MAT513 Homework 9
Due Wednesday, April 12

1. Here are several invented definitions which are variations on the definition of continuity. In each case, if you give an example you must justify that it meets the stated criteria.

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **onetinuous** at c if for every $\varepsilon > 0$, we have $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < 1$.
Give an example of a function g that is onetinuous on all of $\mathbb{R}$, and another function h that is continuous at every $c \in \mathbb{R}$, onetinuous at $c = 0$, but not onetinuous at $c = 2$, or explain why no such function can exist.

(b) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **equaltinuous** at c if for every $\varepsilon > 0$, whenever $|x - c| < \varepsilon$ we also have $|f(x) - f(c)| < \varepsilon$.
Give an example of a function f which is not onetinuous at any $c \in \mathbb{R}$, but is equaltinuous at every $c \in \mathbb{R}$, or explain why no such function can exist.

(c) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **lesstinuous** at $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there is a δ with $0 < \delta < \varepsilon$ so that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.
Find a function f which is lesstinuous on all of $\mathbb{R}$ but is nowhere equaltinuous, or explain why no such function can exist.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

2. Let A and B be subsets of $\mathbb{R}$, with $f\colon A \to B$ and $g\colon B \to \mathbb{R}$. Prove that if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the set $K = \{\, x \mid f(x) = 0 \,\}$ is a closed set.

4. Is every bounded and continuous function uniformly continuous? If yes, provide a proof. If no, give a counterexample.

5. Observe that if a and b are real numbers, then we can define $\max(a,b) = \dfrac{(a+b)+|a-b|}{2}$; this can readily be extended to a finite set of numbers $\{\, a_1, a_2, \ldots, a_n \,\}$ via

$$\max\{\, a_1, a_2, \ldots, a_n \,\} = \max(a_1, \max\{\, a_2, a_3, \ldots, a_n \,\}).$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous, then $g(x) = \max(f_1(x), f_2(x), \ldots, f_n(x))$ is also continuous.

(b) For each positive integer n, define

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n \end{cases}$$

For each n, $f_n\colon \mathbb{R} \to \mathbb{R}$ is continuous.
Write an explicit formula for the function
$h(x) = \sup\{\, f_1(x), f_2(x), f_3(x), \ldots \,\}$.
Is $h(x)$ continuous?