MAT513 Homework 11
Due Wednesday, April 26

1. Show that if $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c}(f(x)/g(x))$ does not exist as a real number.

Also, show that if $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = L \neq 0$, then $\lim_{x\to c}(f(x)/g(x)) = 0$.

2. In 2005, police in Scotland installed cameras at certain points along the A77 roadway to record license numbers and automatically calculate the average speed of individual cars between certain points along the road, then automatically issue speeding tickets to drivers whose average speed exceeded the limit. Some drivers objected that merely recording their positions at certain times was no proof that they were speeding at any given moment, especially since they slowed down when passing the cameras.

Write a paragraph or so responding to the claim, probably with some appeal to the mean value theorem.

3. Let f be twice differentiable on an open interval containing the point c.

(a) Show that $f'(c) = \lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ and $f'(c) = \lim\limits_{h\to 0} \dfrac{f(c)-f(c-h)}{h}$.

(b) Show that $f'(c) = \lim\limits_{h\to 0} \dfrac{f(c+h)-f(c-h)}{2h}$.

(c) Show that $f''(c) = \lim\limits_{h\to 0} \dfrac{f(c+h)-2f(c)+f(c-h)}{h^2}$. Hint: L'Hospital's rule.

4. Compute the limits below. You can, of course, use your knowledge of standard derivatives from elementary calculus.

(a) $\lim\limits_{x\to 0} x\ln(1+x)$

(b) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{\sin x}{x^3}\right)$

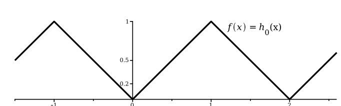
5. As discussed in class, for $x \in \mathbb{R}$, define the function

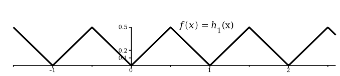
$$h_0(x) = |x|$$

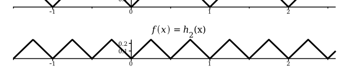
when $-1 \leq x \leq 1$ and extend the function to $\mathbb{R}$ by requiring that $h_0(x) = h_0(x+2)$ for each $x \in \mathbb{R}$. For $n \in \mathbb{N}$, let $h_n(x) = h_0(2^n x)/2^n$.

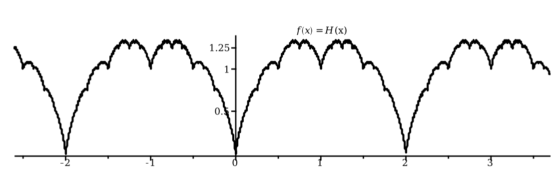
Then let $g(x) = \sum\limits_{n=0}^{\infty} h_n(x)$.

For each $x \in \mathbb{R}$, $g(x)$ converges absolutely, since each term in the series is nonnegative and $g(x) \leq \sum\limits_{n=0}^{\infty} \dfrac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$.









Show that $g(x)$ is continuous on $\mathbb{R}$ and not differentiable at any dyadic point $x = p/2^k$, $p \in \mathbb{Z}$, $k = 0, 1, 2, \ldots$.

(Hint: for continuity, observe that h_k is continuous for all k, so any finite sum these is also continuous – you still need to account for the tail of the series, however.)