and let $\mathscr{M}$ be a proper closed subspace of $\mathscr{X}$. For $x \in \mathscr{X}$, denote by $[x]$ the coset in $\mathscr{X}/\mathscr{M}$ containing x, i.e., $[x] := x + \mathscr{M}$. Prove the following statements.

a. $||[x]|| := \inf\{||x+m|| \ ; \ m \in \mathscr{M}\}$ defines a norm on $\mathscr{X}/\mathscr{M}$.

b. For every $\varepsilon > 0$ there exists $x \in \mathscr{X}$ such that $||x|| = 1$ and $||[x]|| \geq 1 - \varepsilon$.

c. The projection map $\pi : \mathscr{X} \to \mathscr{X}/\mathscr{M}$ has norm 1.

d. If $\mathscr{X}$ is complete then so is $\mathscr{X}/\mathscr{M}$.

e. The quotient-norm topology for $\mathscr{X}/\mathscr{M}$ is the coarsest topology for which the projection map $\pi : \mathscr{X} \to \mathscr{X}/\mathscr{M}$ is continuous.