MAT 513 Midterm Spring 2017 Solutions

Problem 1. Definition.

Problem 2. Convergence of sequence using definition.

Problem 3. Suppose that $\{a_n\}$ is a sequence with a subsequence $\{a_{n_j}\}$ that converges. Prove that $\{a_n\}$ converges.

Proof. Suppose that $\{a_n\}$ is increasing; the same proof applies when it is decreasing. By the Monotone Convergence Theorem, it suffices to show that $\{a_n\}$ is bounded above. Since $\{a_{n_j}\}$ converges, it is bounded, so there exists $M \in \mathbb{R}$ such that $a_{n_j} \leq M$ for all $j \in \mathbb{N}$. Let $k \in \mathbb{N}$ be arbitrary. Note that $n_k \geq k$. Since $\{a_n\}$ is increasing, we have

$$a_k \leq a_{n_k} \leq M.$$

This completes the proof. □

Problem 4. Let S and T be non-empty, bounded subsets of $\mathbb{R}$ such that $\sup S \in S$ and $\sup T \in T$.

(a) Let $V = S \cup T$. Prove that V is bounded and $\sup V \in V$.

Proof. By assumption, there exist $A, B \in \mathbb{R}$ such that $|s| \leq A$ for every $s \in S$ and $|t| \leq B$ for every $t \in T$. Thus, for every $v \in V$ we have $|v| \leq \max\{A, B\}$, so V is bounded. Since V is non-empty, $\sup V$ exists. We will show that $\sup V = \max\{\sup S, \sup T\}$, in which case, $\sup V \in V = S \cup T$ by assumption. Let $a = \sup S$, $b = \sup T$, and suppose that $a \geq b$. So, we will show that $\sup V = a$.

First note that a is an upper bound of V. Indeed, if $v \in V$, then $v \in S$ or $v \in T$. If $v \in S$, then $v \leq a$. If $v \in T$, then $v \leq b \leq a$.

Second, let c be another upper bound of V. By assumption, $a \in T \subset V$. Therefore, $a \leq c$. This shows that a is the least upper bound of V. □

(b) Let $W = S \cap T$. Is it true that $\sup W \in W$? If so, give a proof. If not, give a counterexample.

Proof. First note that W could be empty, in which case there is no supremum. Even if W is non-empty the statement is false. For example, let $S = (0, 1]$ and observe that $\sup S = 1 \in S$. Then, let $T = (0, 1/2) \cup \{2\}$. Note that $\sup T = 2 \in T$. However, $W = S \cap T = (0, 1/2)$, and $\sup W = 1/2$, which does not lie in $(0, 1/2)$. □

Problem 5. Convergence of series using convergence criteria.