**MAT513 Homework 3**
Due Wednesday, February 15

**1.** Explain why the following argument for showing that $\mathbb{Q}$ is uncountable is incorrect.

Assume for contradiction that $\mathbb{Q}$ is countable. Thus, we can write $\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$. For each $n \in \mathbb{N}$ we find a closed interval $I_n$ with distinct endpoints such that $r_n \notin I_n$ and $I_{n+1} \subset I_n$. The nested intervals property implies that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ so the infinite intersection contains a rational number $r_m$. However, $r_m \notin I_m$, a contradiction.

**2.** Find a countable collection of disjoint open intervals. Can you find an uncountable collection of disjoint open intervals?

*Hint:* you may use the fact that an infinite subset of a countable set is countable.

**3.** Let $I = \{\, x : 0 < x < 1 \,\}$ be the open unit interval $(0,1)$, and let $S$ be the open unit square, that is, $S = \{\, (x,y) : 0 < x < 1 \text{ and } 0 < y < 1 \,\} = (0,1) \times (0,1)$.

**(a)** Find an injective function (that is, a one-to-one function) $f : I \to S$. This should be *very easy*: $f$ does not need to be surjective (onto).

**(b)** Use the fact that every real number $x$ has a decimal expansion to produce an injective function $g : S \to I$. Is your function $g$ a surjection (onto)?
It might be helpful to remember that every real number which has a "terminating" decimal expansion (such as 0.25) can also be written as an infinite decimal (e.g., $0.2499\overline{9}\cdots$ or $0.2500\overline{0}\cdots$).

As a consequence of the Schröder-Bernstein Theorem (which says that if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$), this shows that the unit interval and the unit square have the same cardinality.

**4.** A real number $x \in \mathbb{R}$ is called **algebraic** if there are integers $a_0, a_1, a_2, \dots, a_n$ so that

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

that is, $x \in \mathbb{R}$ is algebraic if it is a root of a polynomial with integer coefficients (note that rational numbers are algebraic, since each is the root of a degree 1 polynomial). Real numbers which are not algebraic are called **transcendental** numbers.

**(a)** Show that $\sqrt{2}$ and $\sqrt{3} + \sqrt{2}$ are algebraic.

**(b)** Fix $n \in \mathbb{N}$ and let $A_n$ be set of algebraic numbers which are roots of polynomials of degree $n$. Show that each $A_n$ is a countable set.

*Hint:* the Fundamental Theorem of Algebra is relevant here; you may assume it.

**(c)** Prove that the set of algebraic numbers is a countable set.

**5.** In both parts below, justify your answer fully by establishing an bijection between the set in question and a set of known cardinality.

**(a)** Let $\mathcal{F}$ be the set consisting of all functions from $\{\,0,1\,\}$ to $\mathbb{N}$. What is the cardinality of $\mathcal{F}$?

**(b)** Let $\mathcal{G}$ be the set consisting of all functions from $\mathbb{N}$ to $\{\,0,1\,\}$. What is the cardinality of $\mathcal{G}$?

**6.** * Study but do not submit this problem. Suppose that $A_1, A_2, A_3, \ldots$, are countable sets.

**(a)** Show that $A_1 \cup A_2$ is countable. More generally, $A_1 \cup A_2 \cup \cdots \cup A_n$ is countable for each $n \in \mathbb{N}$.

**(b)** Show that $\bigcup_{i=1}^{\infty} A_i$ is countable.

**(c)** Show that $A_1 \times A_2 = \{(a_1, a_2) : a_1 \in A_1,\ a_2 \in A_2\}$ is countable. More generally, $A_1 \times A_2 \times \cdots \times A_n$ is countable for each $n \in \mathbb{N}$.

**(d)** Is the set $\{(a_1, a_2, a_3, \ldots) : a_i \in A_i,\ i \in \mathbb{N}\}$ countable?