MAT 533 S25
PROBLEM SET 10

1. (Folland, Exercise 8.8) Let $f \in L^p(\mathbb{R})$. A function $h \in L^p(\mathbb{R})$ is called the L^p-*derivative* of f if

$$\lim_{y \to 0} \left|\left| \frac{\tau_{-y} f - f}{y} - h \right|\right|_p = 0.$$

For f in $L^p(\mathbb{R}^n)$ the L^p-partial derivatives are similarly defined.

Fix $p \in [1, \infty]$ and let $q = \frac{p}{p-1}$ denote its conjugate exponent. Show that if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ and the L^p partial derivative $\partial_j f$ exists then $\partial_j(f * g)$ exists in the ordinary sense, and equals $(\partial_j f) * g$.

2. (Folland, Exercise 8.15) Let $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ for $x \neq 0$ and $\operatorname{sinc}(0) := 1$.

a. For $a > 0$ show that $\hat{\chi}_{[-a,a]}(x) = \check{\chi}_{[-a,a]}(x) = 2a\operatorname{sinc}(2ax)$.

b. Show that

$$\mathscr{H}_a := \left\{ f \in L^2(\mathbb{R}) \ ; \ \hat{f}(\xi) = 0 \text{ for almost every } \xi \in [-a, a]^c \right\}$$

is a Hilbert subspace of $L^2(\mathbb{R})$ and that $\{\sqrt{2a}\operatorname{sinc}(2ax - k) \ ; \ k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathscr{H}_a$.

c. Show that if $f \in \mathscr{H}_a$ then there exists $f_o \in \mathscr{C}_0(\mathbb{R})$ such that $f = f_o$ a.e., and

$$f_o(x) = \sum_{k \in \mathbb{Z}} f_o(k/(2a))\operatorname{sinc}(2ax),$$

with the latter series convergent both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$.

3. (Folland, Exercise 8.18) Let $f \in L^2(\mathbb{R})$.

a. Show that the L^2 derivative f' (in the sense of Exercise 1) exists if and only if $\xi \mapsto \xi \hat{f}(\xi)$ is in L^2, and in this case $\widehat{(f')}(\xi) - 2\pi\sqrt{\text{-}1}\xi\hat{f}(\xi)$.

b. Show that if the L^2 derivative f' exists then

$$\frac{1}{2}\int |f(x)|^2 dx \leq \left(\int |xf(x)|^2 dx \right)^{1/2} \left(\int |f'(x)|^2 dx \right)^{1/2}.$$

(Hint: Note that $\int |f|^2 = \int \left(\frac{d}{dx}(x) \right) |f|^2$, and if the integrals on the right hand side are finite then one can integrate by parts.)

c. (**Heisenberg's Inequality**) For any $x_o, \xi_o \in \mathbb{R}$

$$\int (x - x_o)^2 |f(x)|^2 dx \int (\xi - \xi_o)^2 |\hat{f}(\xi)|^2 d\xi \geq \frac{||f||_2^2}{16\pi^2}.$$

(Hint: Use the function $g(x) = e^{-2\pi\sqrt{\text{-}1}\xi_o x} f(x + x_o)$ to reduce to the case $x_o = \xi_o = 0$.)

4. (Folland, Exercise 8.30) Suppose that $f \in L^1(\mathbb{R}^n)$, that f is continuous at 0, and that $\hat{f} \geq 0$. Show that $\hat{f} \in L^1(\mathbb{R}^n)$