MAT 533 S25
PROBLEM SET 1

1. (Folland, Exercise 4.46.) Prove the locally compact version of the Tietze Extension Theorem.

2. If X is Hausdorff and Y is locally compact then a continuous map $\phi : X \to Y$ is called *proper* if $\phi^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y.

a. Show that if X is Hausdorff, Y is locally compact and there is a proper map $\phi : X \to Y$ then X is locally compact.

b. Show that if X and Y are locally compact Hausdorff spaces then every continuous proper map $\phi : X \to Y$ has closed image.

c. (Folland, Exercise 4.51) Show that a continuous map $\phi \in X \to Y$ is proper if and only if the extension $\phi^* : X^* \to Y^*$ defined by setting $\phi^*(\infty_X) := \infty_Y$ is continuous.

3. (Folland, Exercise 4.57) An open cover $\mathscr{U}$ of a topological space X is called **locally finite** if each $x \in X$ has a neighborhood that intersects only finitely many members of $\mathscr{U}$. If $\mathscr{U}$ and $\mathscr{V}$ are open covers of X then $\mathscr{V}$ is said to be a **refinement** of $\mathscr{U}$ if for each $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ such that $V \subset U$. X is called **paracompact** if every open cover has a locally finite refinement.

a. Show that if X is a σ-compact locally finite Hausdorff space then X is paracompact. In fact, every open cover $\mathscr{U}$ has locally finite refinements $\{V_\alpha\}$, $\{W_\alpha\}$ such that $\overline{V}_\alpha$ is compact and $\overline{W}_\alpha \subset V_\alpha$ for all α. (See Folland for a guiding hint.)

b. Show that if X is a σ-compact locally finite Hausdorff space then for every open cover $\mathscr{U}$ of X there is a partition of unity subordinate to $\mathscr{U}$ and consisting of compactly supported functions.

4. Let $B_r := \{x \in \mathbb{R}^n \; ; \; |x| < r\}$, let $\{\alpha_j\}_{j\in\mathbb{N}} \subset (0, 1]$ and let $0 < C_1 < C_2 < ....$ Consider the set of functions

$$\mathcal{H} := \left\{ f : \mathbb{R}^n \to \mathbb{C} \; ; \; ||f||_{B_j,\alpha_j} < C_j \text{ for all } j \geq 1 \right\},$$

where

$$||f||_{K,\alpha} := \sup_{x\in K} |f(x)| + \sup_{x,y\in K, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

a. Is $\mathcal{H}$ a vector space?

b. Is $\mathcal{H}$ relatively compact?

c. Is $\mathcal{H}$ compact?

In all cases, justify your answer.