MAT513 Homework 8

Due Wednesday, April 5

1. For a function $f\colon A \to \mathbb{R}$ and a limit point c of A, recall that $\lim_{x\to c} f(x) = L$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$ and $x \in A$.

Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x (for example, $\lceil 0.5 \rceil = 1 = \lceil 1 \rceil$).

(a) Suppose we take $\varepsilon = 1$. What is the largest value of δ we can use in the definition of $\lim_{x\to\pi} \lceil x/2 \rceil$?

(b) Suppose we take $\varepsilon = .01$. What is the largest value of δ we can use in the definition of $\lim_{x\to\pi} \lceil x/2 \rceil$?

(c) Write a proof, using the definition of limit above, that $\lim_{x\to\pi} \lceil x/2 \rceil = 2$.

(d) Consider $g(x) = \dfrac{1}{\lceil x \rceil}$. A student makes the (false) claim that $\lim_{x\to 4} g(x) = \dfrac{1}{4}$. Give an explanation of why this cannot be true by exhibiting the largest ε for which there is no δ that satisfies the definition.

2. Calculate each of the limits below or show they do not exist, fully justifying each of your answers. You can use the definition or other facts we know from the definition.

(a) $\lim_{x\to 2} x^3$

(b) $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$

(c) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

3. Introductory calculus courses typically refer to the limit of a function *from the right* (or the left) as the limit obtained by "letting x approach a from the right (or left) side".

Adapt the definition of $\lim_{x\to a} f(x)$ to give a proper definition for the statements

$$\lim_{x\to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^-} f(x) = M \,.$$

Then prove that

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x) \,.$$

4. Thomae's function is defined as follows:

$$T(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ with } p \in \mathbb{Z} \setminus \{0\},\, q \in \mathbb{N} \text{ and } \gcd(p,q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that $\lim_{x\to c} T(x) = 0$ for any $c \in \mathbb{R}$. Conclude that T is continuous at irrationals and discontinuous at rationals.