MAT513 Homework 7

Due Wednesday, March 29

1. Let $A = \left\{ (-1)^n + \frac{1}{n+1} \;\middle|\; n \in \mathbb{N} \right\}$. Answer the following questions about A, justifying your answers fully.

(a) Is A an open set?

(b) Is A a closed set?

(c) Does A contain any isolated points? (If so, identify them.)

(d) What are the limit points of A?

(e) What is $\bar{A}$?

2. In class we showed that if $K \subseteq \mathbb{R}$ is compact, then it must be closed and bounded. Prove the converse: If $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

3. A notion dual to the closure of a set is the interior of a set. Specifically, given a set E, the **interior of E** is denoted by $\mathring{E}$ and is defined as

$$\mathring{E} = \{ x \in E \mid \textit{there exists a neighborhood } V_\varepsilon(x) \subseteq E \}.$$

There is a symmetry between many properties regarding closure and the interior of sets.

(a) Recall that we have showed that $\bar{E}$ is the smallest closed set containing E. Show that a set E is closed if and only if $E = \bar{E}$.

(b) Show that $\mathring{U}$ is the largest open set contained in U.

(c) Show that a set U is open if and only if $U = \mathring{U}$.

(d) Recall that E^c denotes the complement of E, that is, $E^c = \mathbb{R} \setminus E$.
Show that $(\bar{E})^c$ is the interior of (E^c). Also show that $(\mathring{E})^c$ is the closure of E^c.

4. Decide whether each of the following statements is true or false. Prove the true ones, and give a counterexample for the false ones.

(a) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(b) $\overline{A \cap B} = \bar{A} \cap \bar{B}$.

(c) The interior of $A \cup B$ is $\mathring{A} \cup \mathring{B}$.

(d) The interior of $A \cap B$ is $\mathring{A} \cap \mathring{B}$.

5. Below are several statements about compact sets; some are true and some are not. Prove the true ones, and give a counterexample for the false ones.

(a) Let K_n be a compact set for each $n \in \mathbb{N}$. Then $K = \bigcup_{n=1}^{\infty} K_n$ must be compact.

(b) Let F_λ be a compact set for each $\lambda \in \Lambda$. Then $F = \bigcap_{\lambda \in \Lambda} F_\lambda$ must be compact.

(c) Let A be an arbitrary subset of $\mathbb{R}$, and let K be compact. Then $A \cap K$ is compact.

(d) Let K be compact and F be closed. Then $K \setminus F = \{ x \in K \mid x \notin F \}$ is open.