MAT513 Homework 10

Due Wednesday, April 19

1. Suppose that $f\colon [a,b] \to [a,b]$ is continuous. Prove that f has a **fixed point**; that is, that there is a $c \in [a,b]$ so that $f(c) = c$.

2. Assume that the temperature $T(x)$ of a point x on the equator of the Earth is a continuous function. As a corollary to the Intermediate Value Theorem, at every moment there is a point x on the equator with the property that its antipodal point (the point $-x$ which is immediately opposite it on a line through the center of the Earth) has exactly the same tempertature, that is $T(x) = T(-x)$.

Write a paragraph or two explaining this in a way that it can be understood by a high school student.

3. Suppose f is differentiable on an interval A. Prove that if $f'(x) \neq 0$ on A, then f must be one-to-one on A. Give an example that shows the converse does not always hold.

4. Let $f\colon [a,b] \to \mathbb{R}$ be a one-to-one function, and let $B = f([a,b])$. Then there is an inverse function $f^{-1}\colon B \to [a,b]$ given by $f^{-1}(y) = x$ where $f(x) = y$. You may assume that if f is a continuous function, then so is f^{-1}.

Assume f is differentiable on $[a,b]$ with $f'(x) \neq 0$ for every $x \in [a,b]$. Show that f^{-1} is differentiable on B with $\left(f^{-1}\right)'(y) = 1/f'(x)$ where $y = f(x)$.

5. By analogy with the definition of uniform continuity, let's say that a function $f\colon A \to \mathbb{R}$ is **uniformly differentiable** on A if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon \quad \text{whenever} \quad 0 < |x - y| < \delta \text{ with } x, y \in A.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? What about $g(x) = x^3$?

(b) Show that if a function f is uniformly differentiable on an interval A, then the derivative of f must be continuous on A.

6. Let $h : [0,3] \to \mathbb{R}$ be differentiable with $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Show there must be a point c with $h'(c) = 1/3$.

(b) Show there is another point b with $h'(b) = 1/4$.