MAT513 Homework 5
Due Wednesday, March 1

1. Let $y_1 = 1$ and for each $n \in \mathbb{N}$, let $y_{n+1} = \dfrac{3y_n + 4}{4}$.

(a) Show that $y_n < 4$ for all $n \in \mathbb{N}$. *Hint:* use induction.

(b) Now show that $y_1, y_2, y_3, \ldots$ forms an increasing sequence.

(c) Conclude that the sequence converges. What is the limit?

2. (**Cesàro Means**) Given a sequence $\{x_n\}$, define a new sequence whose terms are the arithmetic mean of the first n terms. That is, let

$$y_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

(a) Show that if $\{x_n\}$ converges to a limit L, then the sequence of averages $\{y_n\}$ also converges to the same limit.

(b) Give an example where the sequence of averages $\{y_n\}$ converges but the original sequence $\{x_n\}$ does not.

3. (**Calculating square roots**) Consider the sequence defined by $x_1 = 2, \; x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$.

(a) Show that x_n^2 is always greater than 2, and then use this to conclude that $\{x_n\}$ is a decreasing sequence. Then show that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence in part (a) to obtain a sequence that converges to $\sqrt{c}$.

4. An invented definition:
A sequence $\{a_n\}$ is **quasi-increasing** if for all $\varepsilon > 0$, there exists an $K \in \mathbb{N}$ so that whenever $n > m \geq K$ it follows that $a_n > a_m - \varepsilon$.

(a) Give an example of a sequence which is quasi-increasing but is not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? That is, suppose $\{a_n\}$ is bounded and quasi-increasing. Must it also converge? If so prove it; if not, give a counter-example.

5. Prove the **Comparison Test for Series:** Let $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Use the Monotone Convergence Theorem (a bounded monotone sequence has a limit).