MAT513 Homework 2

Due Wednesday, February 8

1. Find the supremum and infimum of each of the following sets, if they exist.

(a) $\{1+3/n : n \in \mathbb{N}\}$

(b) $\{n/(1+n) : n \in \mathbb{N}\}$

2. Show that for any real numbers $a, b \in \mathbb{R}$ with $a < b$ there exists an irrational number s such that $a < s < b$.

Hint: use the fact that $\sqrt{2}$ is irrational and that $r\sqrt{2}$ is irrational for every rational number r (why?).

3. Let $I_1 \supset I_2 \supset I_3 \supset \ldots$ be nested intervals.

(a) If the intervals are bounded but are not assumed to be closed, is it true that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$?

(b) If the intervals are closed but are not assumed to be bounded, is it true that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$?

4. Provide a sketch of an argument showing that there exists a real number $x > 0$ such that $x^2 = 3$. That is, provide the steps that have to be followed but you do not need to provide all the details for each step.

5. Write a paragraph or two responding to the following statement: “Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, every measurement corresponding to a real number can be approximated by a rational number to within the precision of any device we can use measure it with. Thus, in science or engineering, it suffices to work only with real numbers that are fractions (or finite decimals, if you prefer).”

6. Let $x_1, x_2, x_3, \ldots$ be integers among $0, 1, \ldots, 9$. We define $0.x_1 = x_1/10$, $0.x_1x_2 = x_1/10 + x_2/100$, and more generally

$$0.x_1x_2\ldots x_n = \frac{x_1}{10} + \frac{x_2}{10^2} + \cdots + \frac{x_n}{10^n} = \sum_{i=1}^{n} \frac{x_i}{10^i}.$$

In this way, we can make sense of numbers with finitely many decimal digits.

(a) Show that $0.x_1x_2\ldots x_n < 1$ for each $n \in \mathbb{N}$. Conclude that the set

$$S = \{0.x_1, 0.x_1x_2, 0.x_1x_2x_3, \ldots\}$$

is bounded above by the number 1.

(b) Define $0.x_1x_2x_3\ldots$ to be the supremum of the set S. Therefore, we can now make sense of numbers with infinitely many decimal digits. Show that

$$0.999\cdots = 1.$$

Hint: show that for the number $0.999\ldots$ we have $S = \{1 - 1/10^n : n \in \mathbb{N}\}$ and compute the supremum of S.