MAT513 Homework 12

Due Wednesday, May 3

1. Let $f(x) = x^2$ on $[\frac{1}{2}, 3]$.

(a) Let $\mathcal{P}$ be the partition $\left\{ \frac{1}{2}, 1, 2, 3 \right\}$ and compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$, where L and U are the lower and upper sums with respect to the partition.

(b) Let $\mathcal{Q} = \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3 \right\}$ and compute $L(f, \mathcal{Q})$ and $U(f, \mathcal{Q})$.

2. Let f be continuous on $[a,b]$ and suppose that $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

Hint: Argue by contradiction.

3. Prove the **Mean Value Theorem for Integrals**: If f is continuous on $[a,b]$, then there exists $c \in (a,b)$ for which

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

(This value $f(c)$ is called the **average value of f** on the interval $[a,b]$.)

Hint: Use the classical mean value theorem to an appropriate function that is differentiable.

4. Assume that functions $u(x)$ and $v(x)$ have continuous derivatives on $[a,b]$. Derive the formula for **integration by parts**:

$$\int_a^b u(t)v'(t)\, dt = \Big(u(b)v(b) - u(a)v(a) \Big) - \int_a^b v(t)u'(t)\, dt.$$

5. Write an explanation of the intuitive heuristic behind the second part of the Fundamental Theorem of Calculus: Let $G(x)$ be the function that measures the area under the graph of g from a to x. Then the derivative of G at x is the height $y = g(x)$, since the approximate change from x to $x+h$ is essentially the area of the rectangle with base h and height y.

You might want to include a relevant picture, and relate this to the explicit statement of the relevant part of the FTC. Give your explanation so that it can be understood by a beginning calculus student.